On L-shaped Point Set Embeddings of Trees

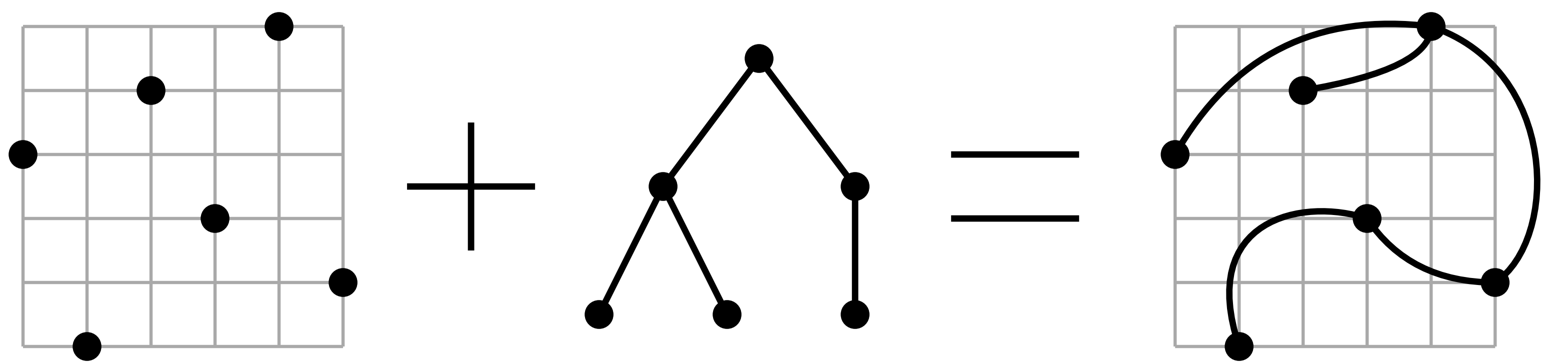
First Non-embeddable Examples

Torsten Mütze and Manfred Scheucher

Point Set Embeddings

T ... tree on n vertices
P ... set of m points

point set embedding ... drawing of T, vertices drawn as points of P



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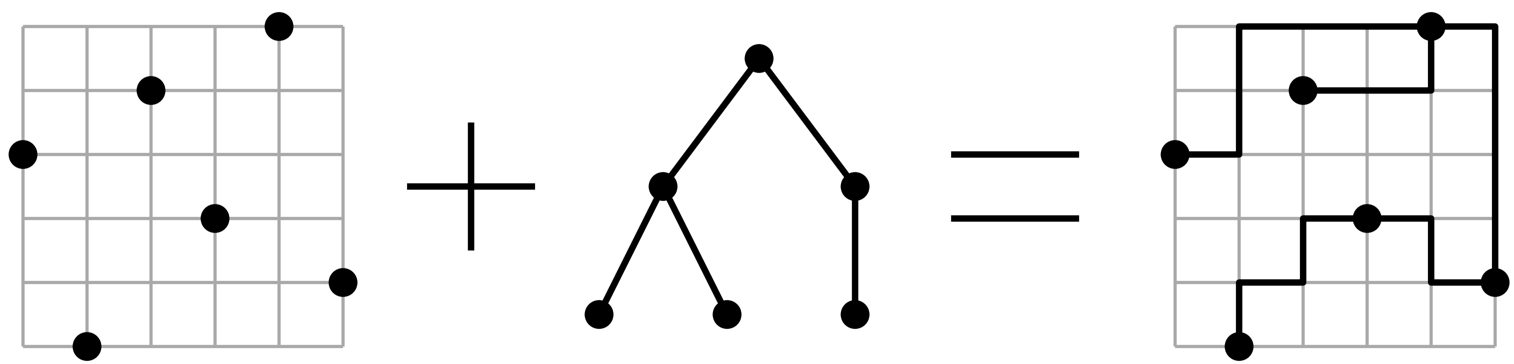

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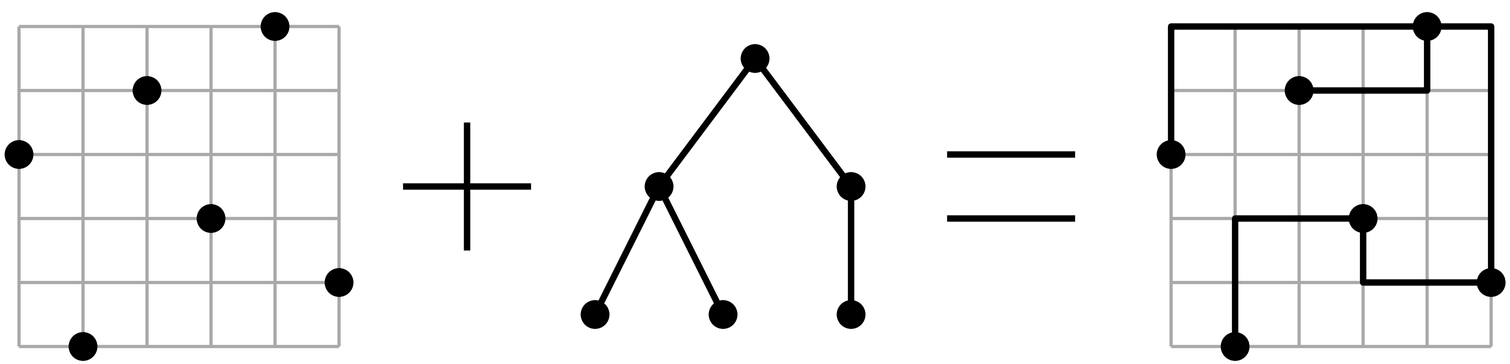

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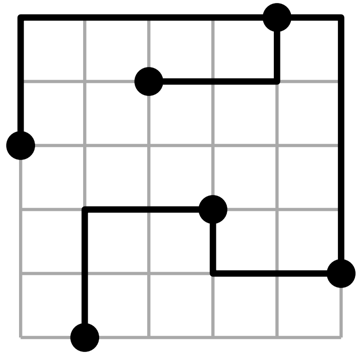
Assumptions:

- distinct x- and y-coordinates
- $P = \{(1, \pi_1), \ldots, (m, \pi_m)\}$



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$f(T)$ . . . minimum number m s.t. tree T admits a planar L-shaped embedding in any set of m points

$$f_d(n) := \max_{\substack{T\text{ : tree on } n \text{ vertices} \\ \text{max. deg. } \Delta(T) \leq d}} f(T)$$

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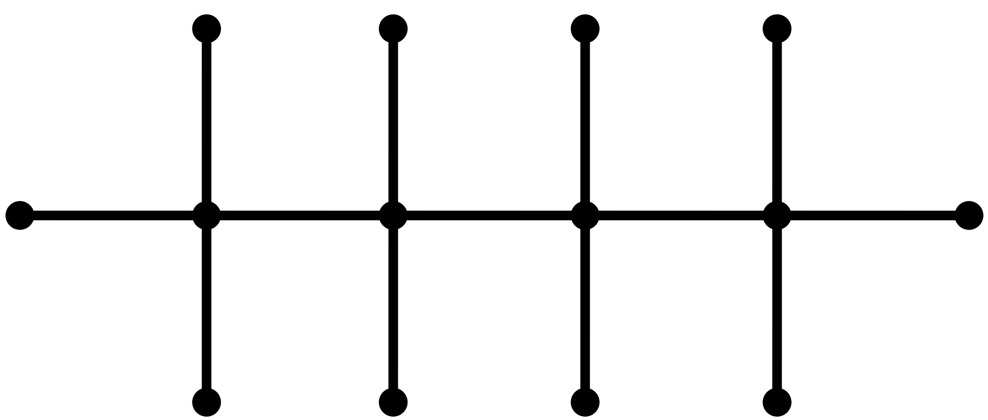
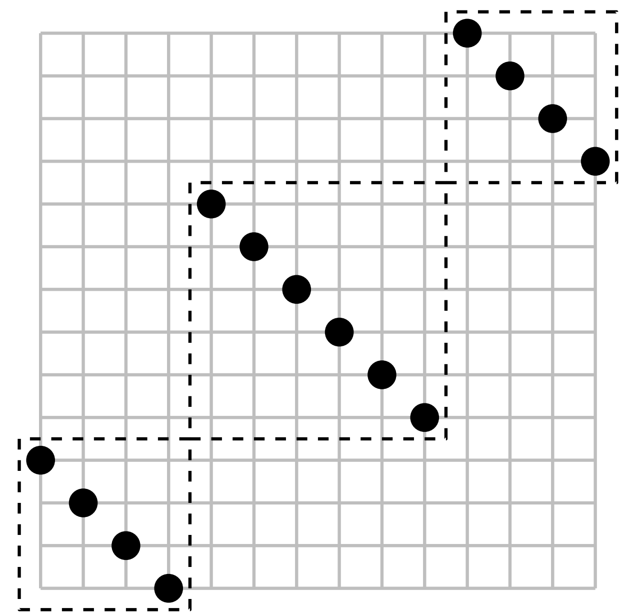
- $f_3(n) \leq O(n^{1.22})$, $f_4(n) \leq O(n^{1.55})$ [Biedl et al.'17]
- no non-trivial lower bound

Embedding Ordered Trees

- Lower bound in a more restrictive setting:
  $\exists$ example which does not always admit an L-shaped embedding if **cyclic order** around each vertex is **fixed** [Biedl, Chan, Derka, Jain, Lubiw '17]:



ordered tree on 14 vertices



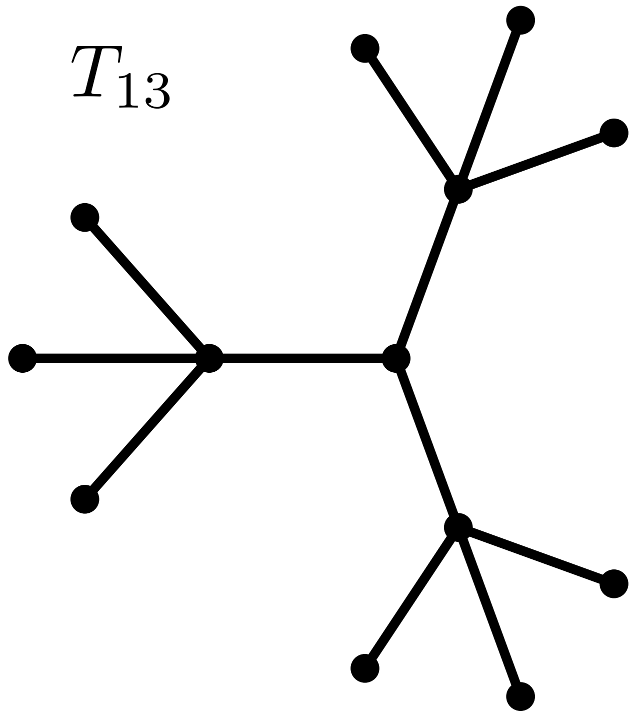
set of 14 points

New Lower Bound

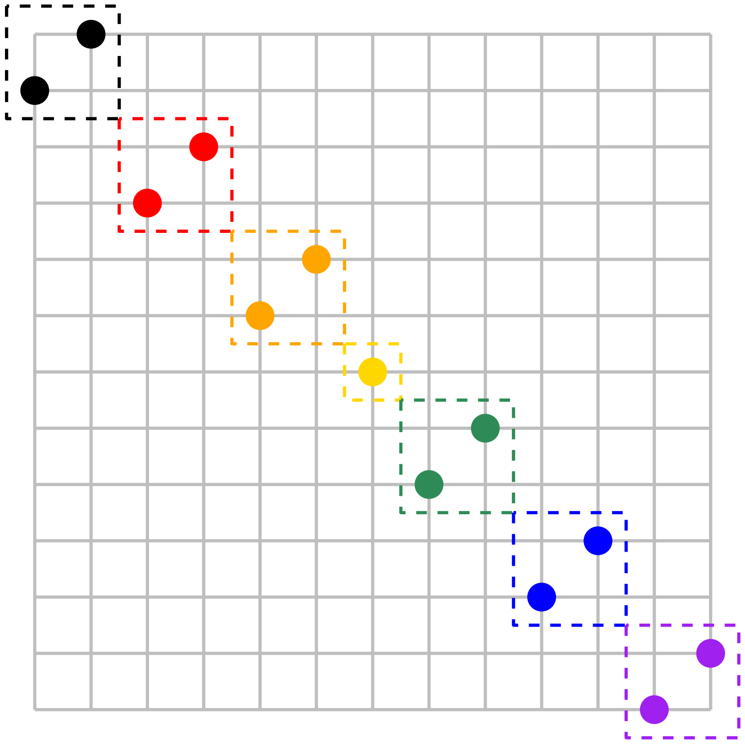
Theorem (Computer-assisted): $f_4(n) = n$ for $n \leq 11$.

Theorem: T_{13} has no L-shaped embedding in P_{13}, hence, $f_4(13) \geq 14$.

T_{13}



P_{13}



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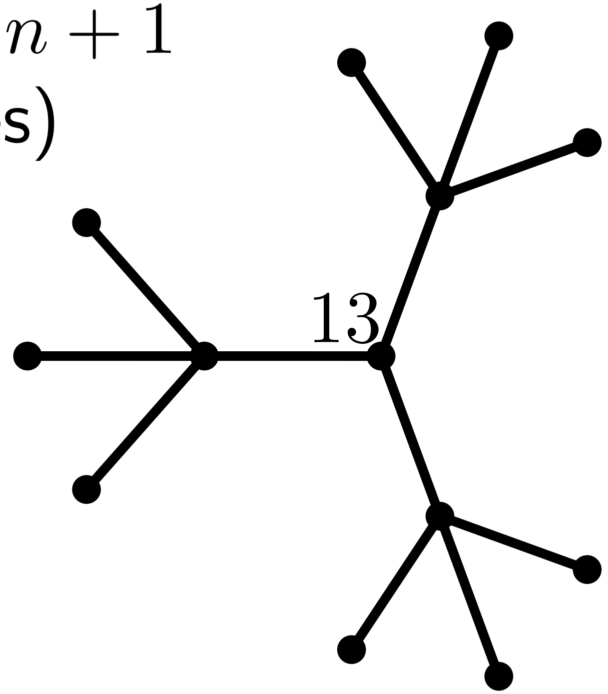
- Further examples for $n \in \{13, 14, 16, 17, 18, 19, 20\}$ (thus $f_4(n) \geq n + 1$ for those values)



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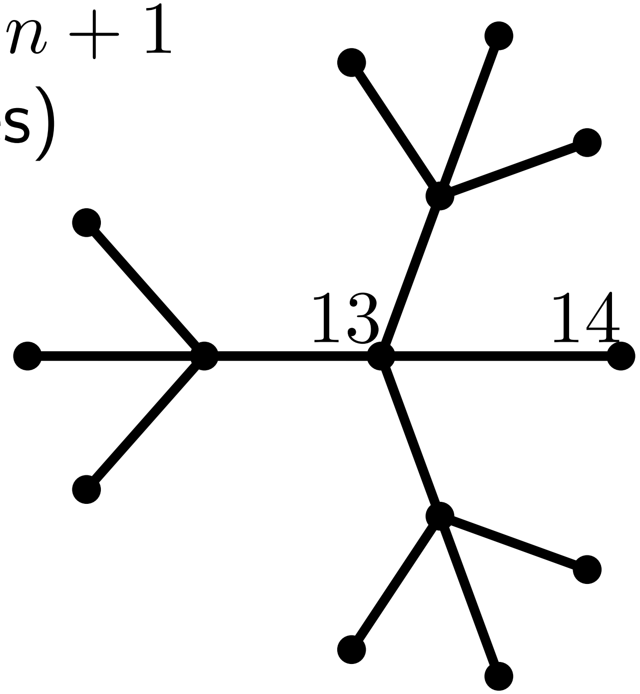

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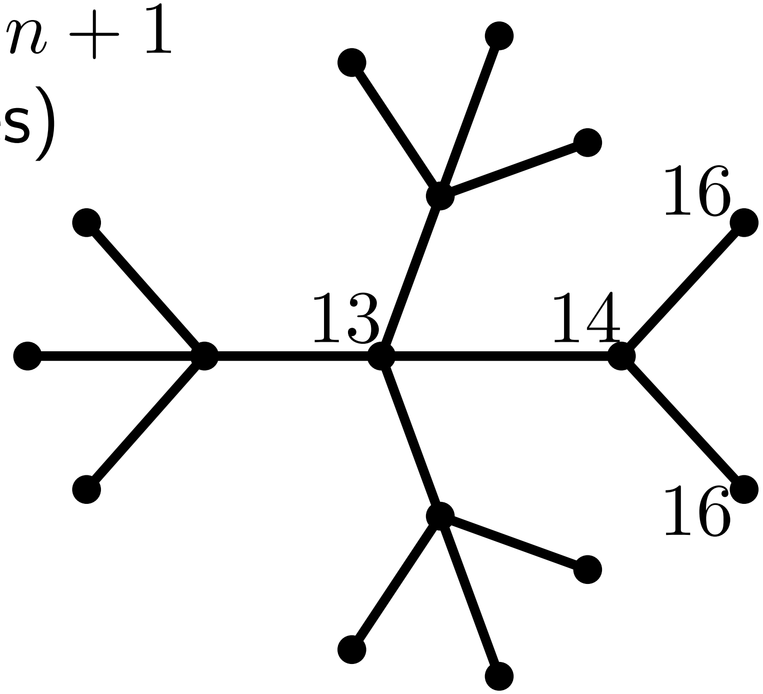

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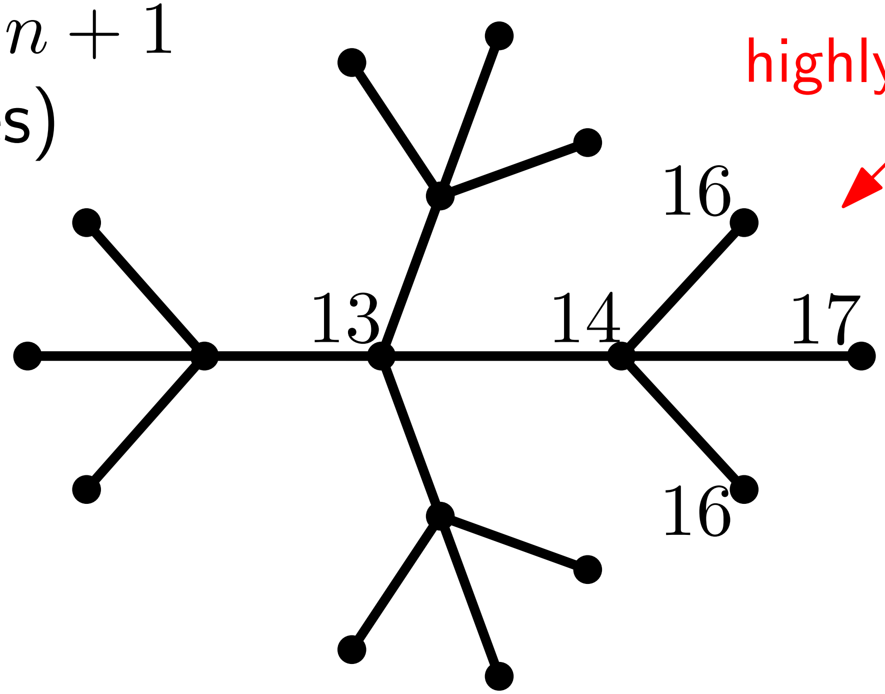
highly symmetric



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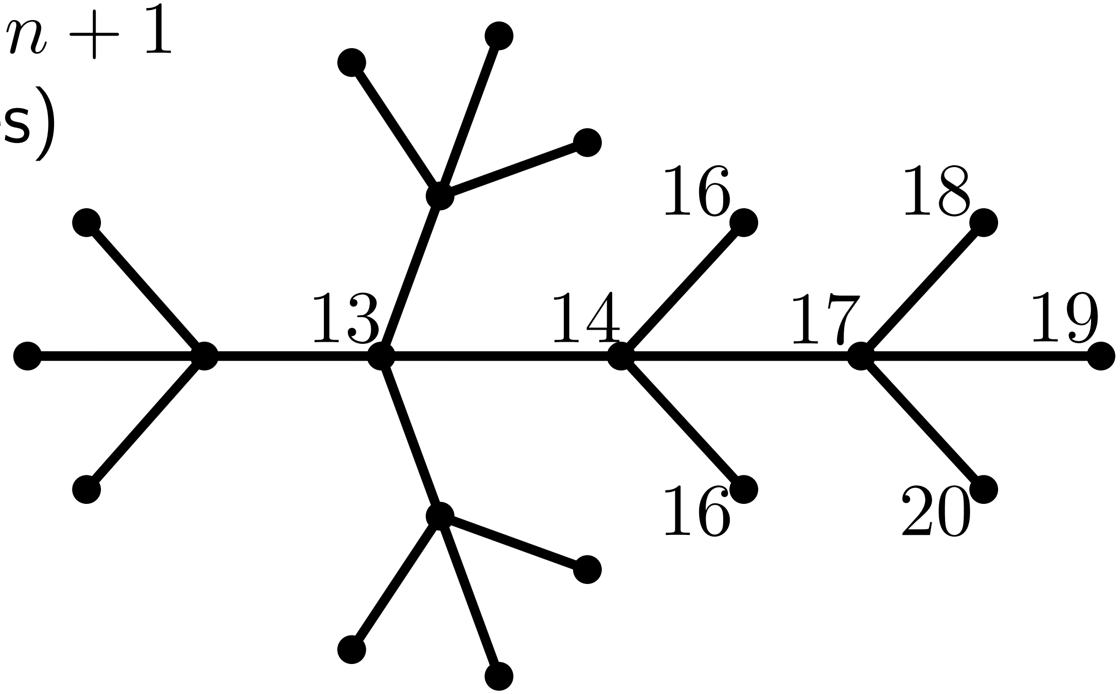

Computer-assisted Proof

- T . . . tree on vertices $\{v_1, \ldots, v_n\}$
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- formulate Boolean satisfiability instance:
  $\exists$ solution iff. T admits an L-shaped embedding in P
- use SAT solver (Picosat, MiniSat, Glucose, ...)
- test all pairs of trees and point sets

$\Theta(c^n)$ $\Theta(n!)$

SAT Model: Variables

- $M_{i,j}$ ... vertex v_i is mapped to point P_j
- $H_{a,b}$ ... edge ab is connected horizontally to a

SAT Model: Clauses

- Injective mapping V to P

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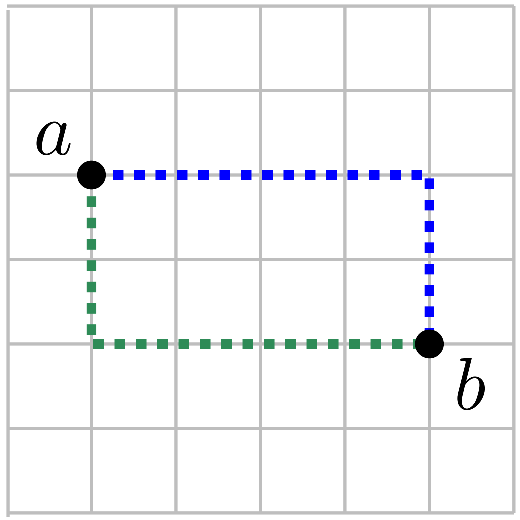

SAT Model: Clauses

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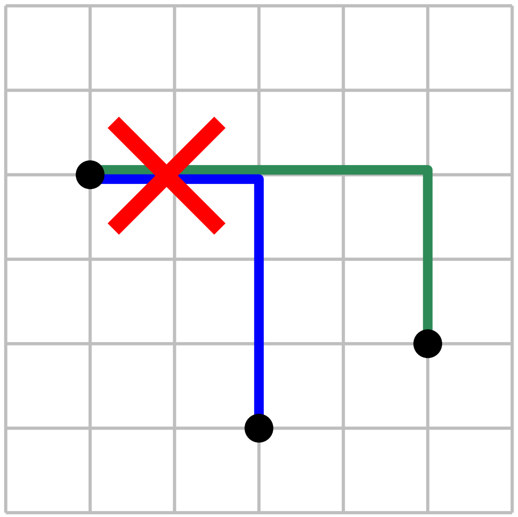

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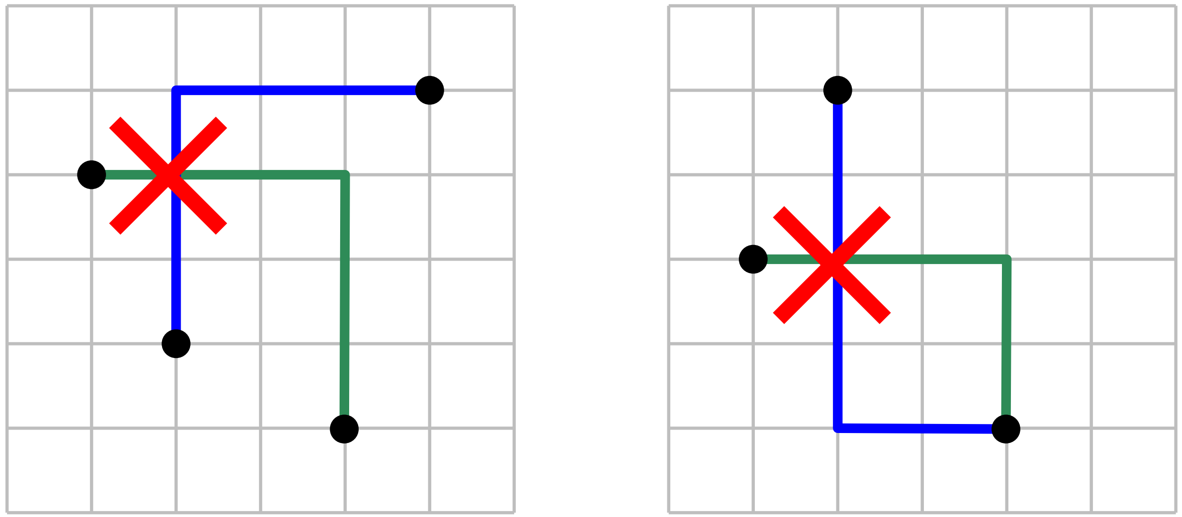

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100+ cpu days

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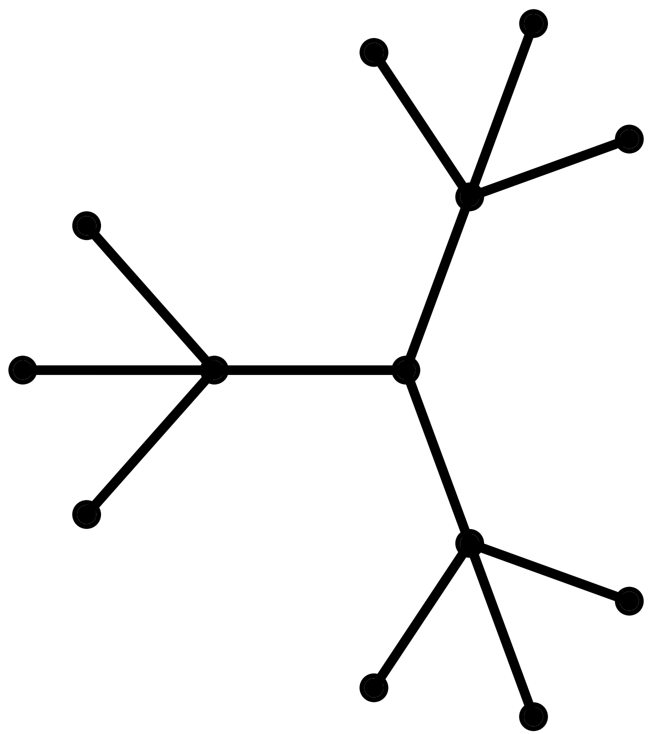
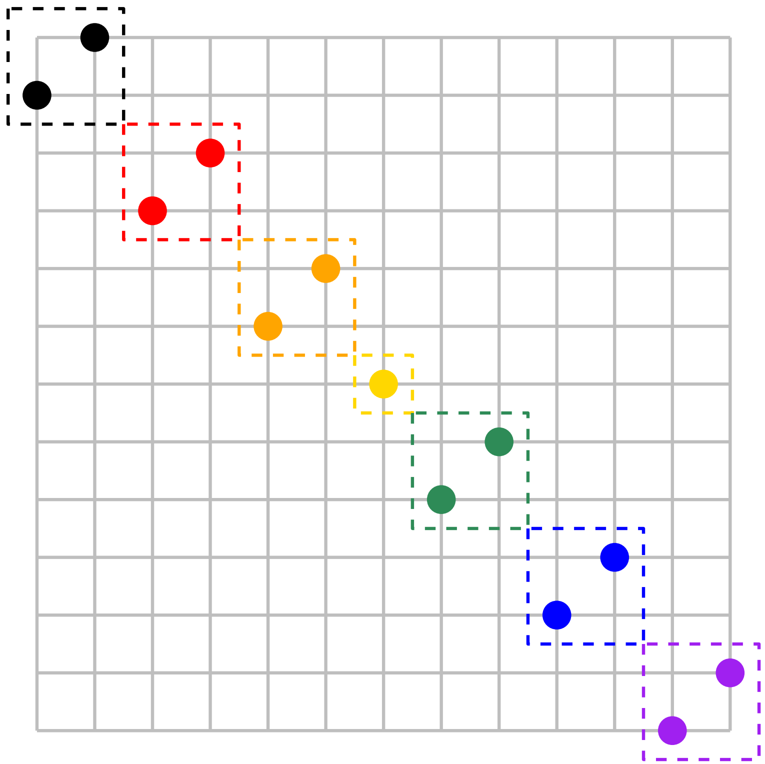
Proof of Theorem 2

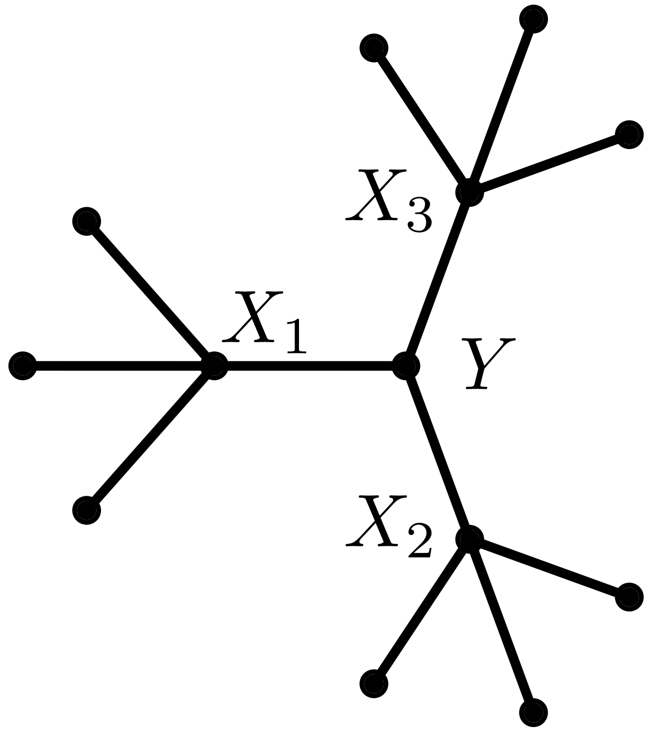
- Assume T_{13} admits an L-shaped embedding in P_{13}

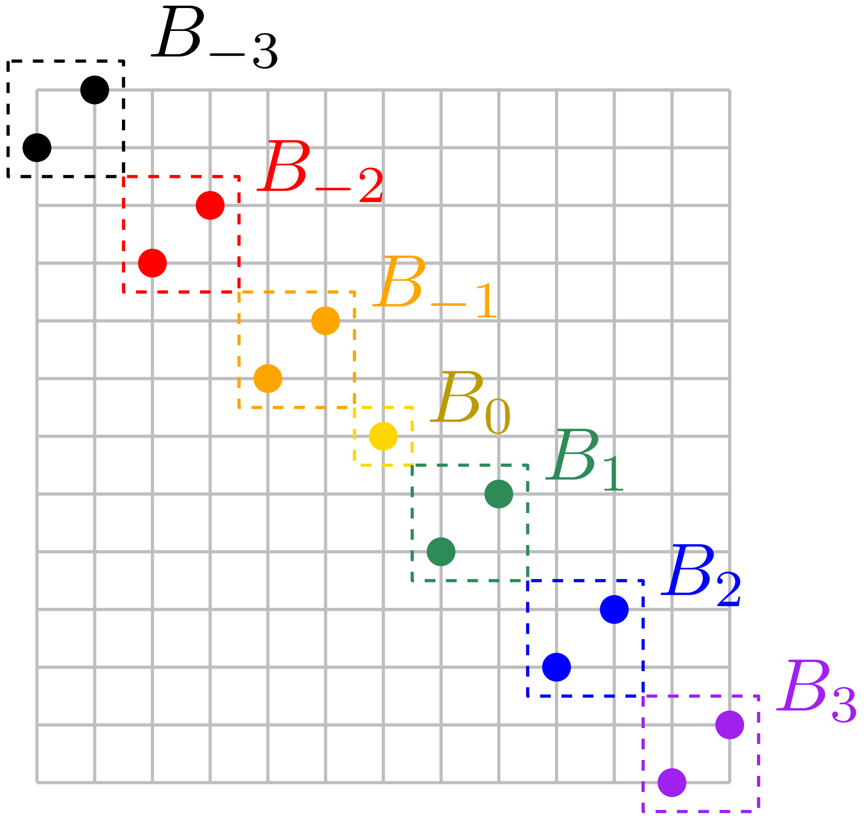




Proof of Theorem 2

- Assume T_{13} admits an L-shaped embedding in P_{13}
- T_{13} and P_{13} have symmetries



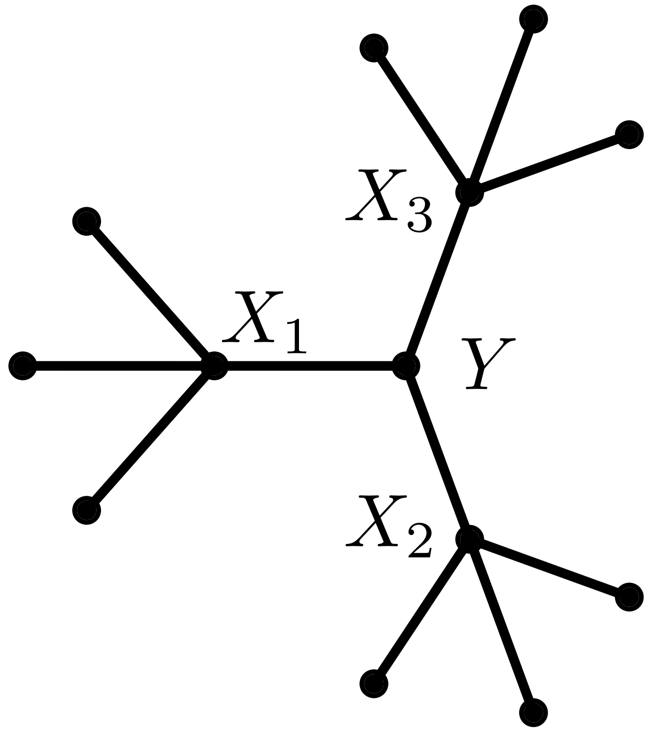


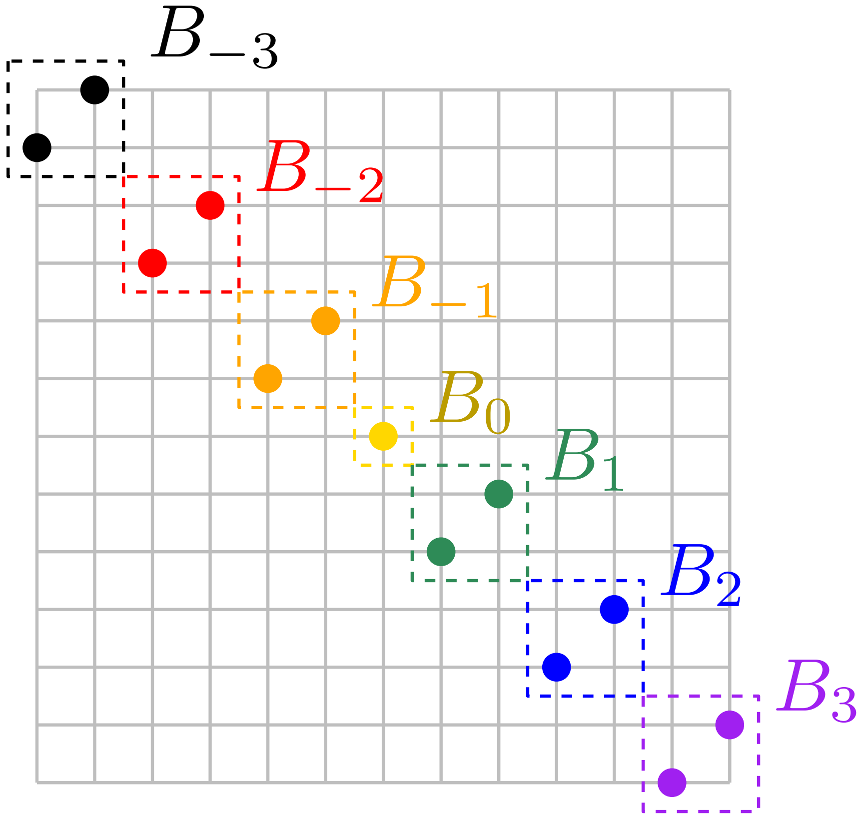
Proof of Theorem 2

- neither of X_1, X_2, X_3, Y is mapped to $B_{\pm 3}$

degree 4 vertices

boundary points





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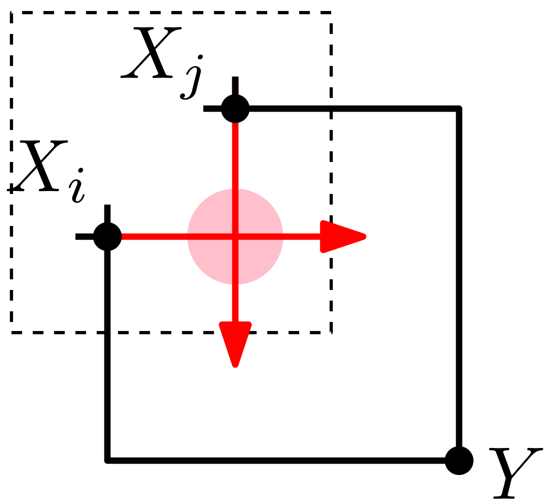

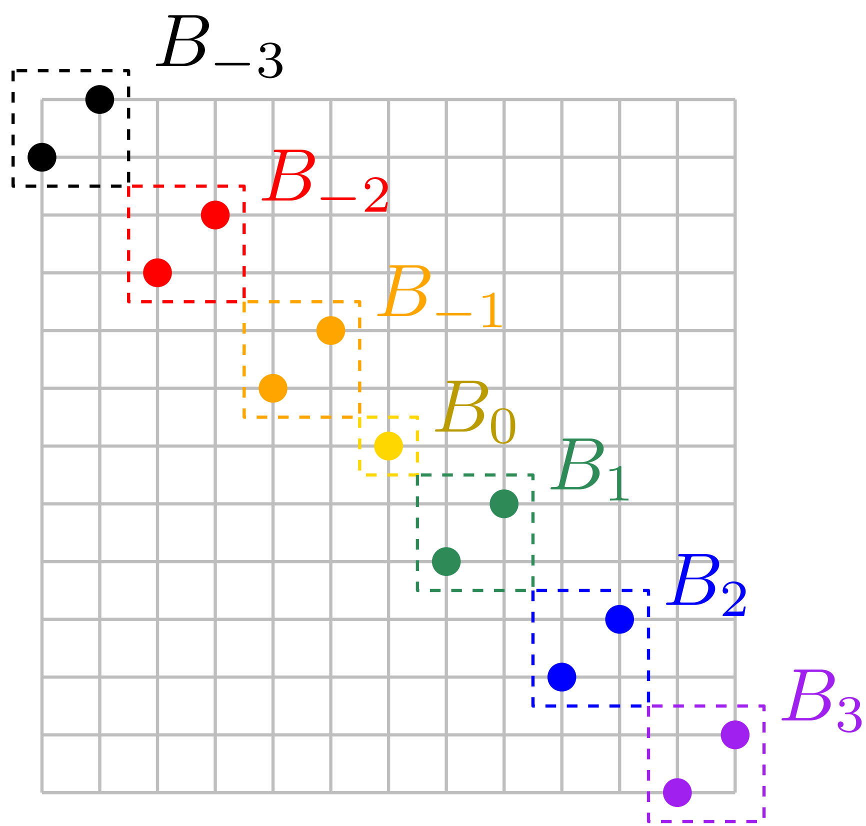

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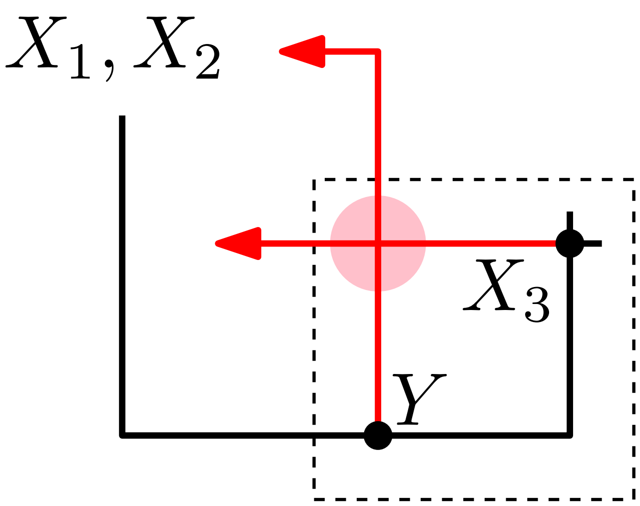

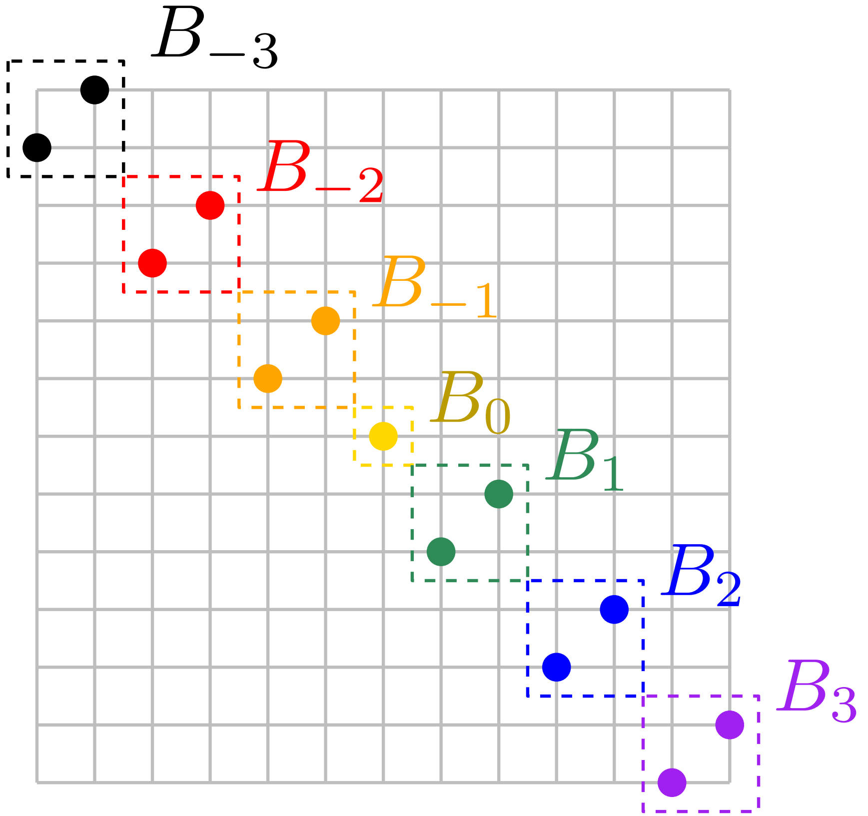

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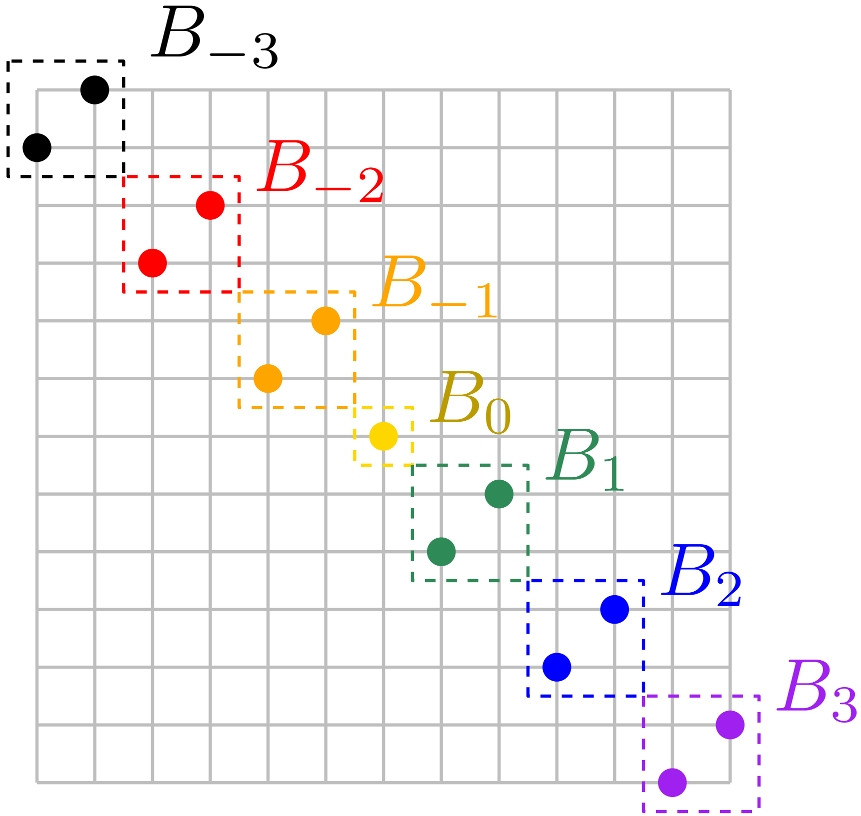

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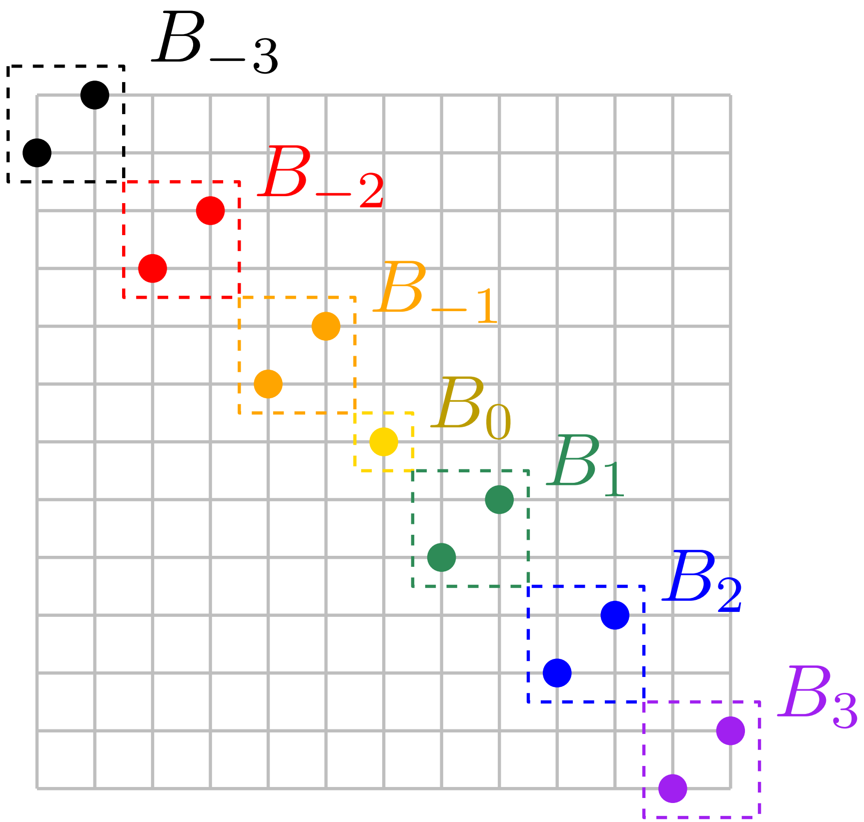
  (w.l.o.g.)



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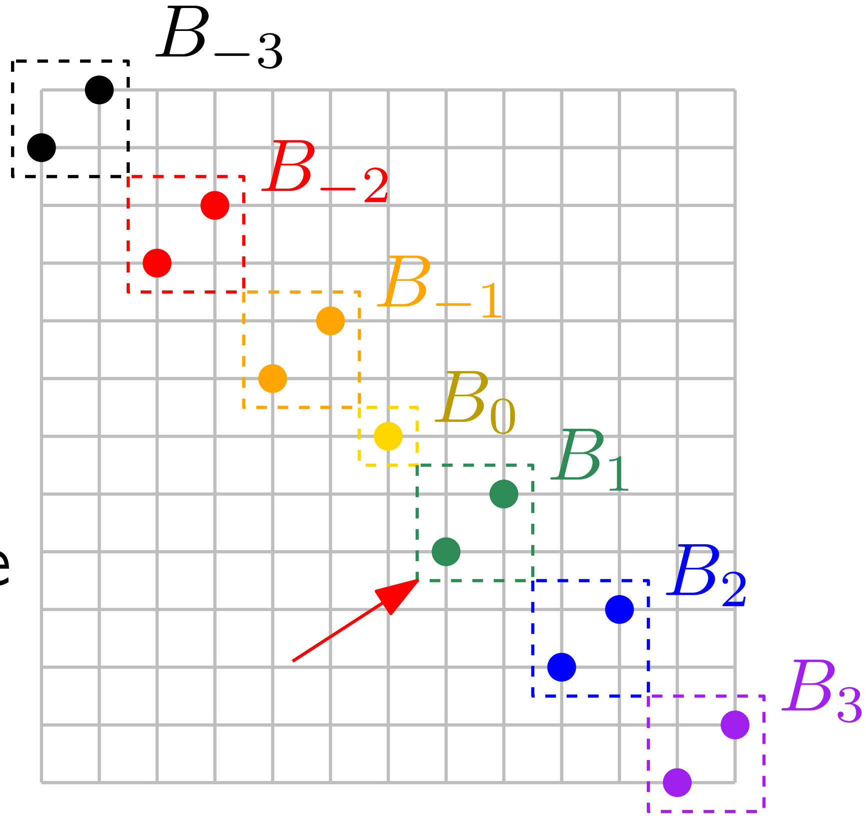
- By symmetry, we may assume they are mapped to B_1



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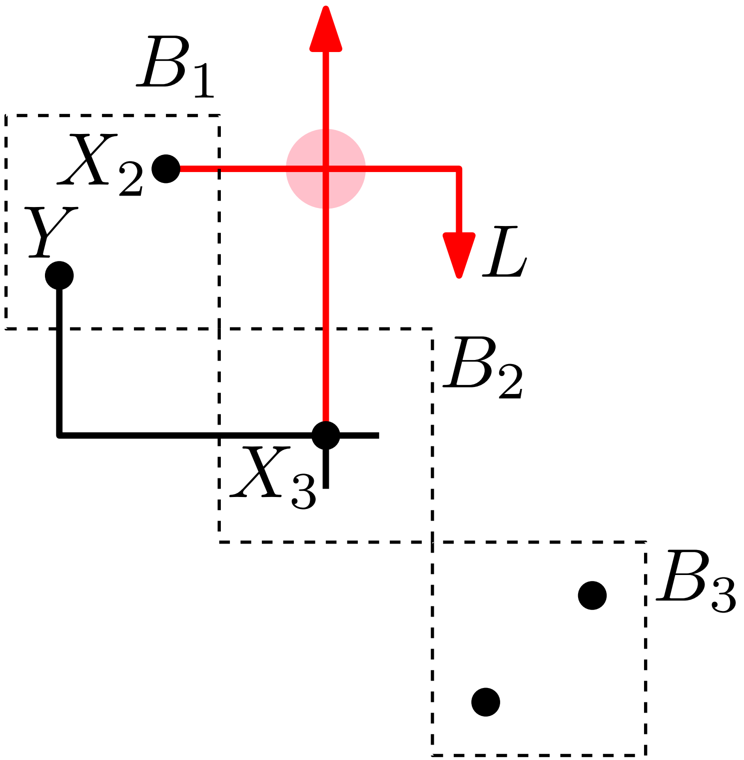

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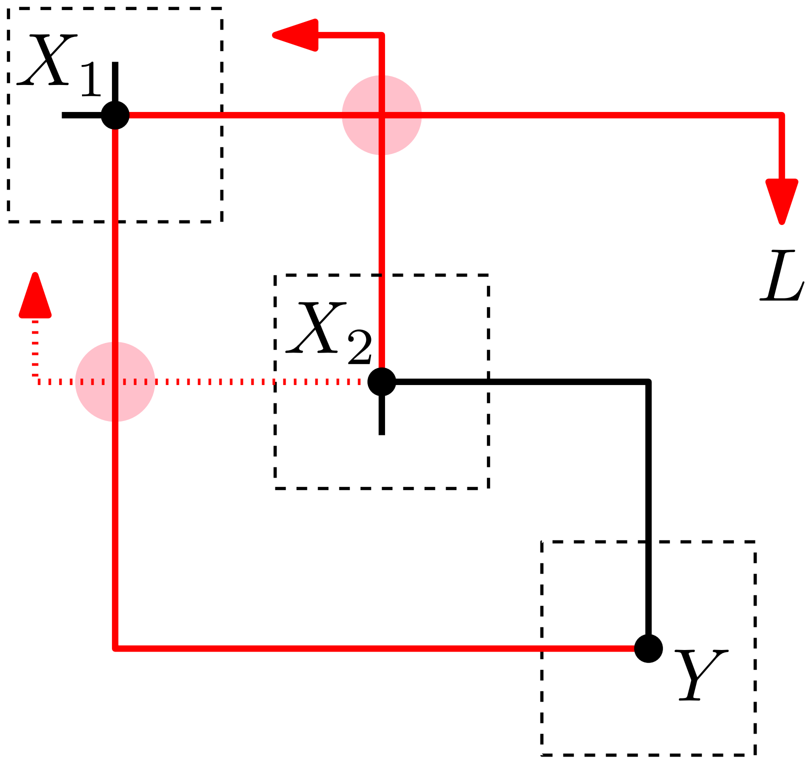
- Case 2: Y and X_2 mapped to distinct blocks

q.e.d.



Discussion

- $n \leq 11$: ✓
- $n = 12$: **???**
- $n = 13, 14, 16, 17, 18, 19, 20$: ✗

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- Q: Is there an infinite family?

Discussion

- n points suffice for paths (pathwidth 0)
- known examples are lobsters (pathwidth 2)
- Q: Do n points suffice for caterpillars (pathwidth 1)?

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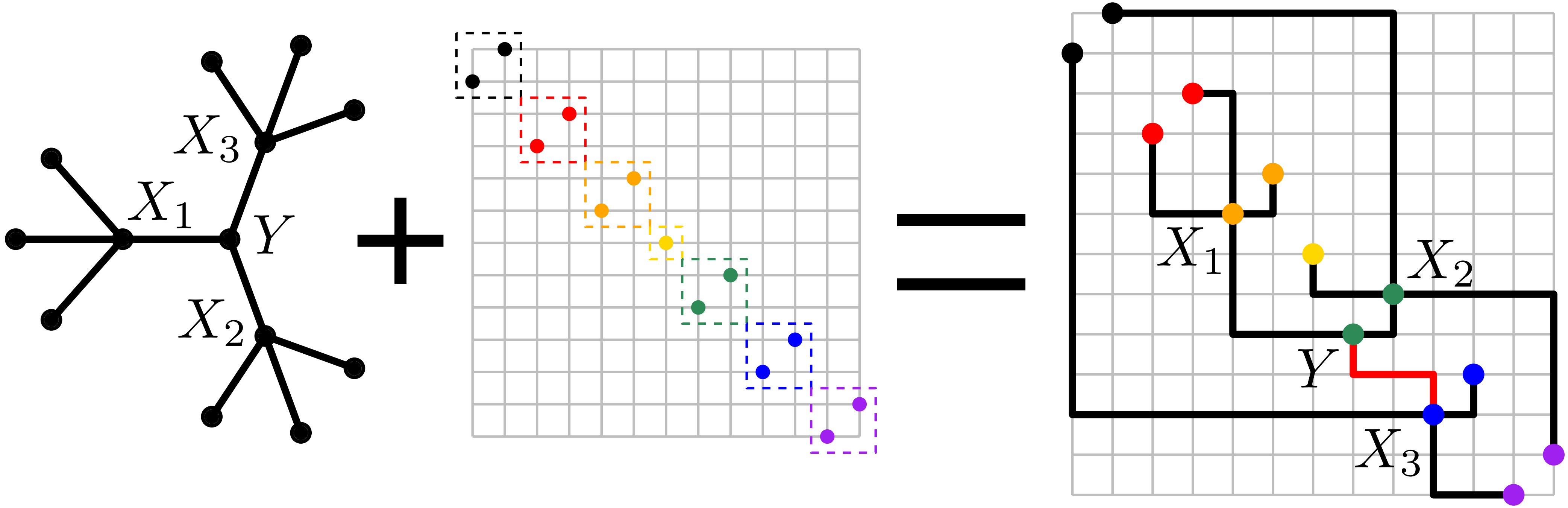
- known examples are lobsters (pathwidth 2)

- Q: Do n points suffice for caterpillars (pathwidth 1)?
- Q: what about trees with maximum degree $\Delta = 3$?

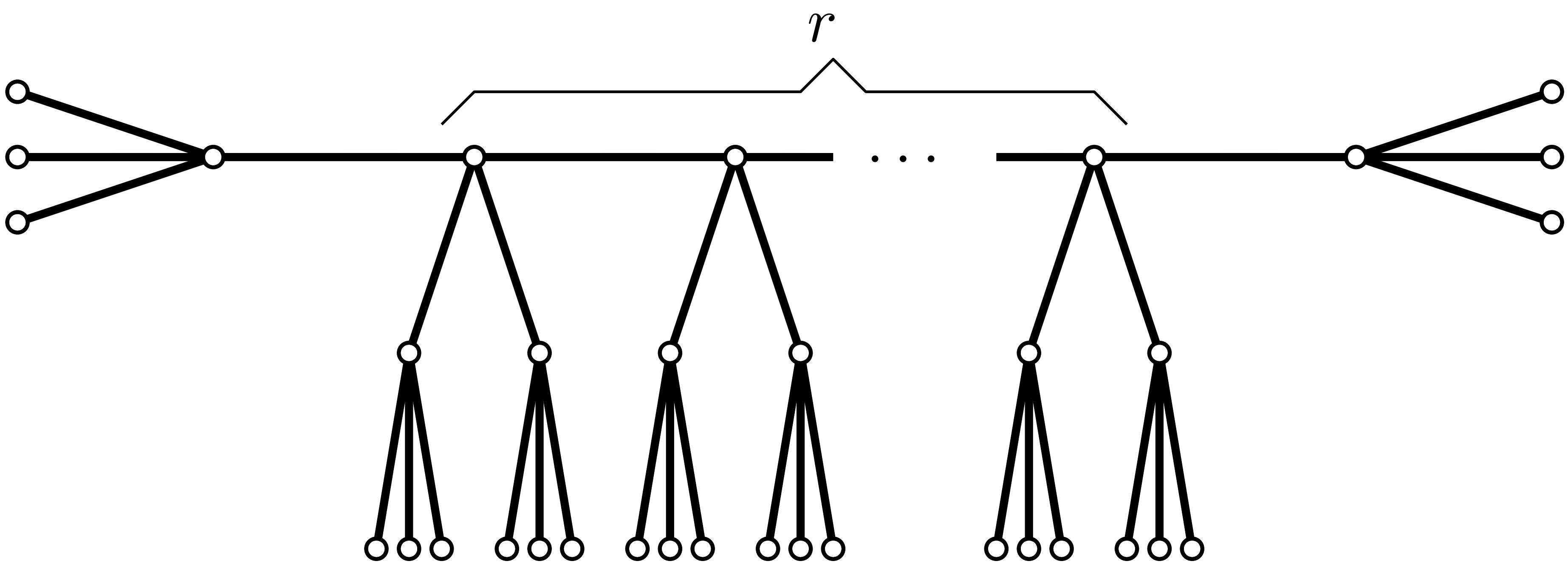
Discussion

- Q: What about orthogeodesic embeddings?



Lower Bound: Ordered Trees (Full Version)

Theorem: ∃ infinite family of **ordered** trees which do not always admit an L-shaped embedding.



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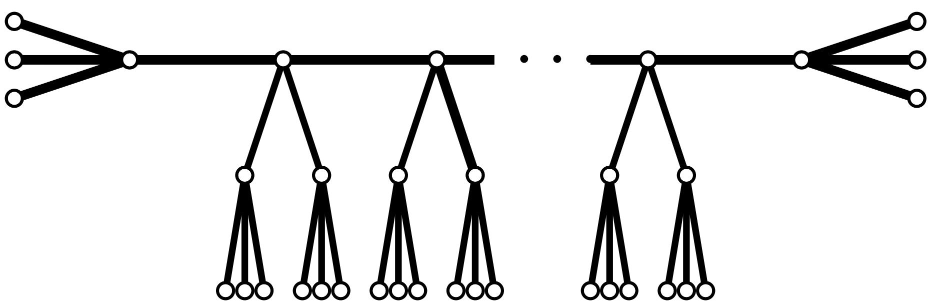

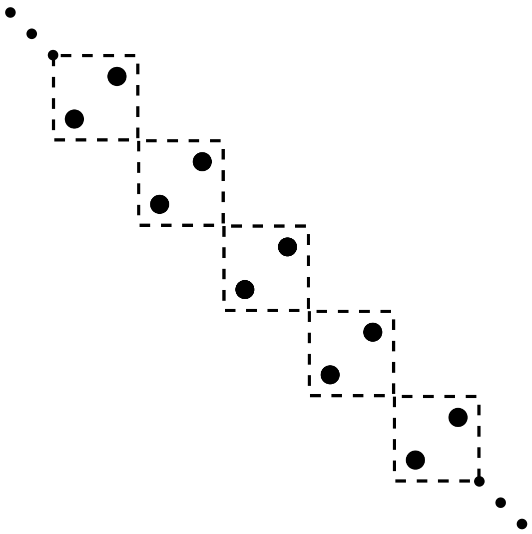

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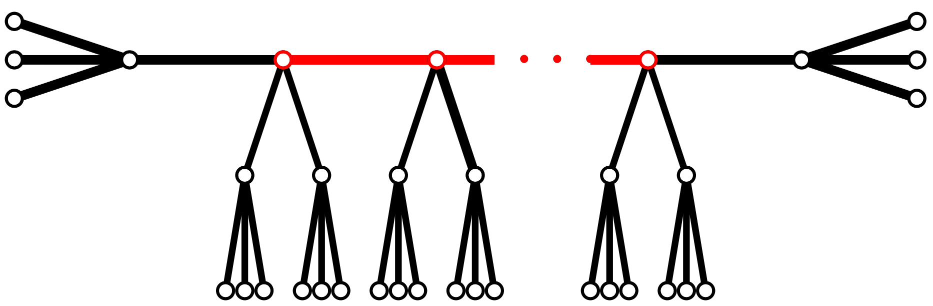
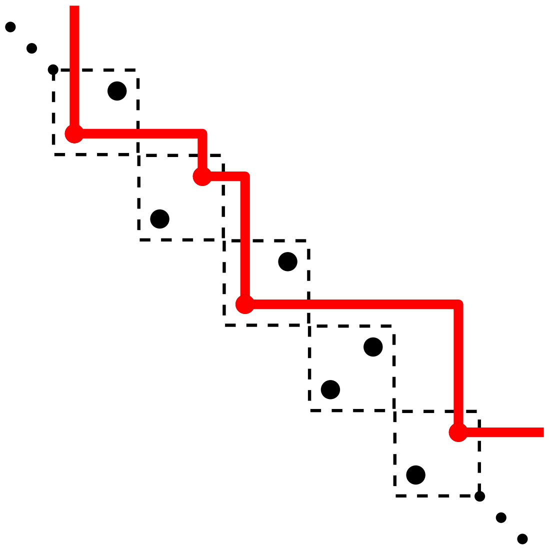
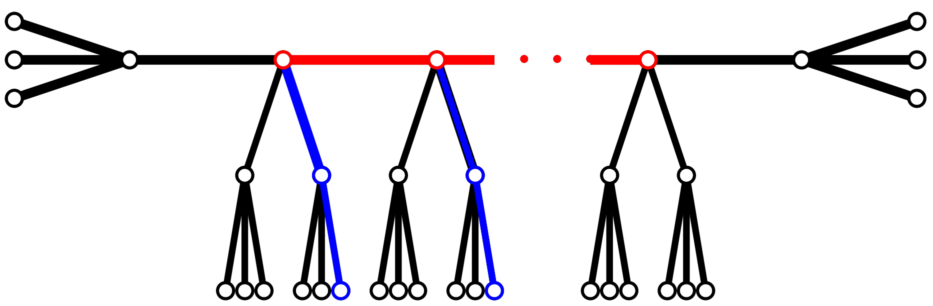
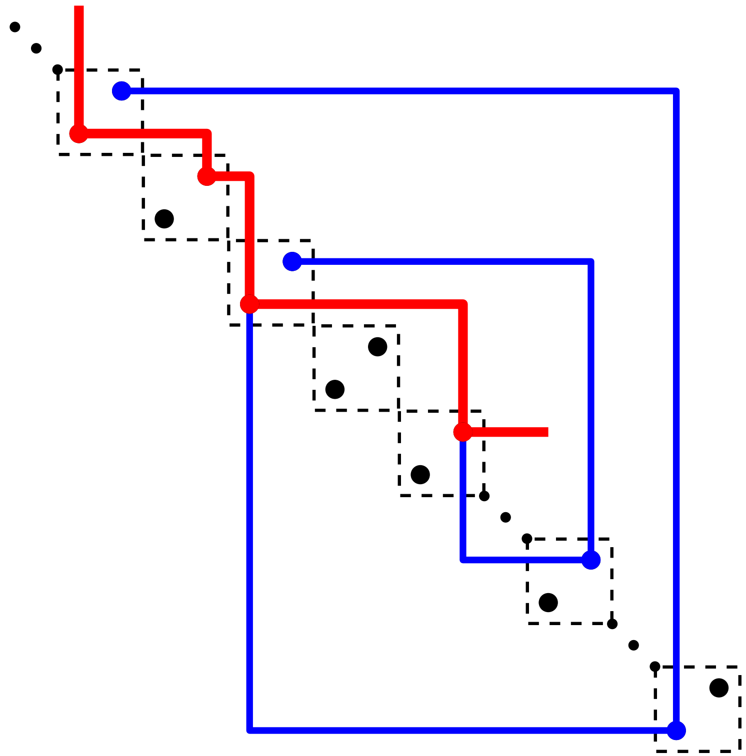




Lower Bound: Ordered Trees (Full Version)

Theorem: $\exists$ infinite family of **ordered** trees which do not always admit an L-shaped embedding.





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This answers question a from Biedl, Chan, Derka, Jain, and Lubiw'17

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Conjecture: Also non-embedable in the original setting.

Thank you
for your
attention!