Same Stats, Different Graphs

(Graph Statistics and Why We Need Graph Drawings)

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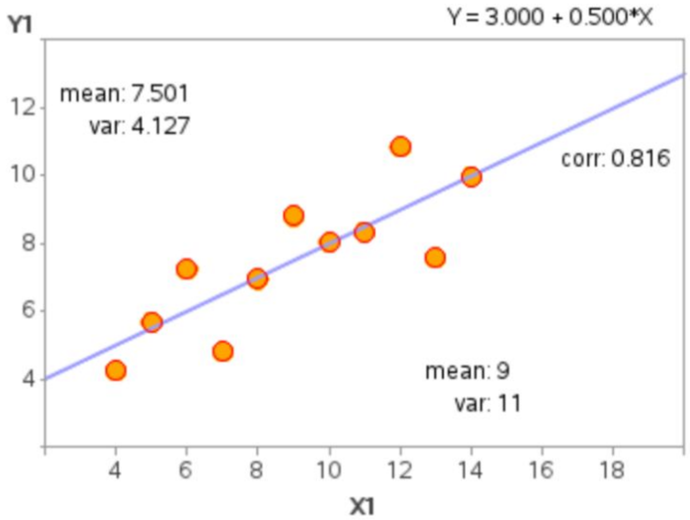
Motivation

Imagine a set of 11 points in 2D with the following summary statistics:

- Mean value of $x = 9$
- Variance of $x = 11$
- Mean value of $y = 7.5$
- Variance of $y = 4.1$
- Correlation between x and $y = 0.8$
- Linear regression line $y = 3 + 0.5x$

Motivation



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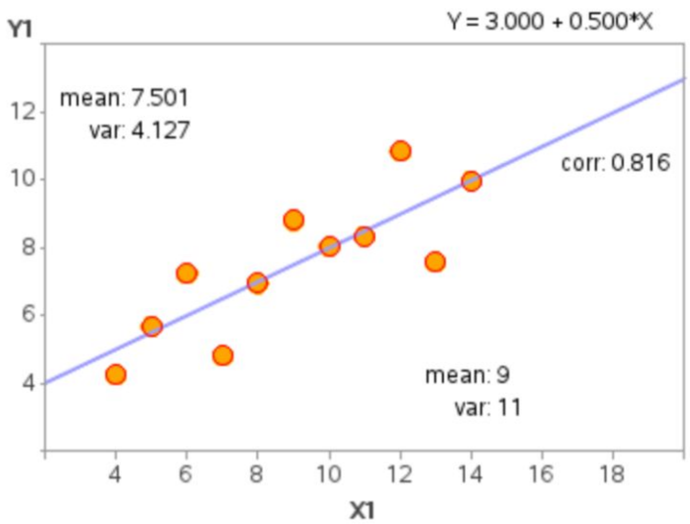

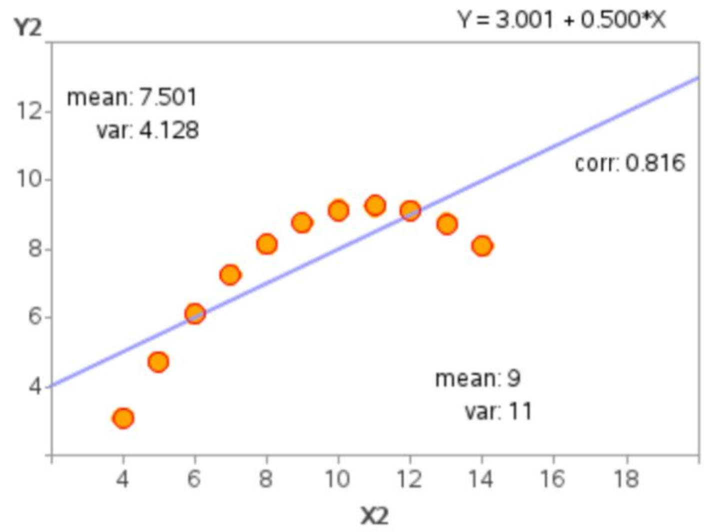

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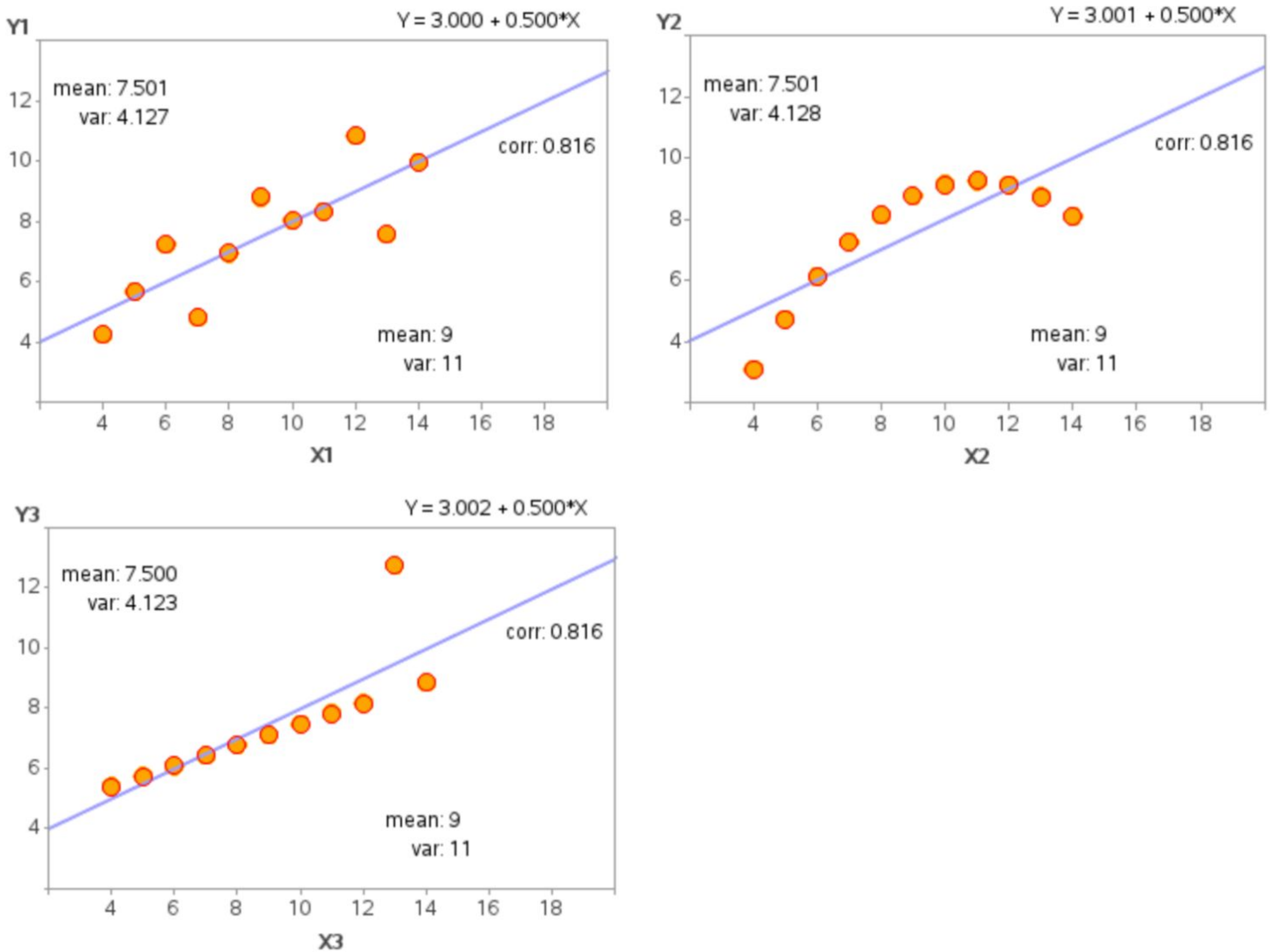

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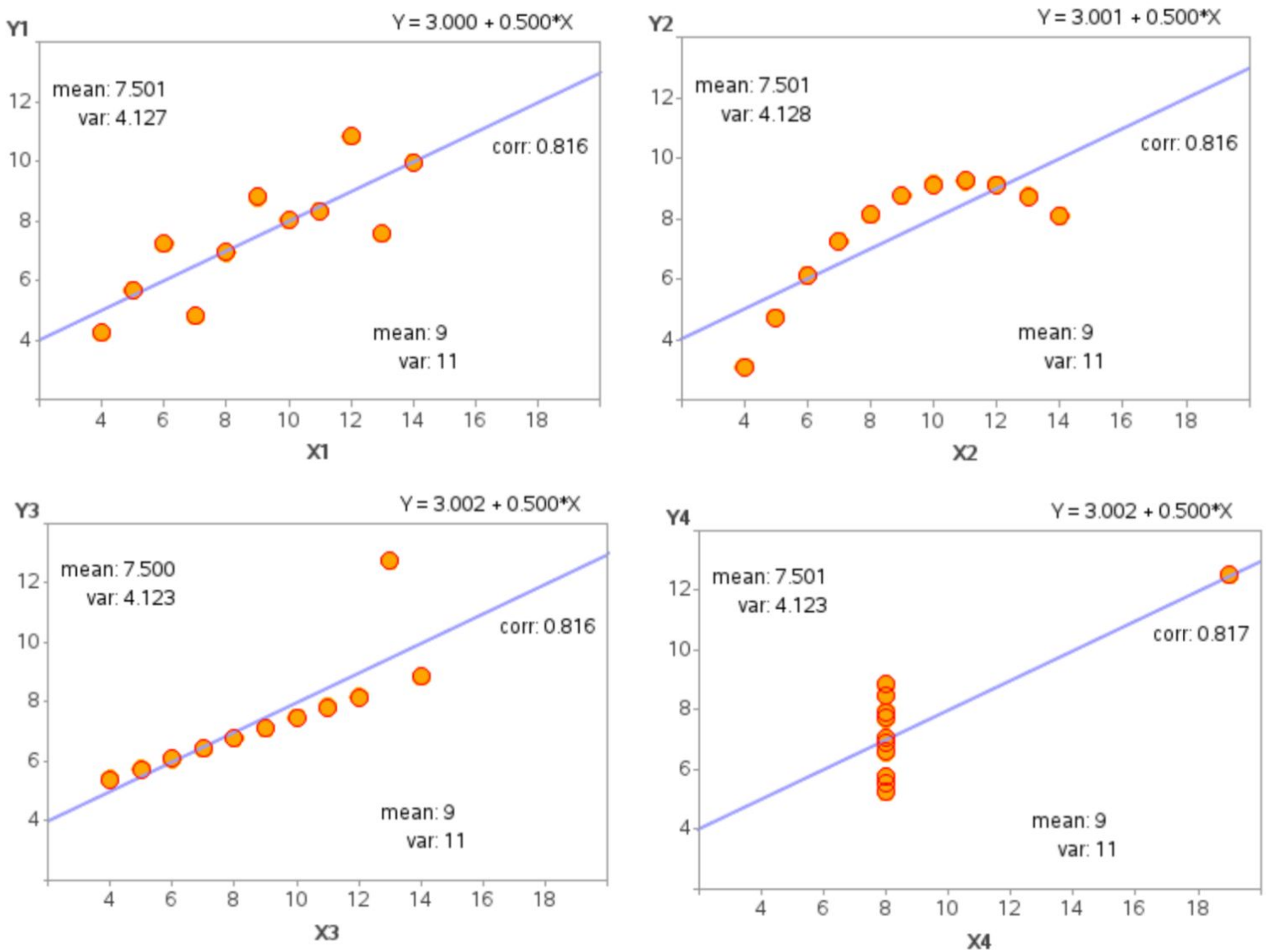

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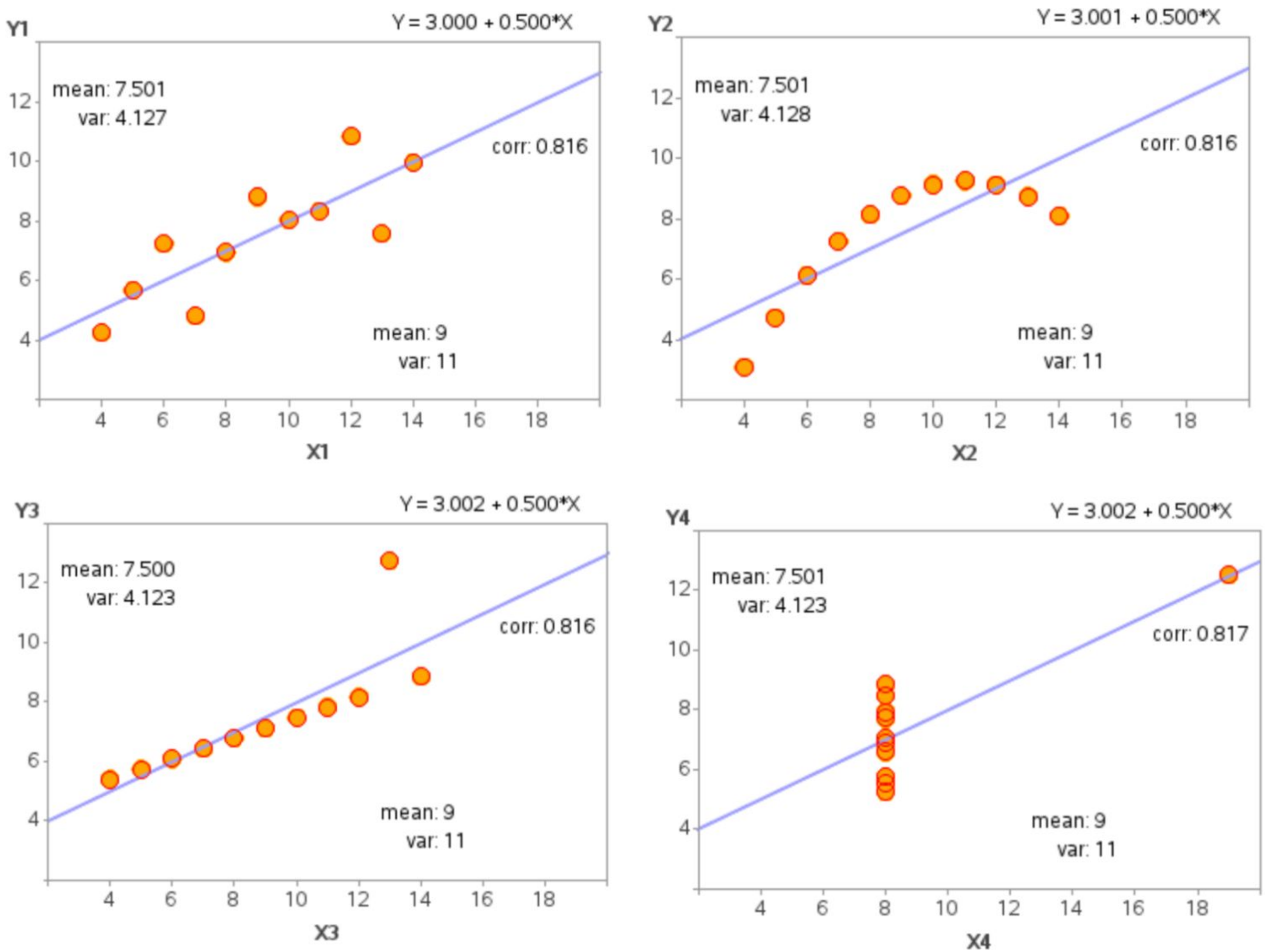
Anscombe's Quartet



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Anscombe, 1973

Anscombe's Quartet

Moral of the story: Summary statistics of a dataset are great, but we should nevertheless look at the data!

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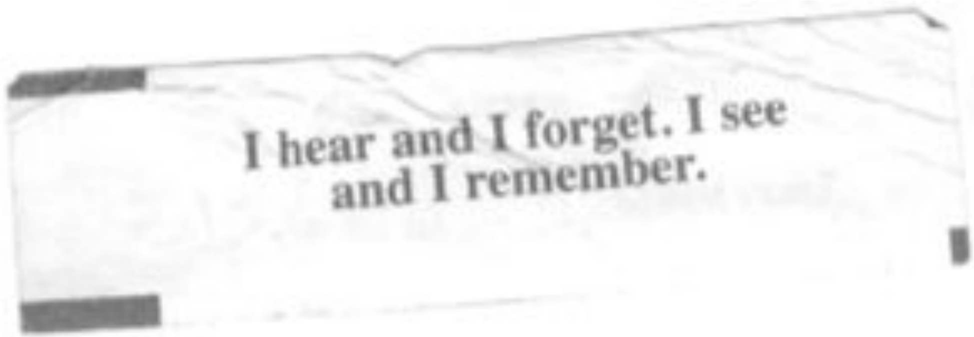
Or in fortune cookie language

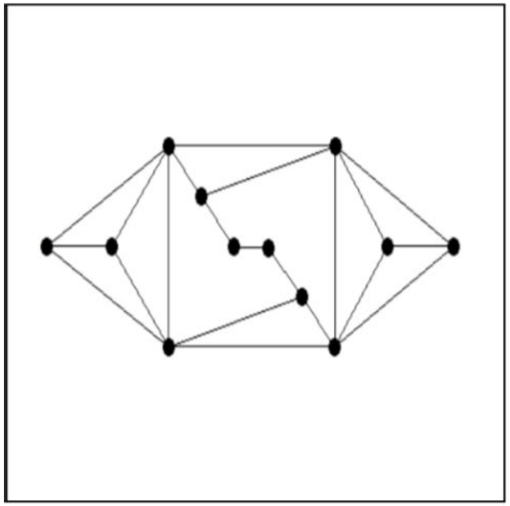


Stephen’s Quartet

Imagine a graph with the following properties (statistics):

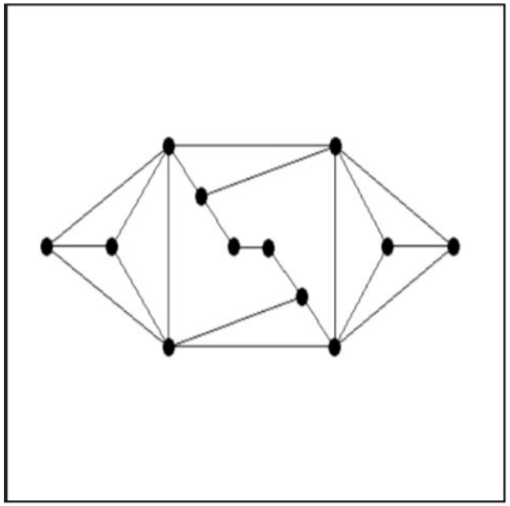
- 12 vertices
- 21 edges
- girth $\gamma = 3$
- number of triangles $\Delta = 10$
- global clustering coefficient = 0.5

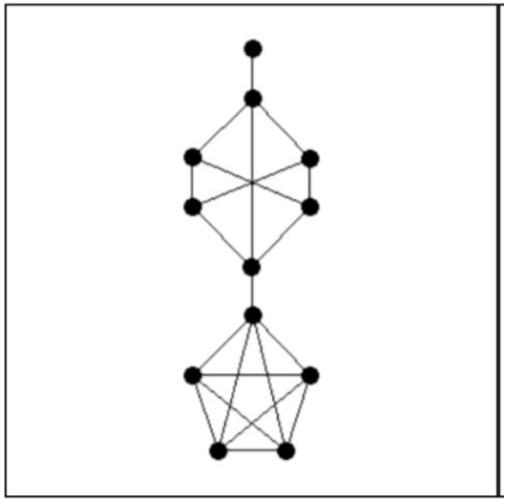
Stephen's Quartet



$|V| = 12$
$|E| = 21$
$\gamma = 3$
$\Delta = 10$
$GCC = 0.5$

Stephen's Quartet





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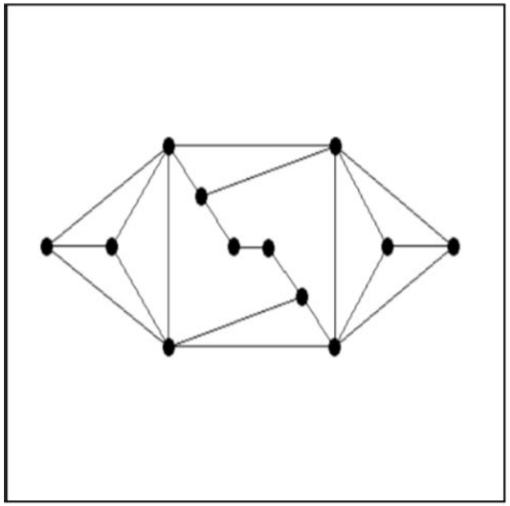

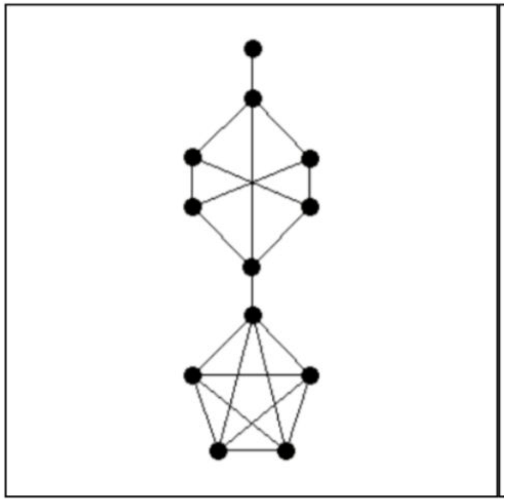

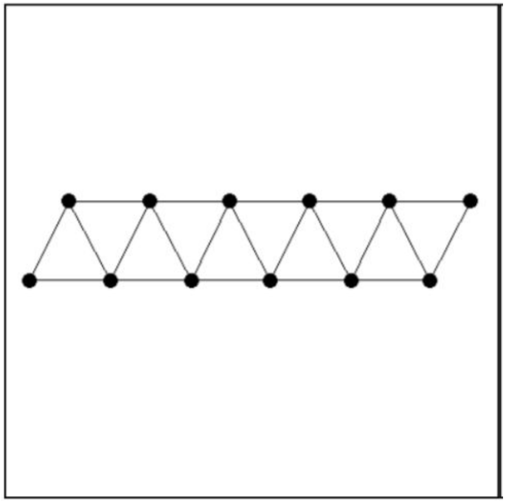

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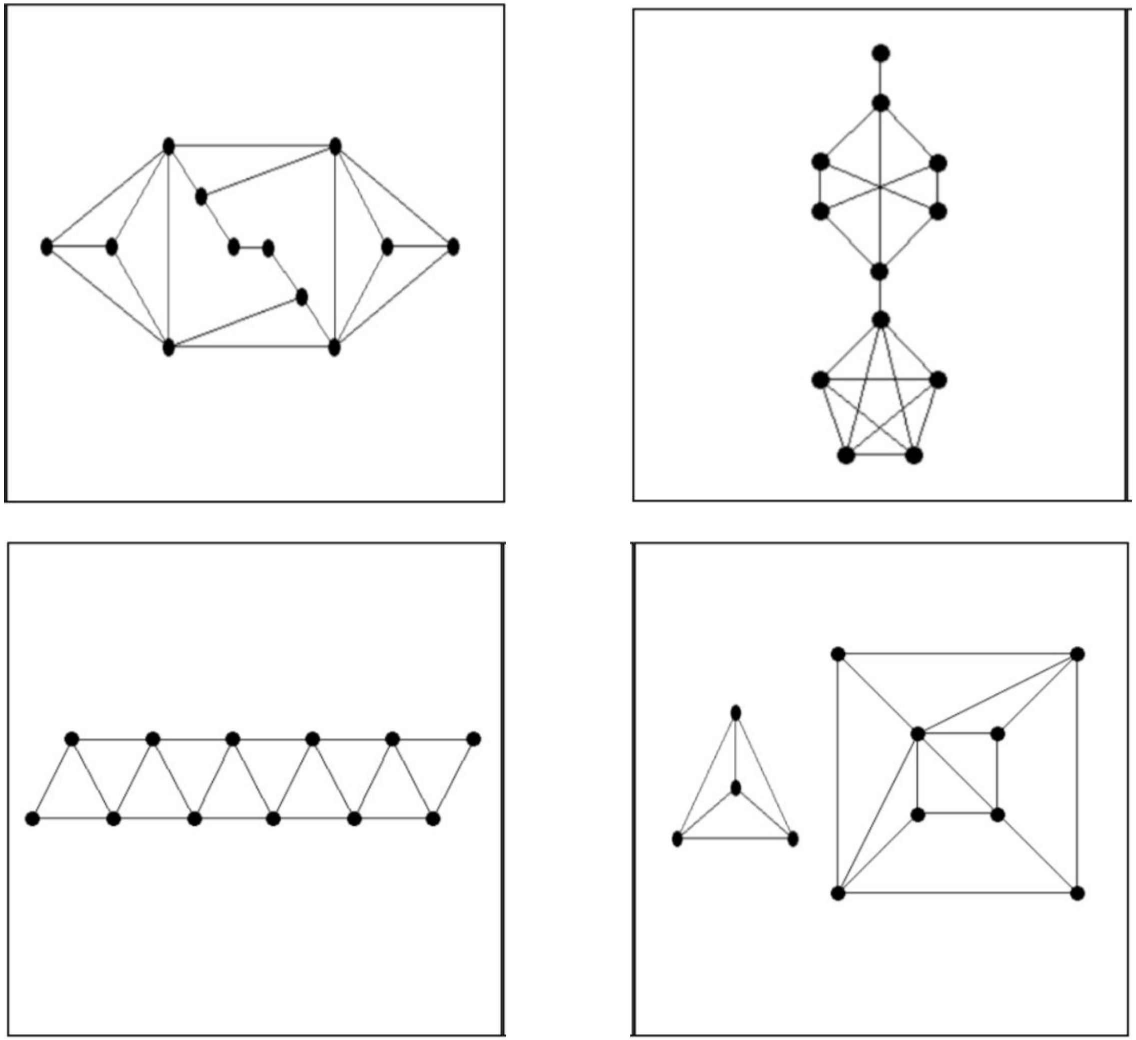

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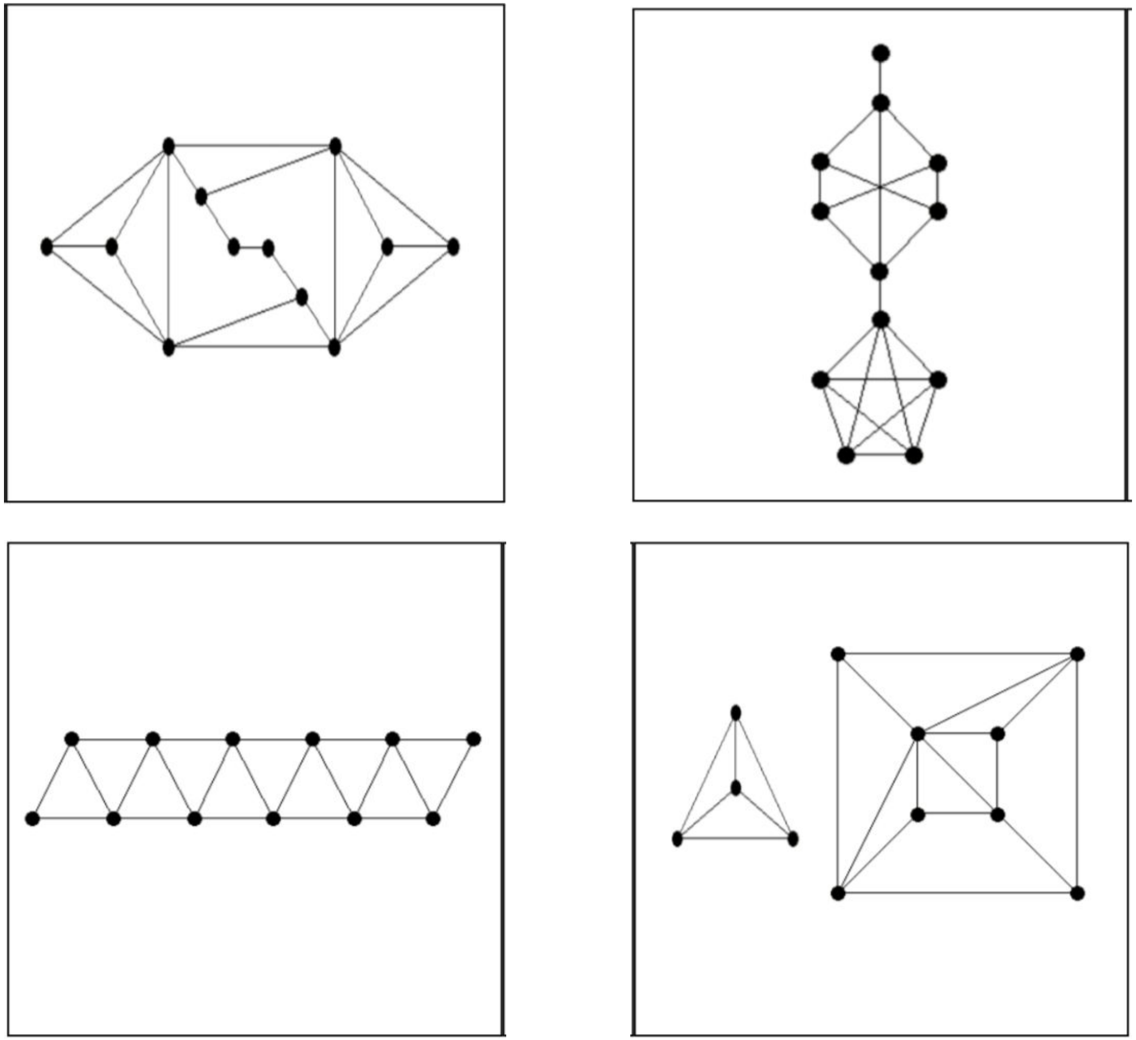

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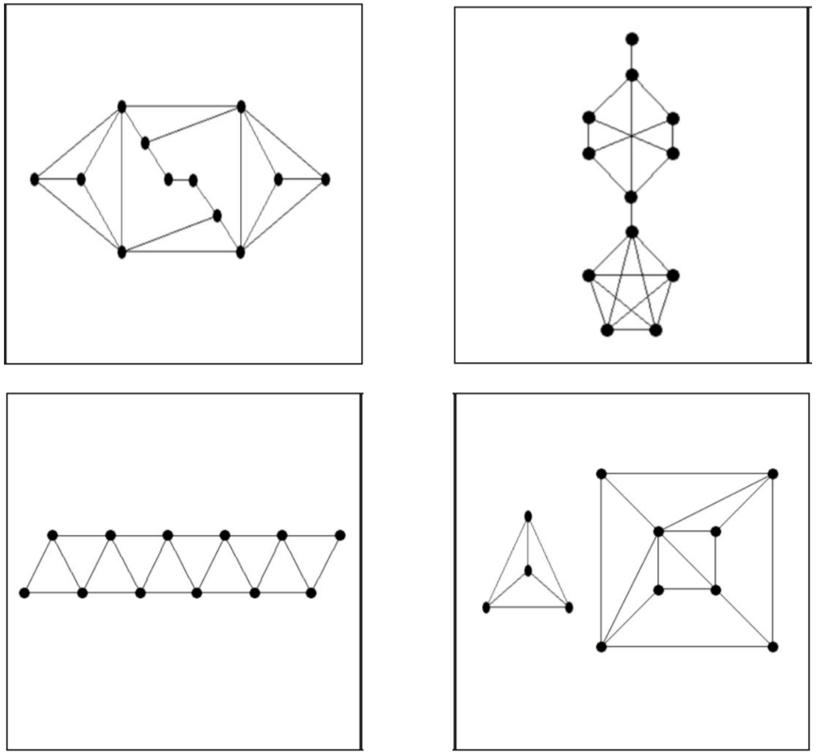
These four graphs have the same 5 statistics but they differ in structure, planarity, connectivity, symmetry, etc.

Stephen's Quartet

Moral of the story: every graph drawing paper could begin with these 4 graphs as the motivation behind "Why We Still Need to Draw our Graphs"



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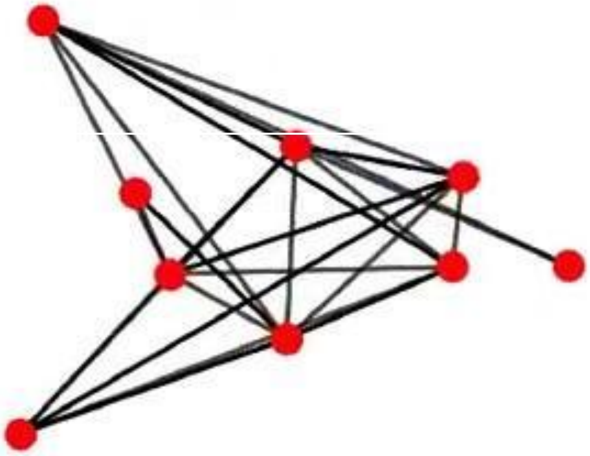

Graph Properties Considered

Average Clustering Coefficient	$ACC(G) = \frac{1}{n}\sum_{i=1}^{n} c(u_i), u_i \in V, n = \|V\|$ $c(v) = \frac{\|\{(u,w)\|u,w \in \Gamma(v),(u,w) \in E\}\|}{\|\Gamma(v)\|(\|\Gamma(v)\|-1)/2}, v, u, w \in V$
Global Clustering Coefficient	$GCC(G) = \frac{3 \times \|triangles\|}{\|connected\ triples\|\ in\ the\ graph}$
Square Clustering	$SCC(G) = \frac{\sum_{u=1}^{k_v}\sum_{w=u+1}^{k_v} q_v(u,w)}{\sum_{u=1}^{k_v}\sum_{w=u+1}^{k_v} [a_v(u,w)+q_v(u,w)]}$
Average Path Length	$APL = ave\{\frac{n-1}{\sum_{v \in V} d(u,v), u \neq v}\}$
Degree Assortativity	$r = \frac{\sum_{xy} xy(e_{xy} - a_x b_y)}{\sigma_a \sigma_b}$
Diameter	$diam(G) = max\{dist(v,w), v, w \in V\}$
Density	$den = \frac{2\|E\|}{\|V\|(\|V\|-1)}$
Ratio of Triangles	$\mathrm{Rt} = \frac{\|triangles\|}{\|V\|(\|V\|-1)/2}$
Node Connectivity	Cv: the minimum number of nodes to remove to disconnect the graph
Edge Connectivity	Ce: the minimum number of edges to remove to disconnect the graph

- normalized to *[0,1]*
- assortativity: *[-1,1]*

Correlations between Graph Properties

	GCC	ACC	SCC	APL	r	diam	den	Rt	Cv	Ce
GCC		1.00	0.86	-0.80	-0.41	-0.61	0.92	0.92	0.99	0.99
ACC	1.00		0.86	-0.80	-0.41	-0.61	0.19	0.92	0.99	0.99
SCC	0.20	0.86		0.10	-0.32	0.13	-0.14	-0.05	-0.13	-0.12
APL	-0.80	-0.80	0.10		0.70	0.94	-0.43	-0.64	-0.76	-0.76
r	-0.41	-0.41	-0.32	0.70		0.71	0.65	-0.29	-0.37	-0.37
diam	-0.61	-0.61	0.13	0.94	0.71		-0.35	-0.38	-0.54	-0.54
den	0.92	0.19	-0.14	-0.43	0.65	-0.35		0.93	0.99	0.99
Rt	0.92	0.92	-0.05	-0.64	-0.29	-0.38	0.93		0.95	0.95
Cv	0.99	0.99	-0.13	-0.76	-0.37	-0.54	0.99	0.95		1.00
Ce	0.99	0.99	-0.12	-0.76	-0.37	-0.54	0.99	0.95	1.00	

Colour scale: 0.9, 0.6, 0.3, 0.0, −0.3, −0.6

* Data from EuroVis'18 paper where we generated 4950 graphs with 100 vertices

Correlations between Graph Properties

Can we trust the numbers from the previous table?

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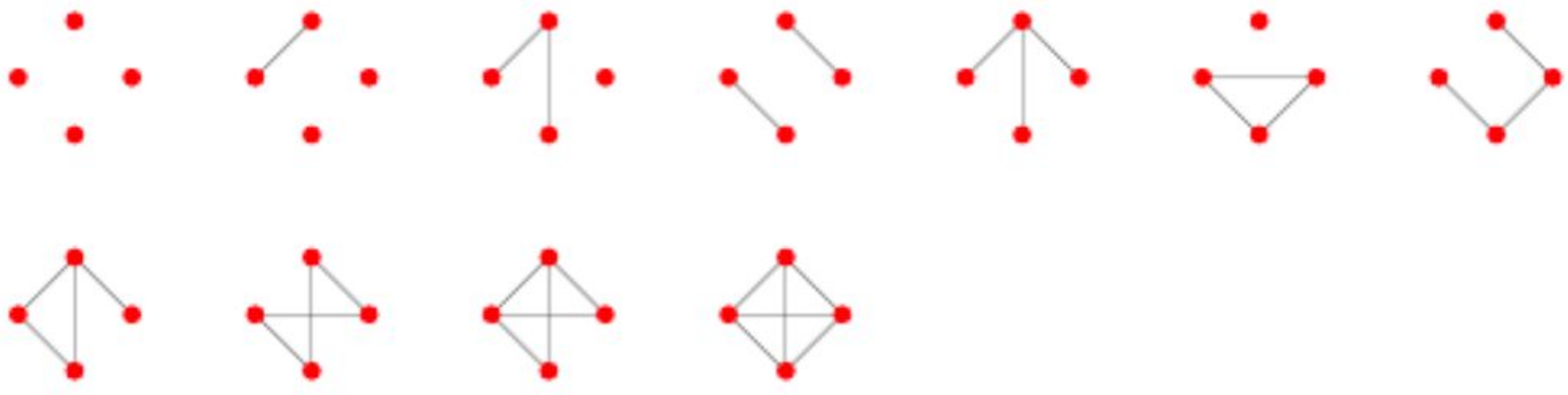

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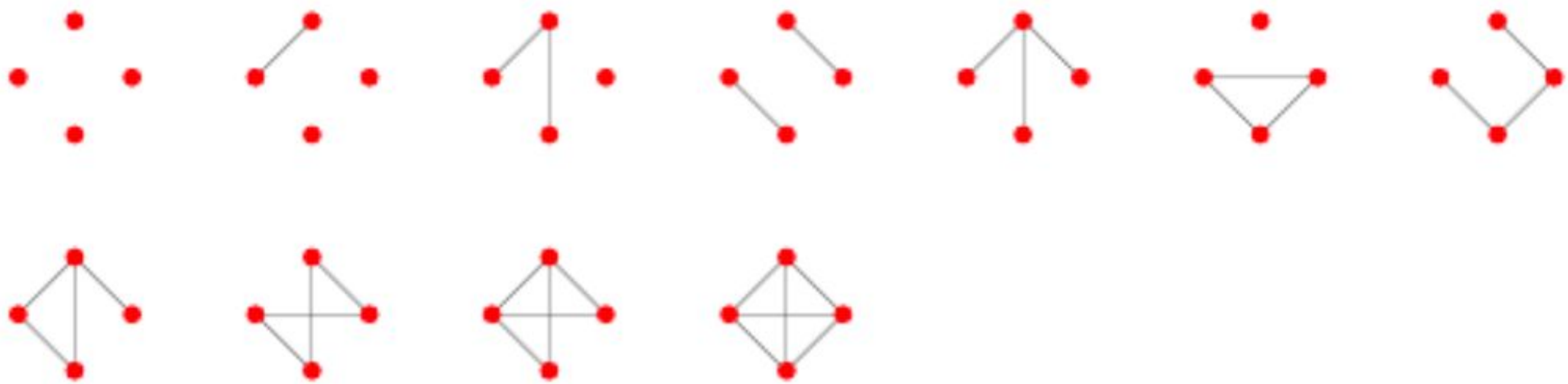

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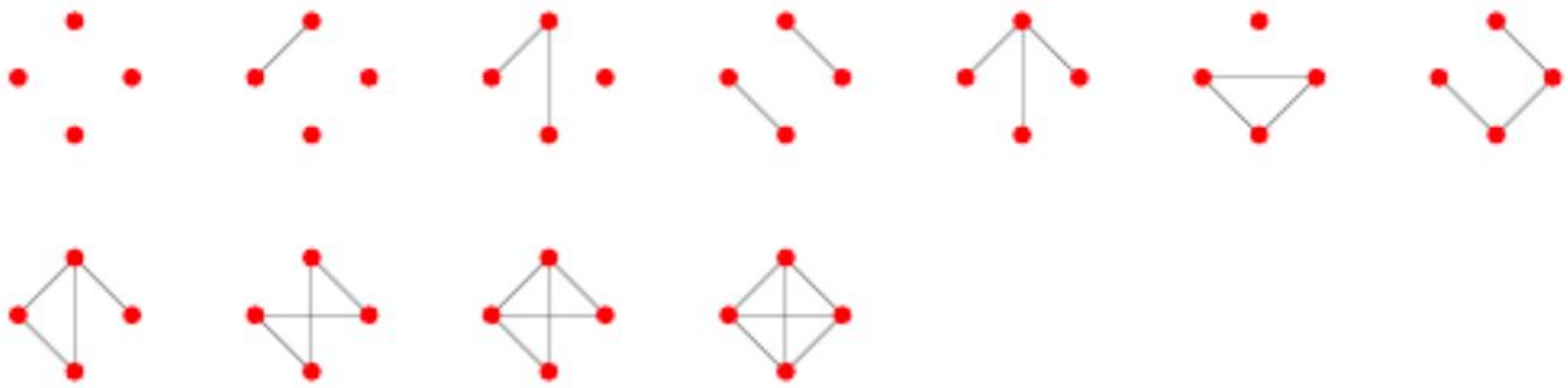
For $|V| = 1, 2 \ldots 9$ the numbers are 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, and for $|V| = 16$ we have 6×10^{22}



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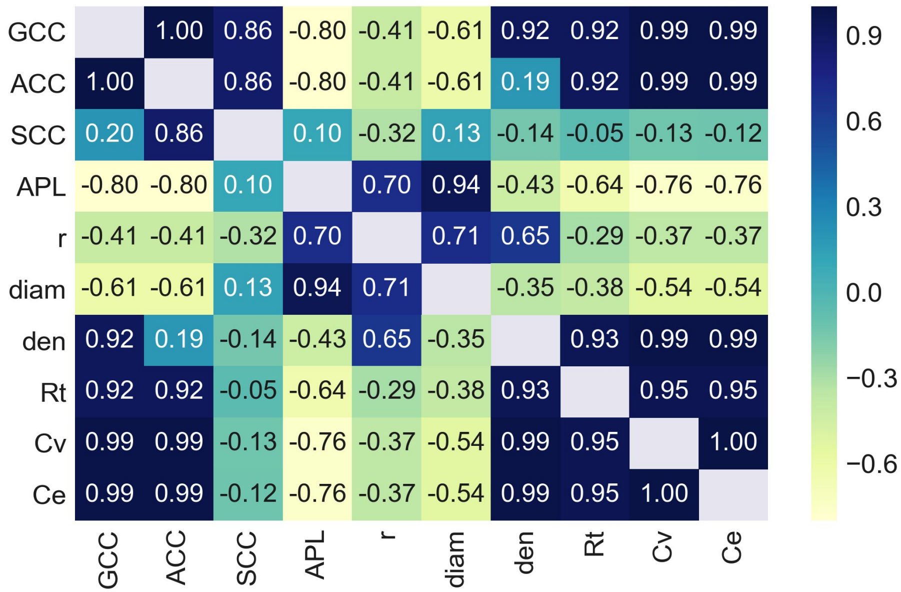
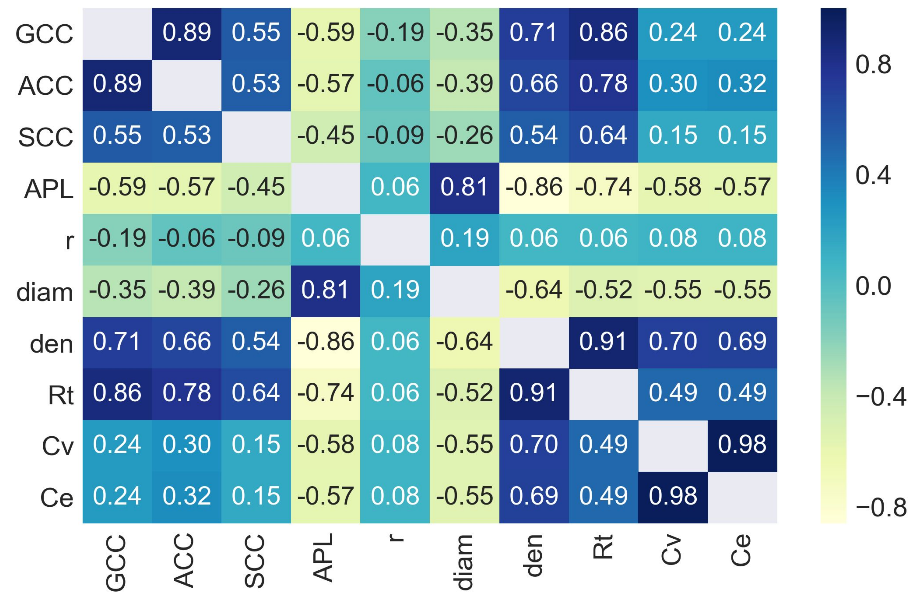
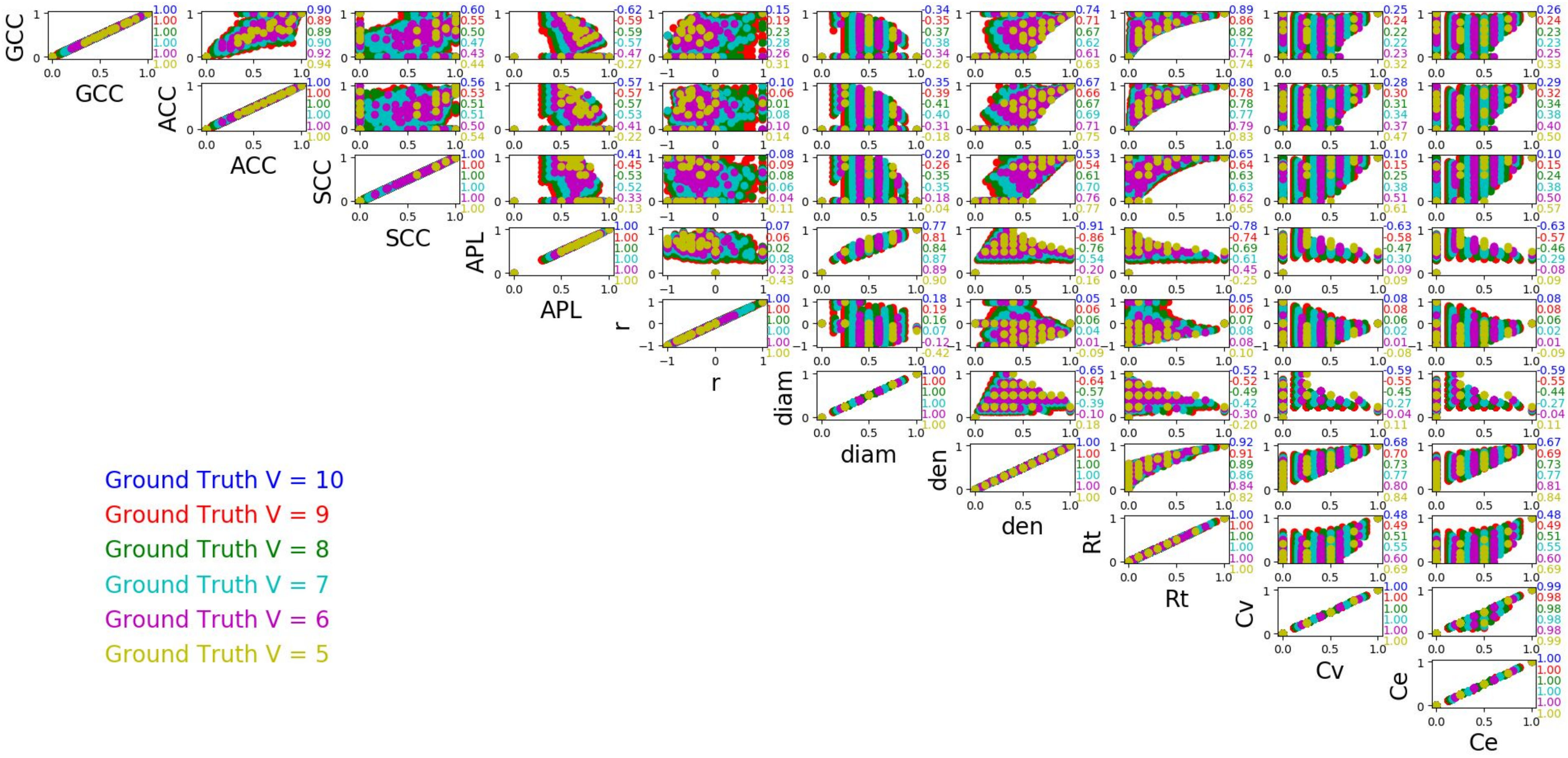


4950 graphs with 100 vertices from EuroVis'17 experiment



ground truth for |V|=9 and the results are different...

Ground Truth Data for Small |V|



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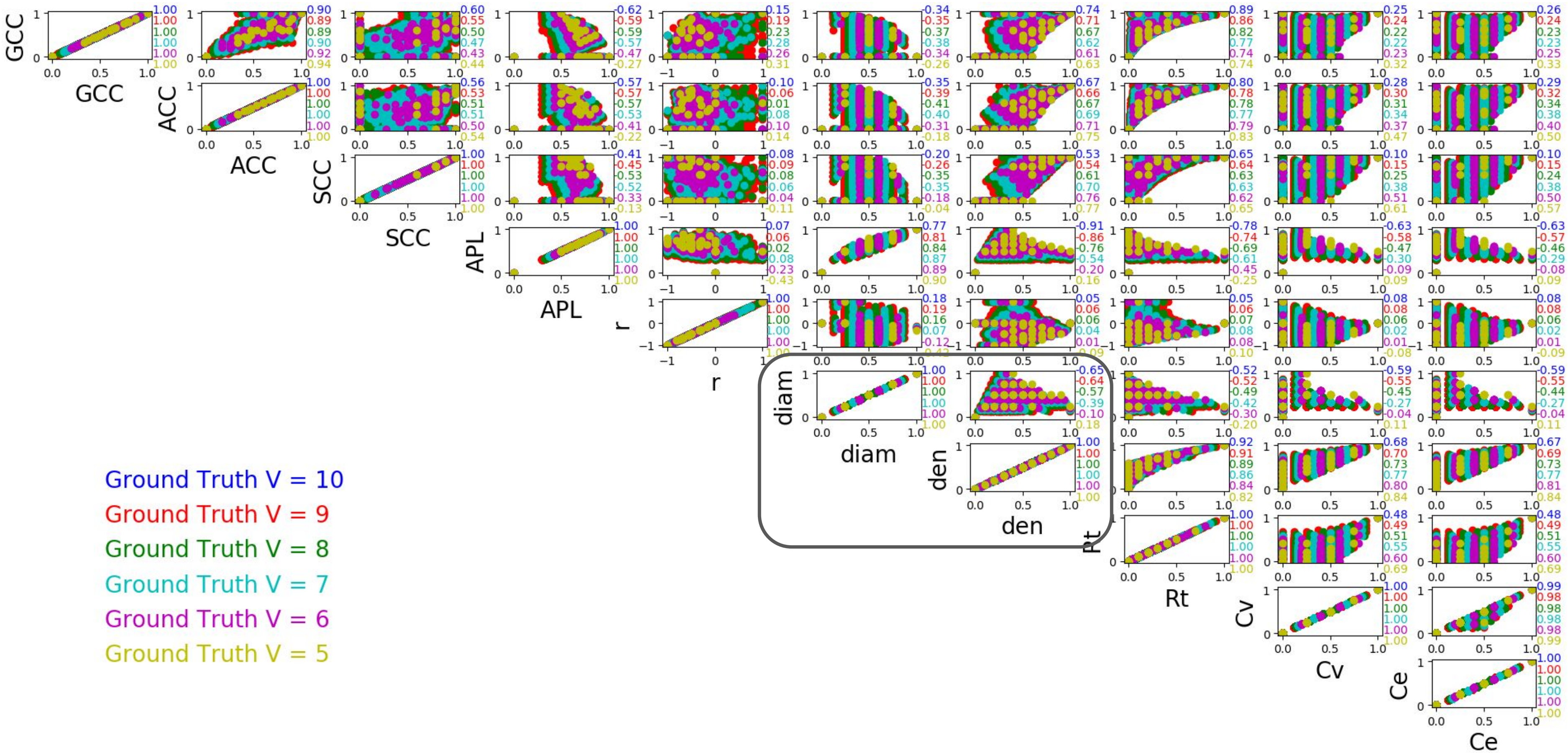
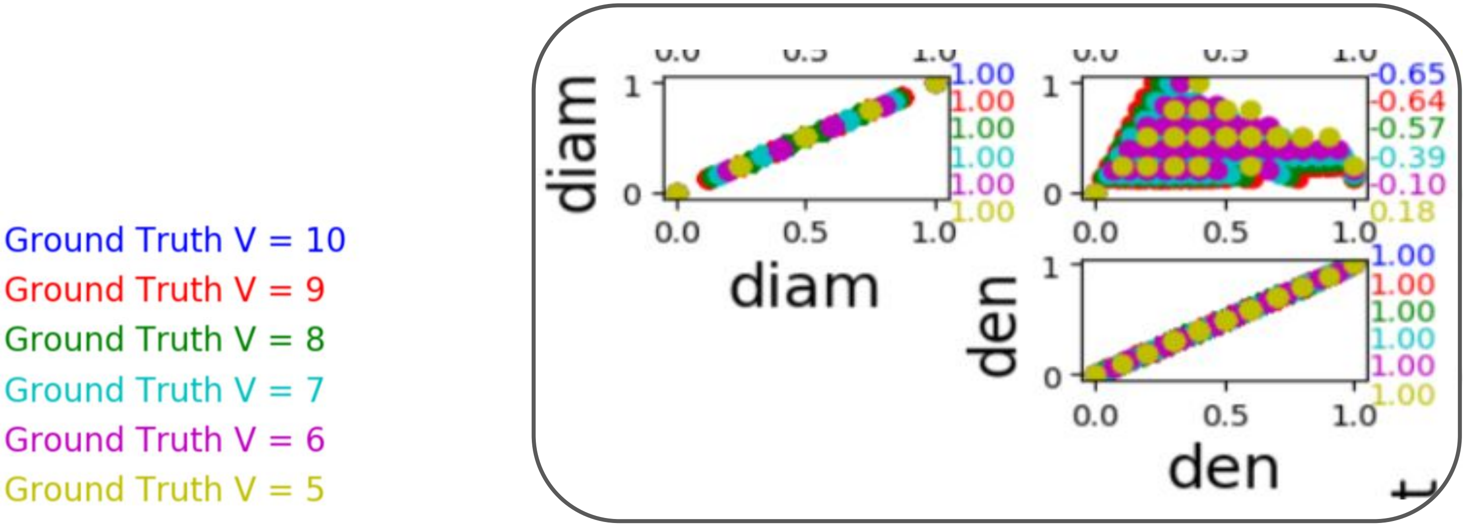


Ground Truth Data for Small |V|

Let’s look at one of these more carefully

As |V| grows the correlation changes!



Graph Generators to the Rescue

We cannot explore the ground data for large values of |V|, so let’s use generators

- Erdos-Renyi
- Watts-Strogatz
- Barabasi-Albert
- geometric

But which generator does a **good job** in this context?

What do we want from a graph generator?

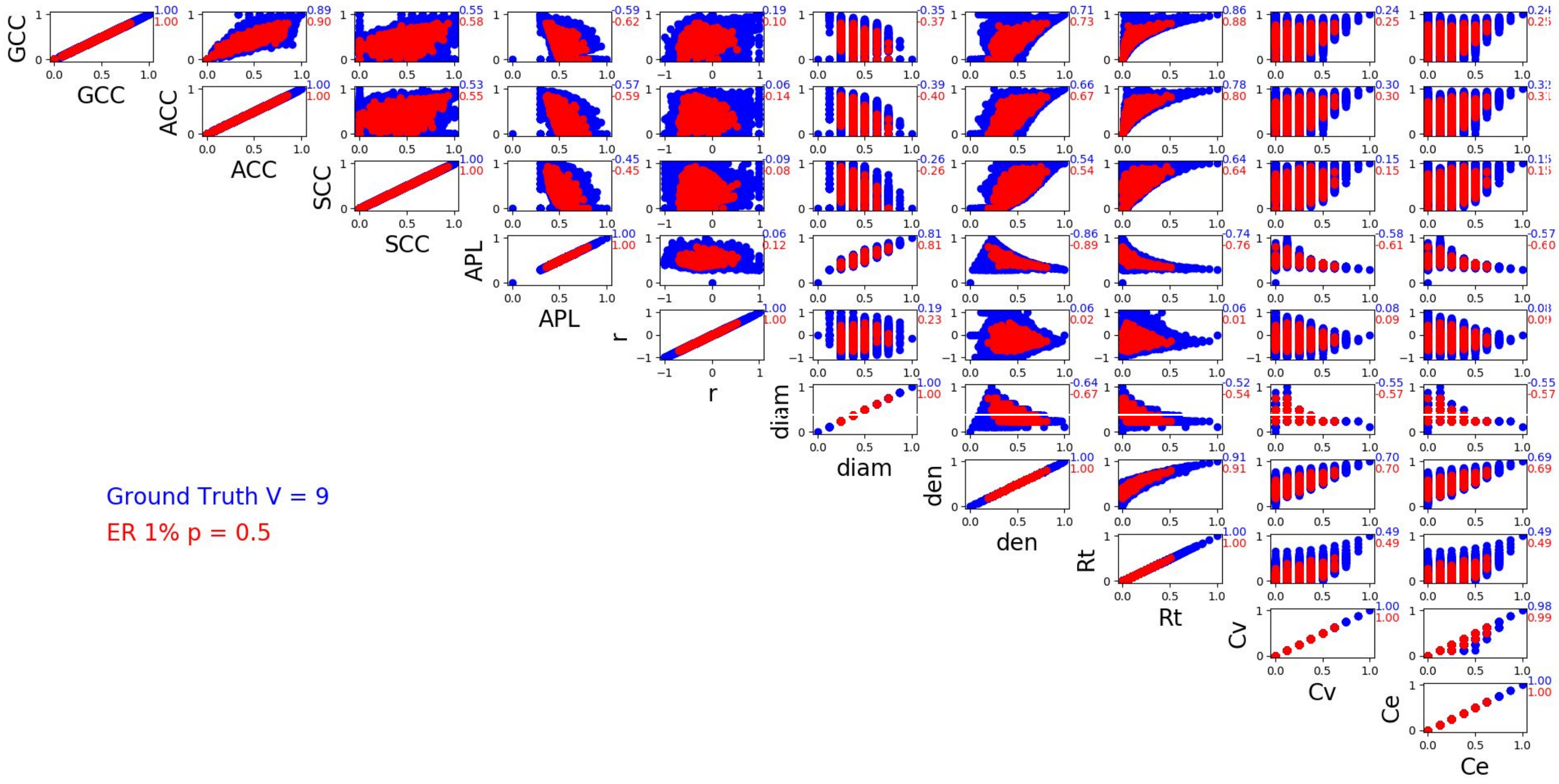
Desirable Generator Properties

Does the graph generator:

- **represent** the ground truth data well, i.e., does the generator yield a sample that with similar properties as those in the ground truth?
- **cover** the complete range of values for the properties found in the ground truth data?

Graph Generator Representativeness

We measure how **representative** a graph generator is by comparing pairwise correlations in the sample and in the ground truth.



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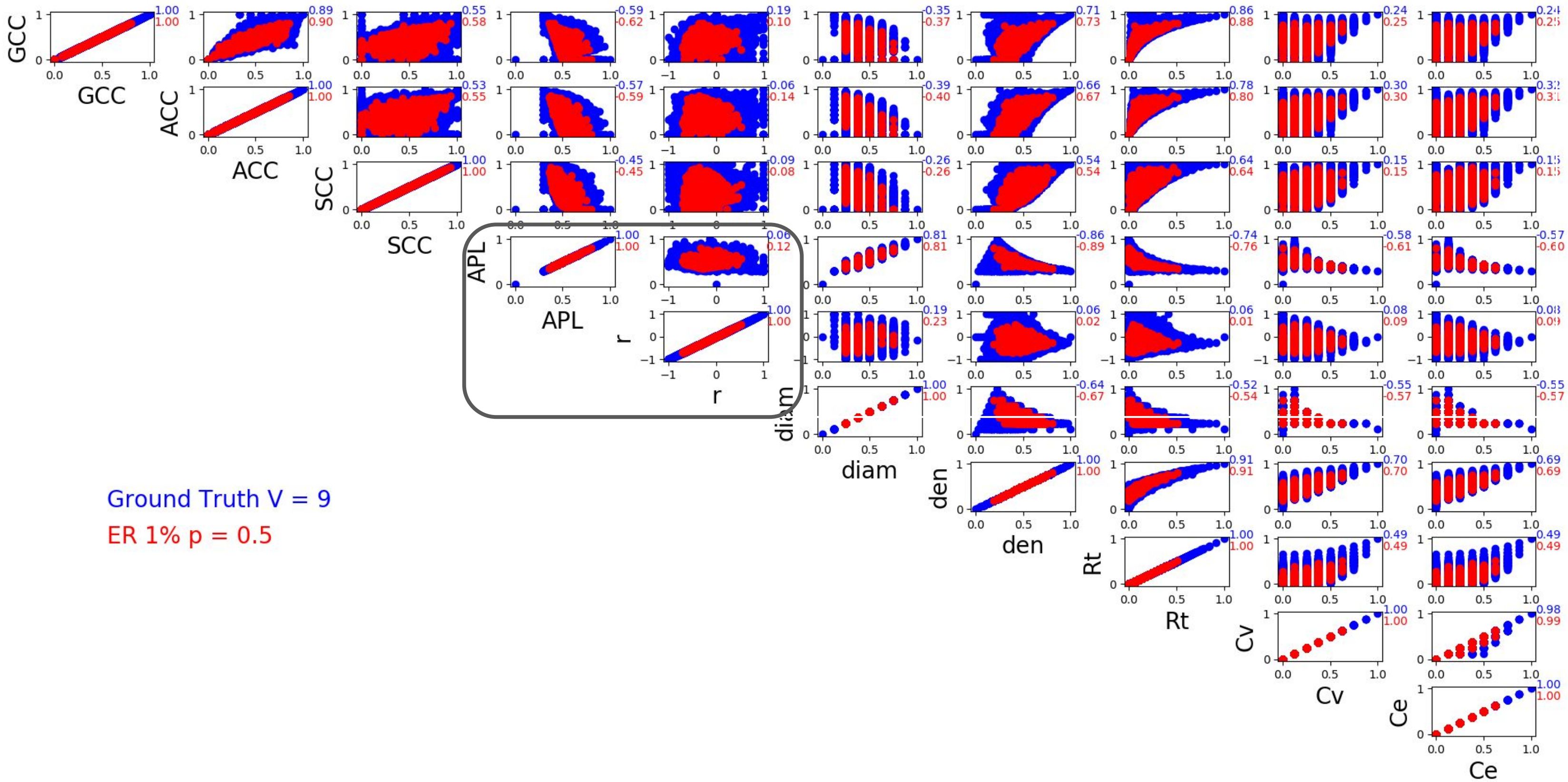

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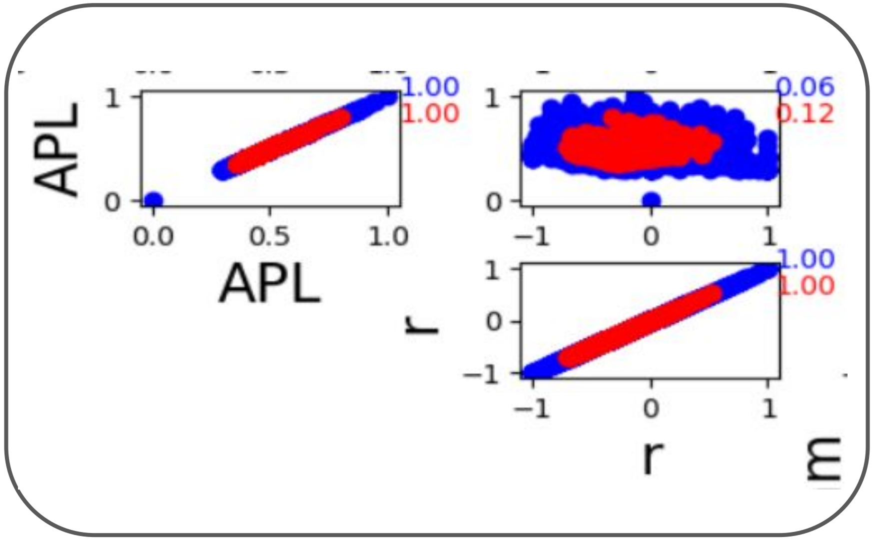


Ground Truth V = 9
ER 1% p = 0.5

Graph Generator Coverage

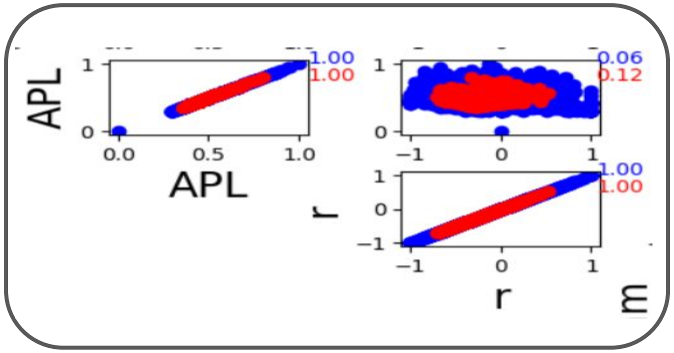
We measure how well a graph generator **covers** the range of values in the ground truth data by comparing the volumes of the generated data and the ground truth

For example, we can compare the ratios of the 10D bounding boxes of the two datasets (generator, ground truth)

WS model	BA model	ER model p = 0.5	ER model p~Uniform	ER model p~Population	Geometric model
22.04%	0.10%	0.98%	90.83%	12.37%	83.87%

Graph Generator Performance?

No graph generator is good at **representing** the ground truth and **covering** it well



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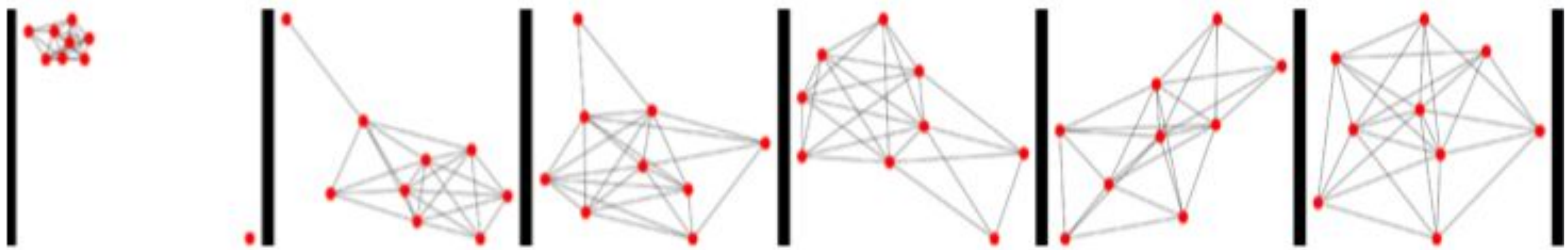
Why?

It seems the answer is that all generators sample the space of isomorphic graphs whereas we are considering the space of *non-isomorphic graphs*

Same Stats, Different Graphs

We can generate graphs with fixed set of statistics that vary in another statistic:



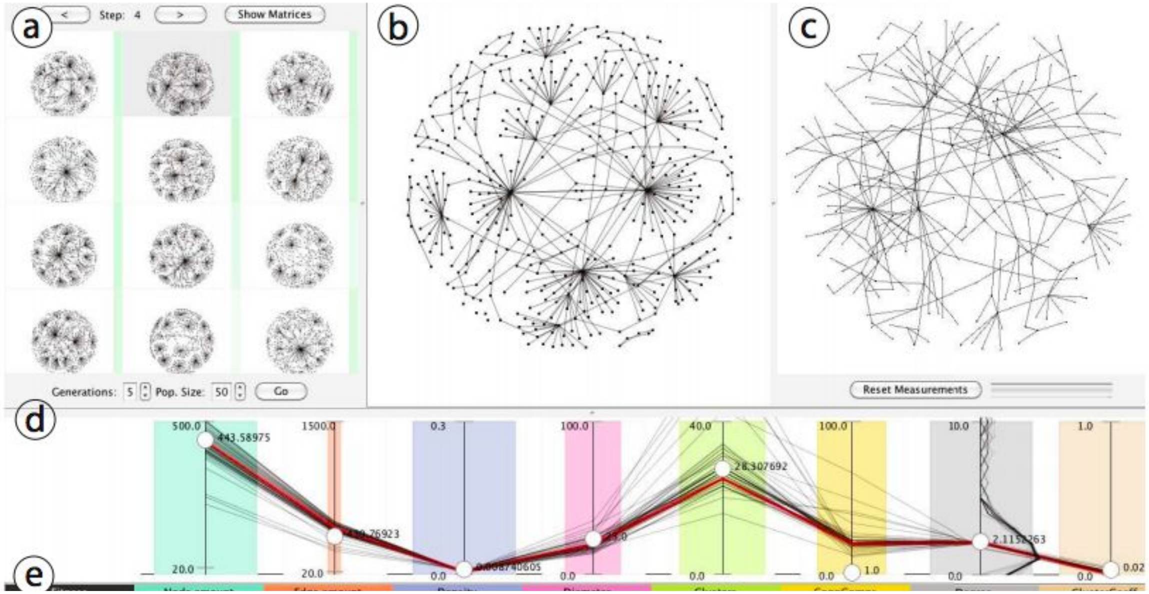
|V| = 9, SCC ∈ (0.75, 0.85), ACC ∈ (0.75, 0.8), r ∈ (-0.3, -0.2), Rt ∈ (0.35, 0.45), and we can find graphs in 6 out of 8 buckets for *connectivity*

for small sizes we can simply look in the ground truth data

for large sizes we must use generators

Related Work

1. F. Anscombe, Graphs in statistical analysis, The American Statistician 27(1), 1973
2. J. Matejka and G. Fitzmaurice, Same stats, different graphs: generating datasets with varied appearance and identical statistics through simulated annealing, CHI, 2017
3. B. Bach, A. Spritzer, E. Lutton, J. D. Fekete, Interactive random graph generation with evolutionary algorithms, GD, 2012
4. U. Soni, Y. Lu, B. Hansen, H. Purchase, S. Kobourov, R. Maciejewski, The perception of graph properties in graph layouts, EuroVis, 2018



Open Problems

1. Some drawing algorithms may not allow us to see differences in statistics between two graphs purely from their drawings; how can we address this?
2. Efficiently generate graphs of the “same stats, different graphs” type?
3. What are the correlations between different graph properties/statistics?
4. Generator that represents and covers the space of non-isomorphic graphs?