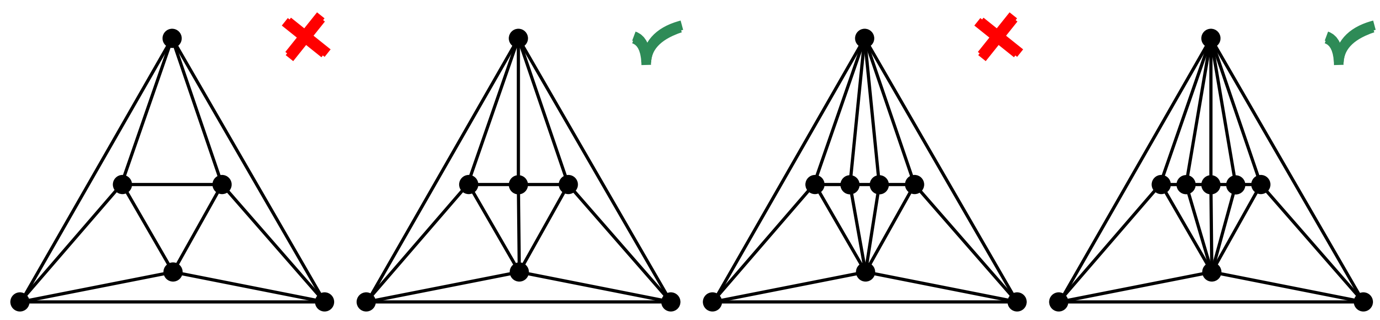
Technische Universität Berlin

On the Area-Universality of Triangulations

Linda Kleist



Planar Graphs and Face Areas

Cartogram

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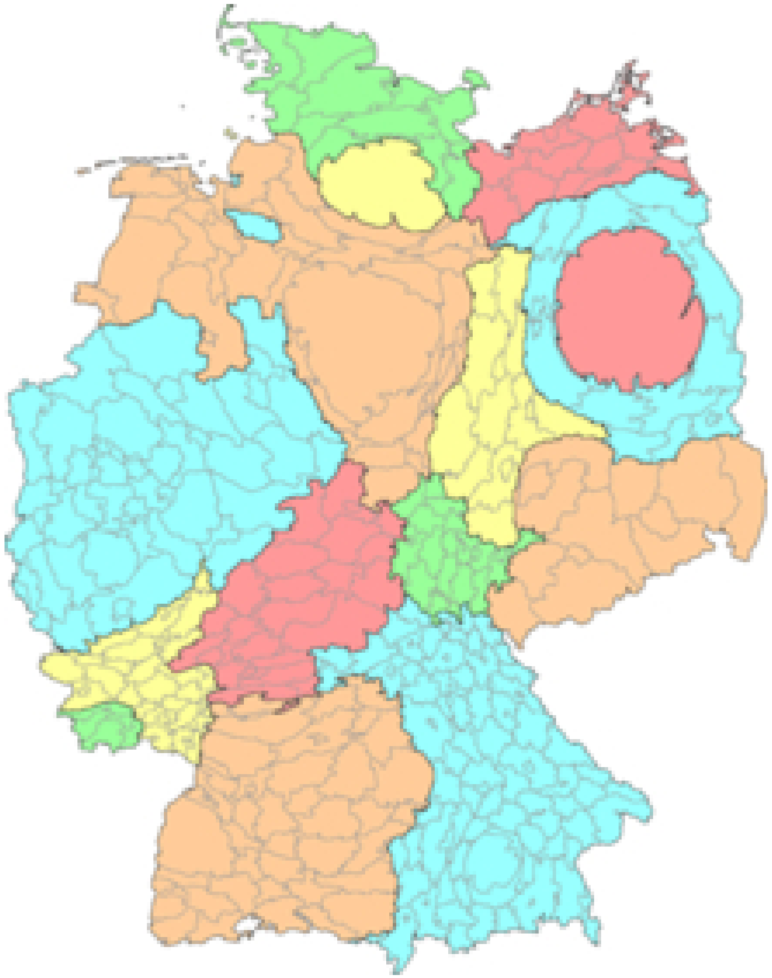
Cartogram



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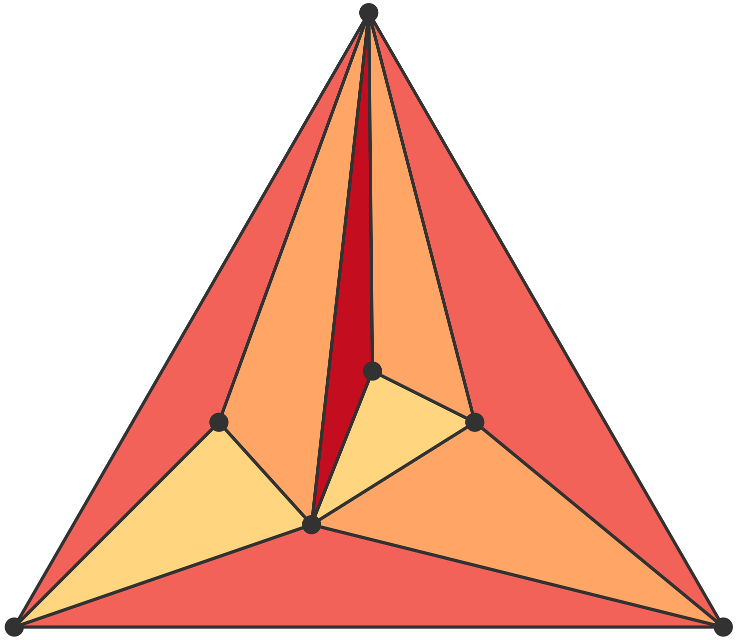
- Germany
- population

Planar Graphs and Face Areas

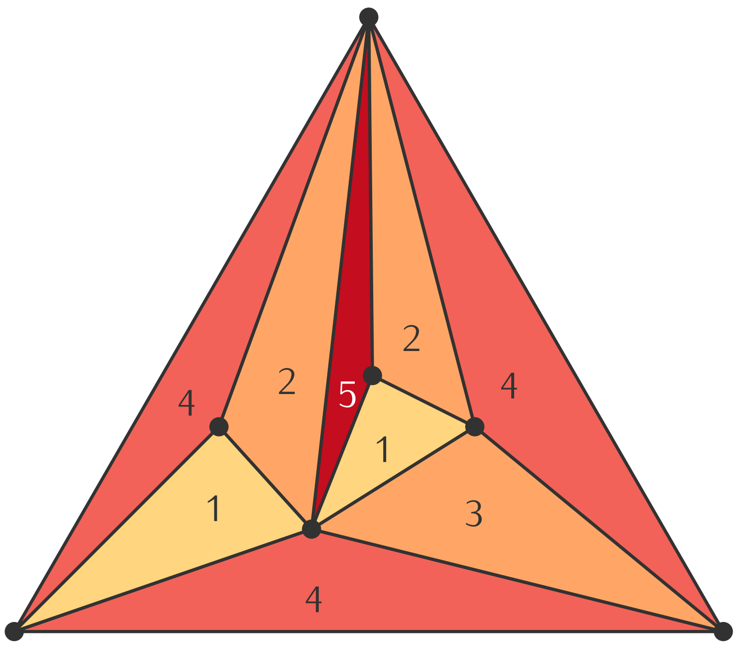
plane graph G – straight-line drawing



Planar Graphs and Face Areas

plane graph G – straight-line drawing

area assignment $\mathcal{A}$ – weights on the inner faces



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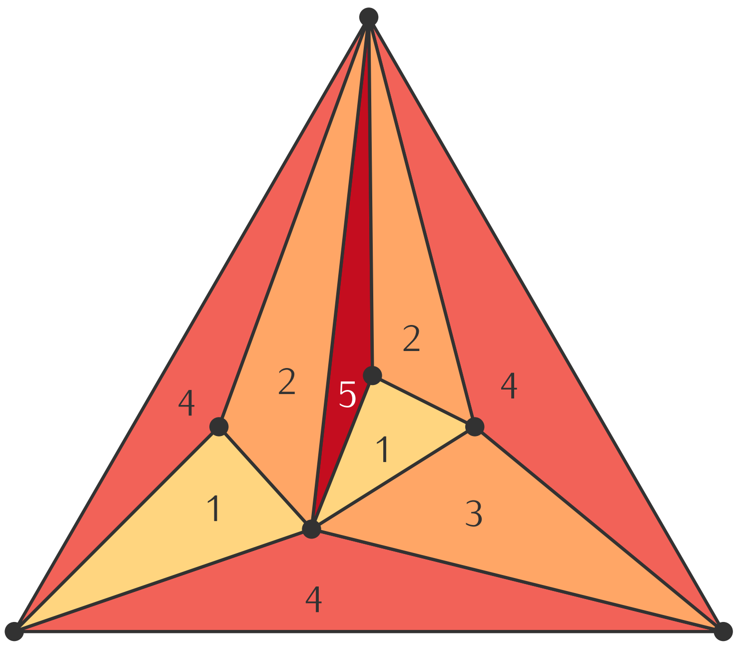
$\rightarrow \exists$ realizing drawing?



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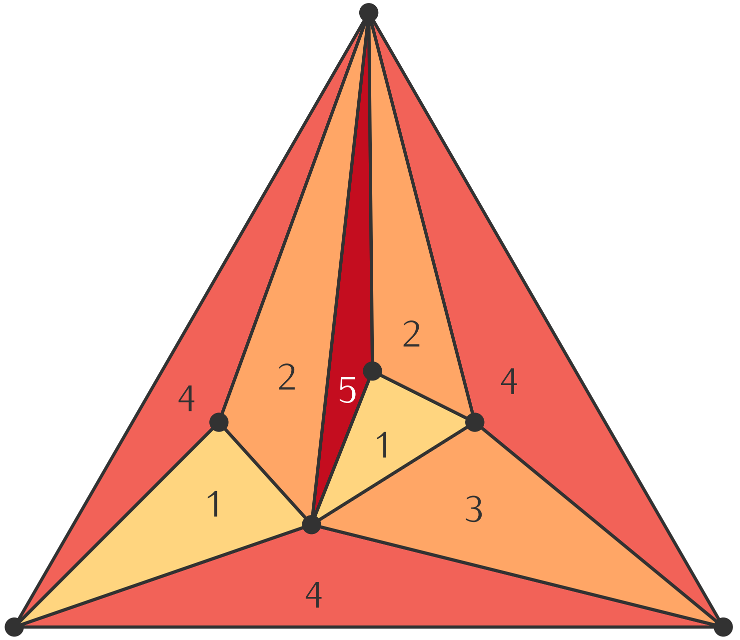

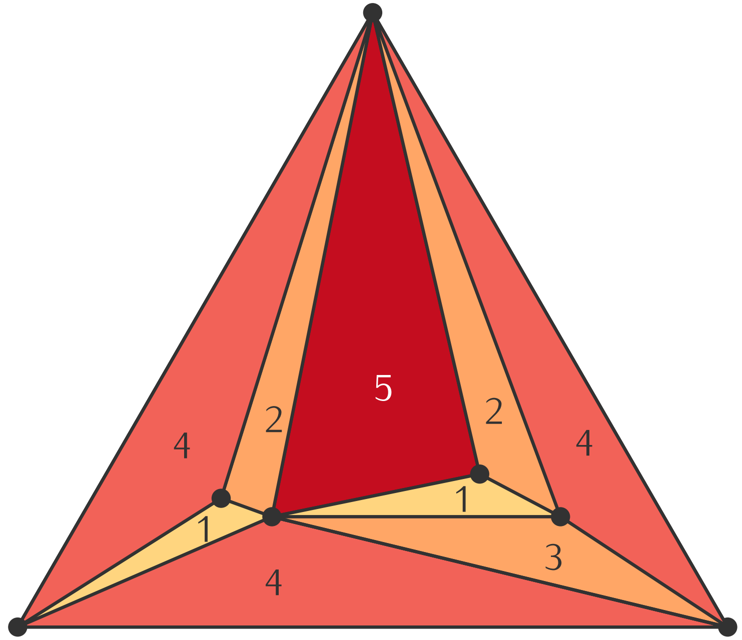

Planar Graphs and Face Areas

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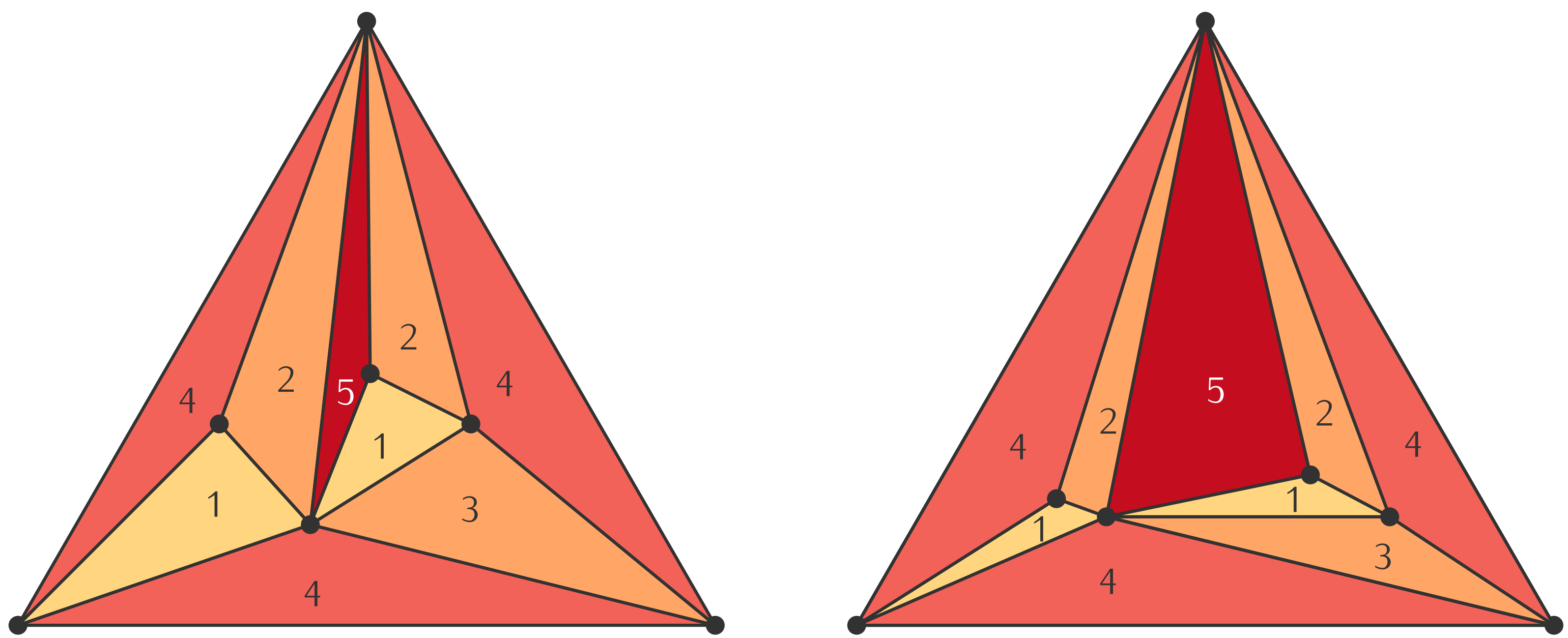
$\rightarrow \exists$ realizing drawing? ‘yes’ $\forall \mathcal{A} \Rightarrow G$ is area-universal



Area-Universality – Results

positive area-universal graphs:

- plane 3-trees [Biedl & Velázquez 2013]

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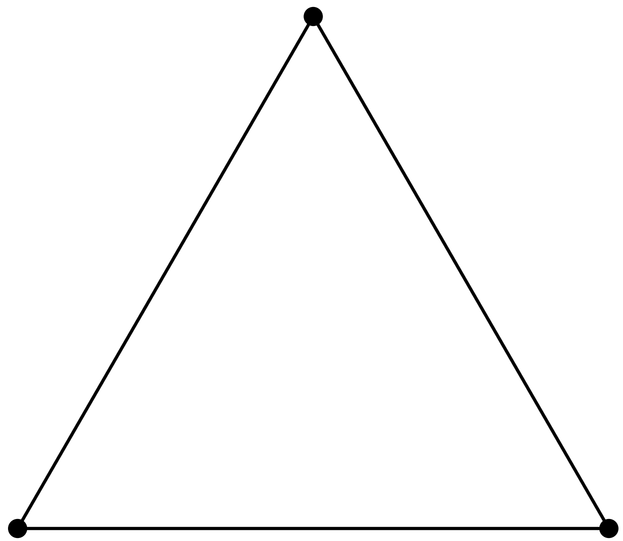

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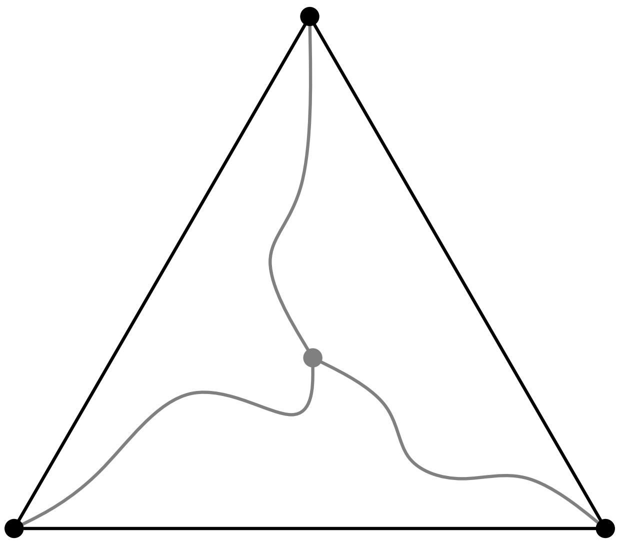

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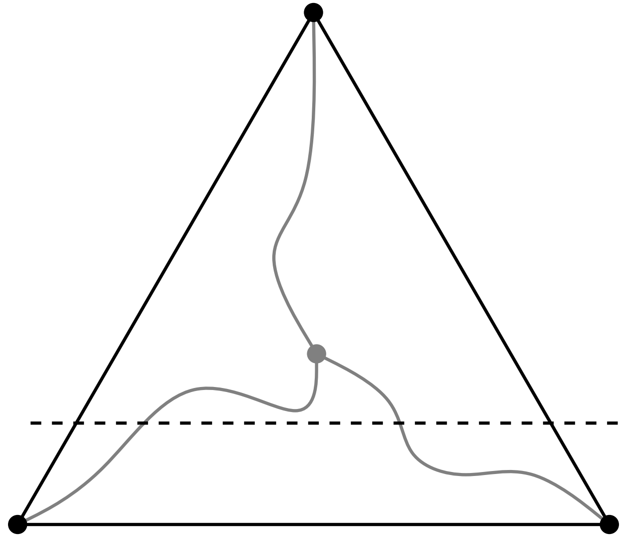

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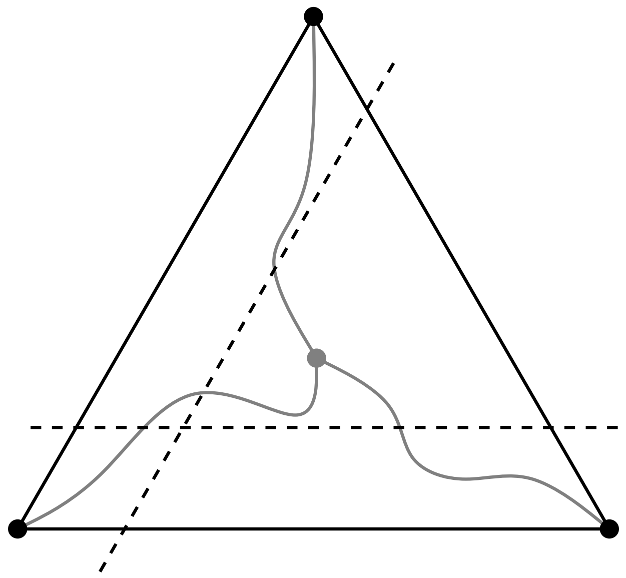

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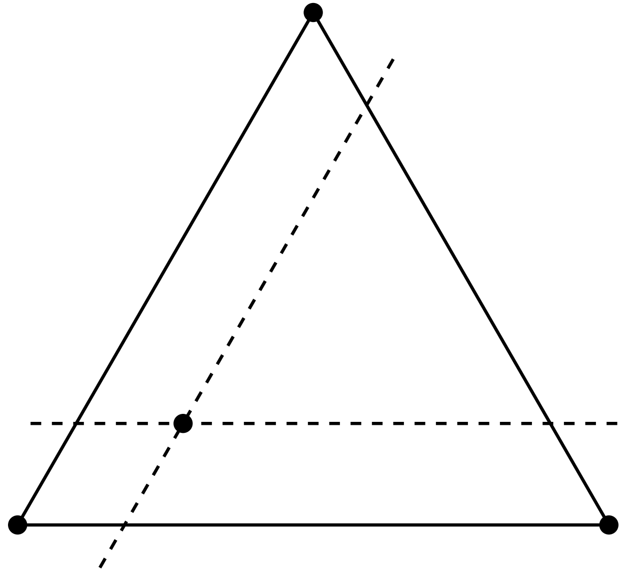

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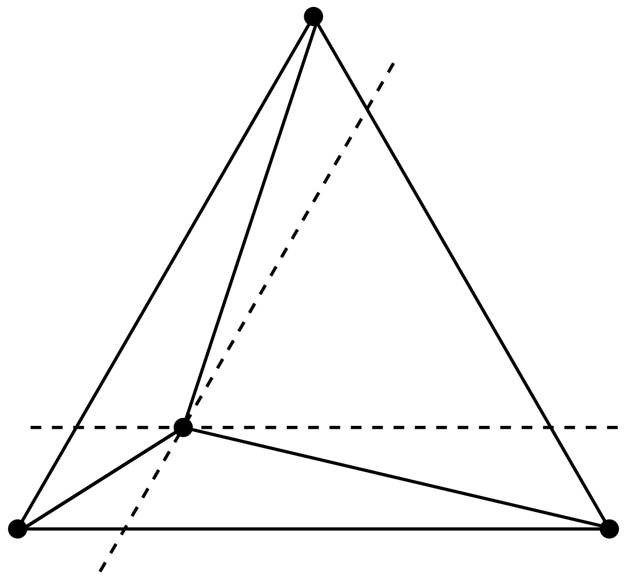

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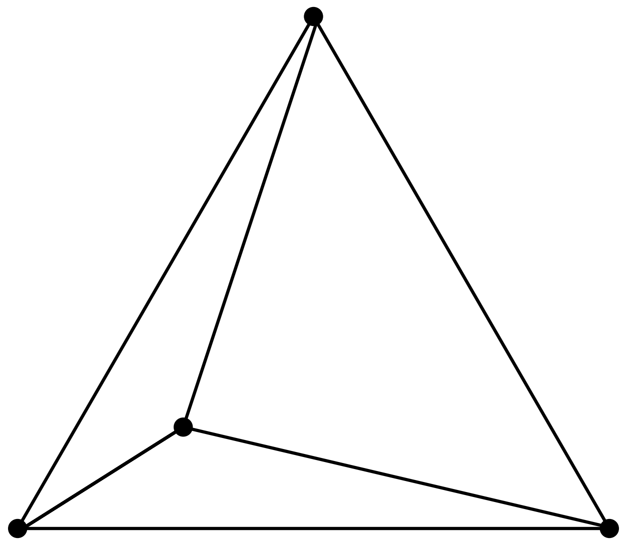

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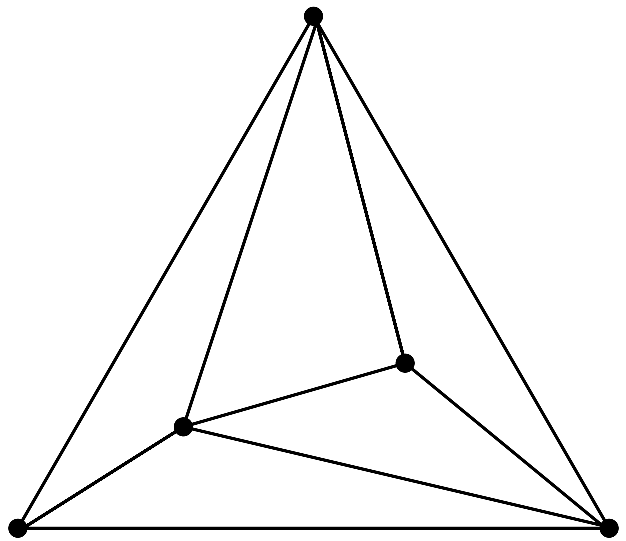

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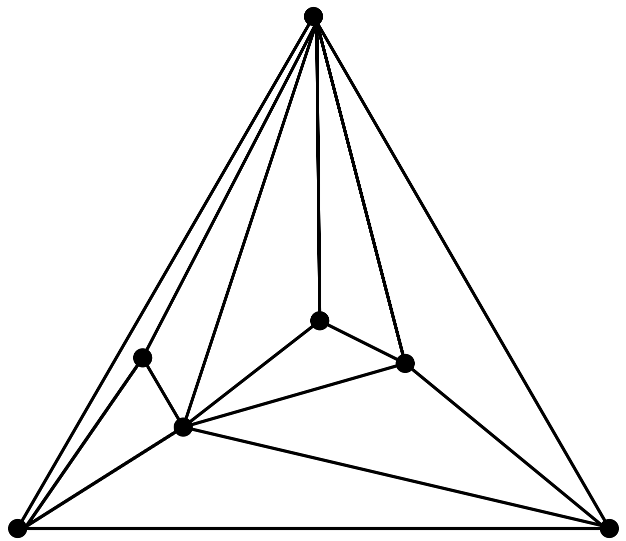

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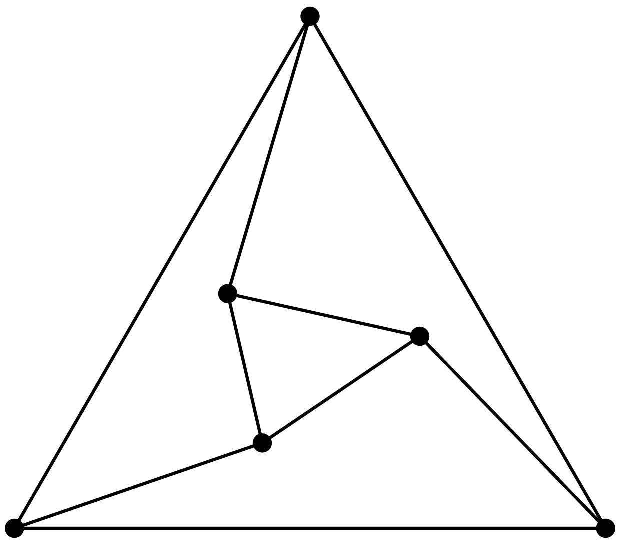

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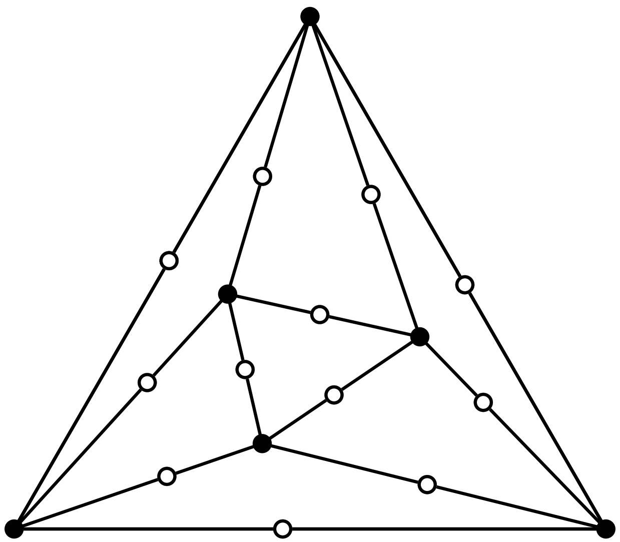

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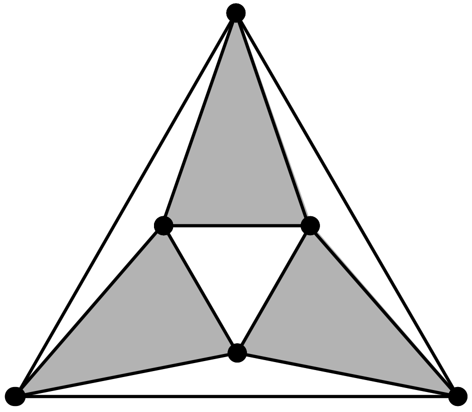

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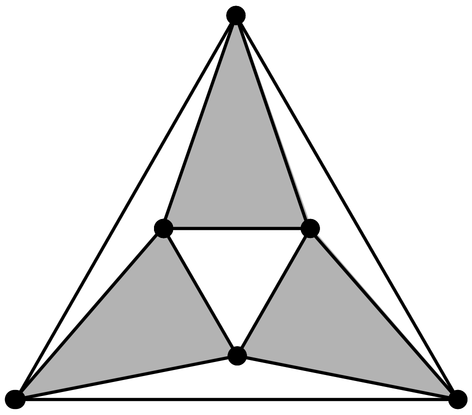

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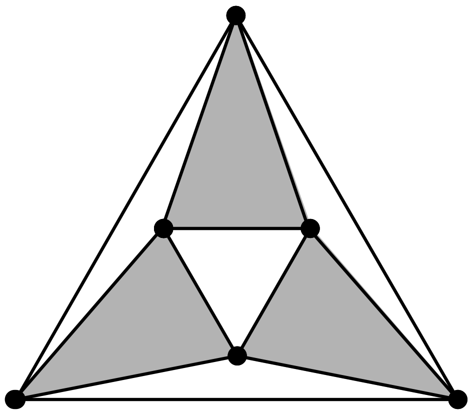

Proving Area-Universality

- Area-Universality is maintained by taking subgraphs.

$\Longrightarrow$ triangulations

Proving Area-Universality

T plane triangulation, $\mathcal{A}$ area assignment

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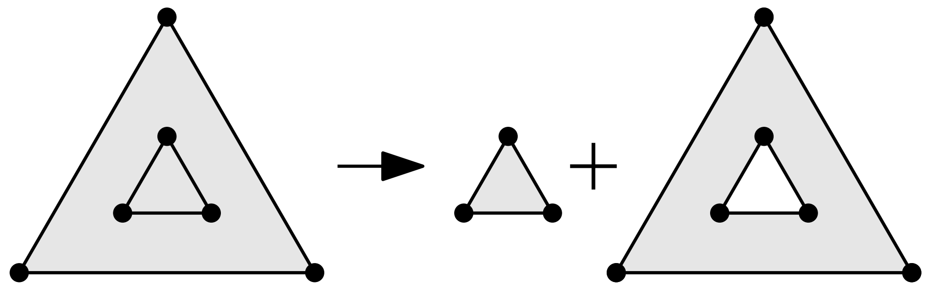
T plane triangulation, $\mathcal{A}$ area assignment

- T area-universal $\iff$ all 4-connected subgraphs of T are area-universal



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- $\mathcal{A}$ realizable $\iff$ AEQ$(T, \mathcal{A})$ has real solution

$$2 \cdot \textsc{area}(v_1, v_2, v_3) = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}$$

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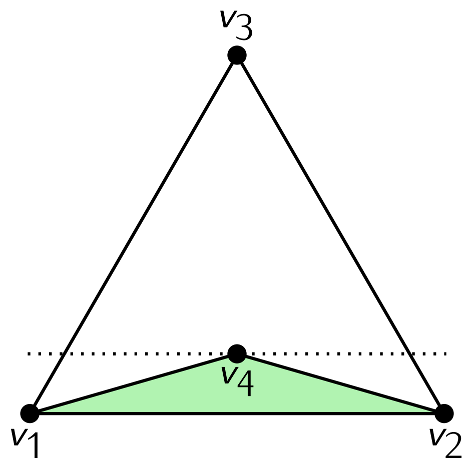

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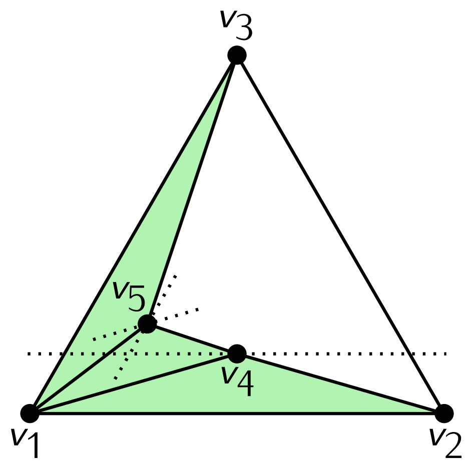

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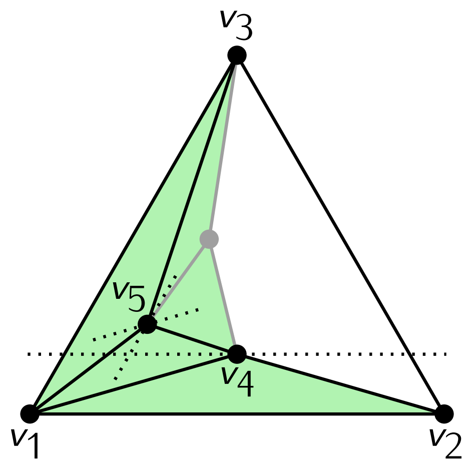

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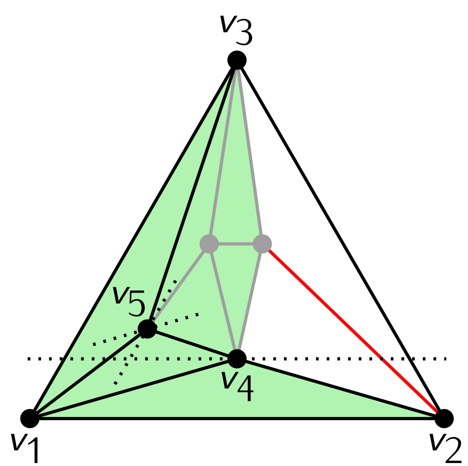

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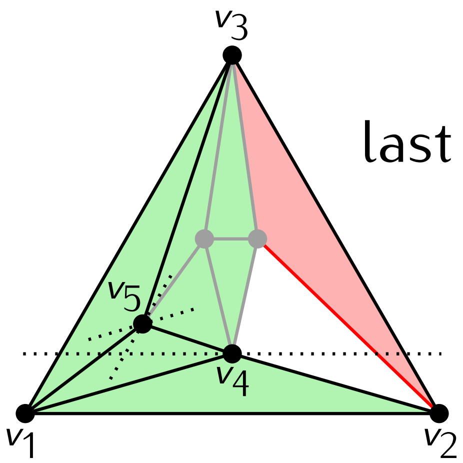

last face function $\mathfrak{f}(x_4)$

Proving Area-Universality

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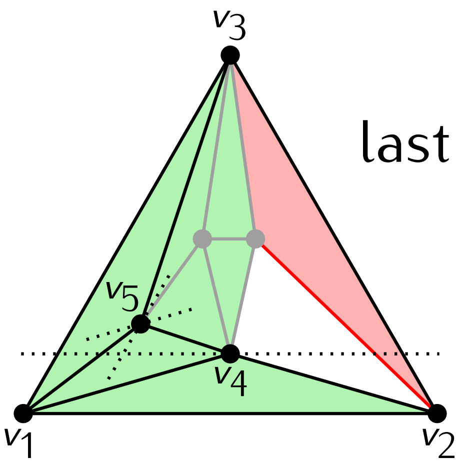

last face function $\mathfrak{f}(x_4) = \frac{p(x_4)}{q(x_4)}$

Proving Area-Universality

T plane triangulation, $\mathcal{A}$ area assignment

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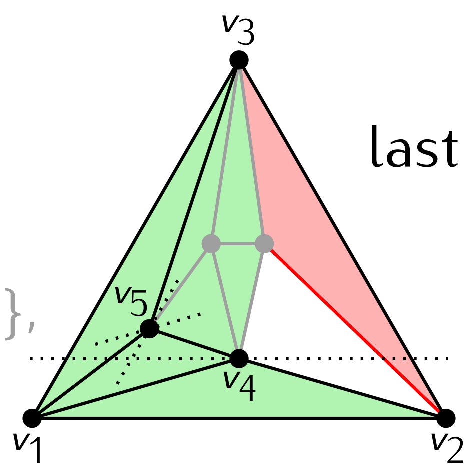
p-order

- pred$(v_i) \subset \{v_1, v_2, \ldots, v_{i-1}\}$,
- pred$(v_1) = \emptyset$,
  pred$(v_2) = \{v_1\}$,
  pred$(v_3) =$ pred$(v_4) = \{v_1, v_2\}$,
- for all $i > 4$: $|\text{pred}(v_i)| = 3$.



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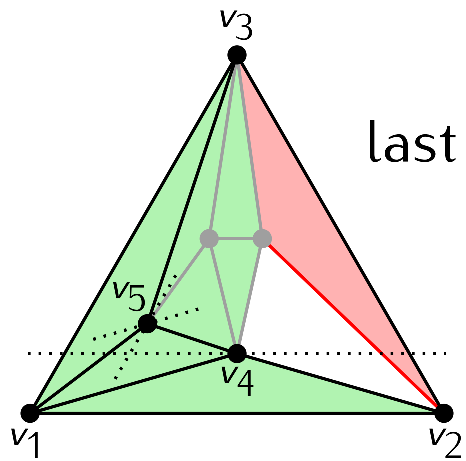

last face function $\mathfrak{f}(x_4) = \frac{p(x_4)}{q(x_4)}$

T plane triangulation with p-order.
If $\mathfrak{f}$ is nice for a dense $\mathbb{A}' \subset \mathbb{A}$,
then T is area-universal.

Proving Area-Universality

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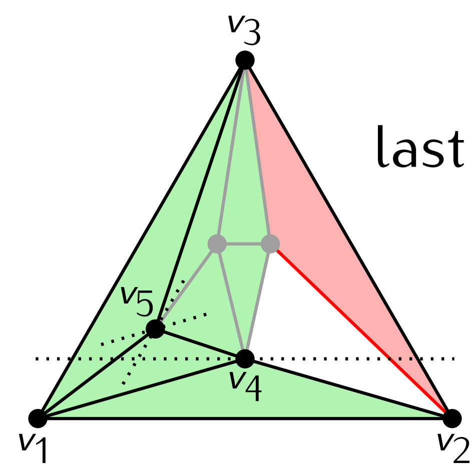

last face function $\mathfrak{f}(x_4) = \frac{p(x_4)}{q(x_4)}$

T plane triangulation with p-order.
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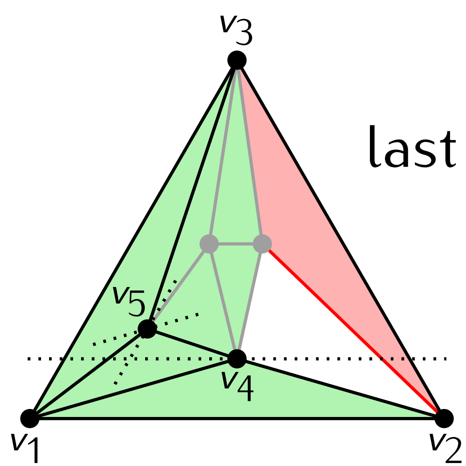

last face function $\mathfrak{f}(x_4) = \frac{p(x_4)}{q(x_4)}$

T plane triangulation with p-order.
If $\mathfrak{f}$ is super nice for generic $\mathcal{A}$,
then every $G \in [T]$ is area-universal.

Proving Area-Universality

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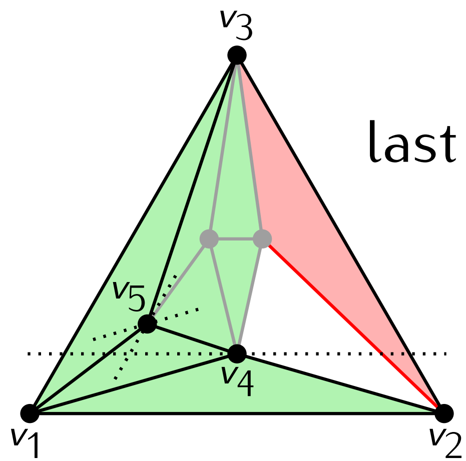

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T plane triangulation with p-order.
If $\mathfrak{f}$ is super nice (crr-free, odd max-degree) for generic $\mathcal{A}$, then every $G \in [T]$ is area-universal.

Proving Area-Universality

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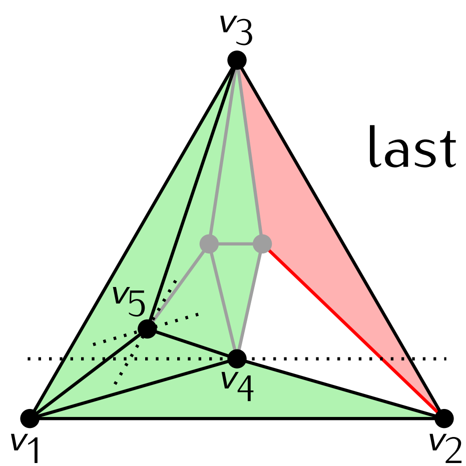

last face function $\mathfrak{f}(x_4) = \frac{p(x_4)}{q(x_4)}$

independent of embedding!

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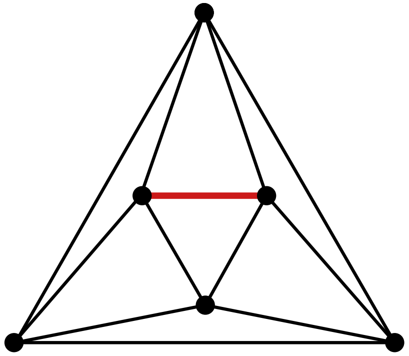
Application

Accordion graphs $\mathcal{K}_\ell$



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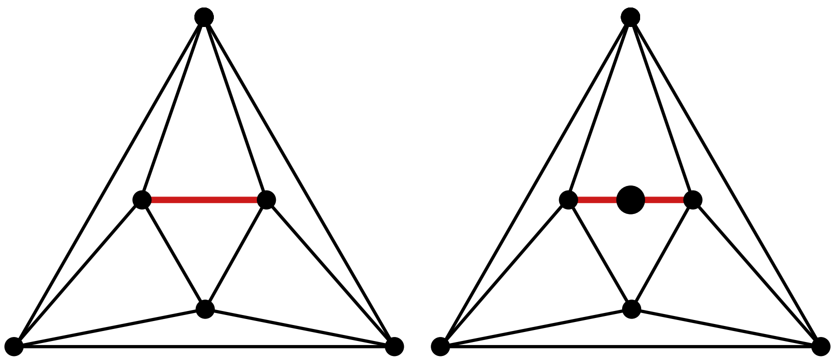

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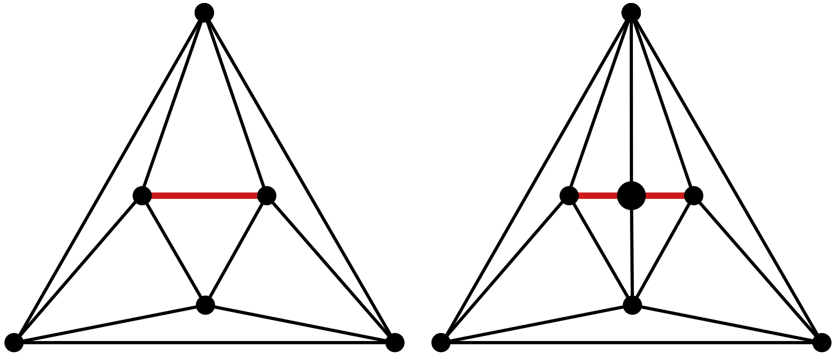

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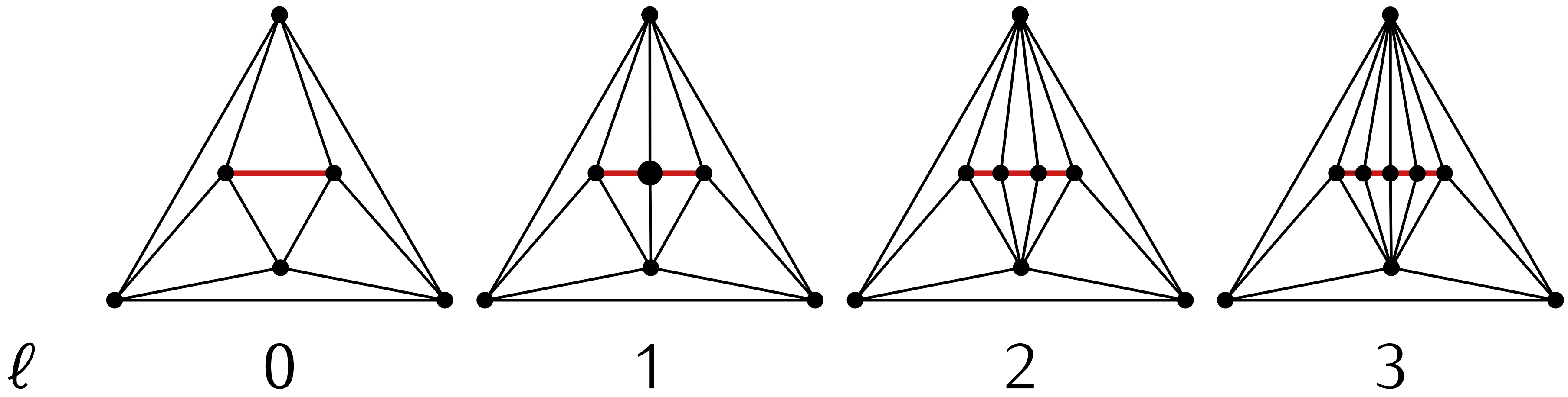

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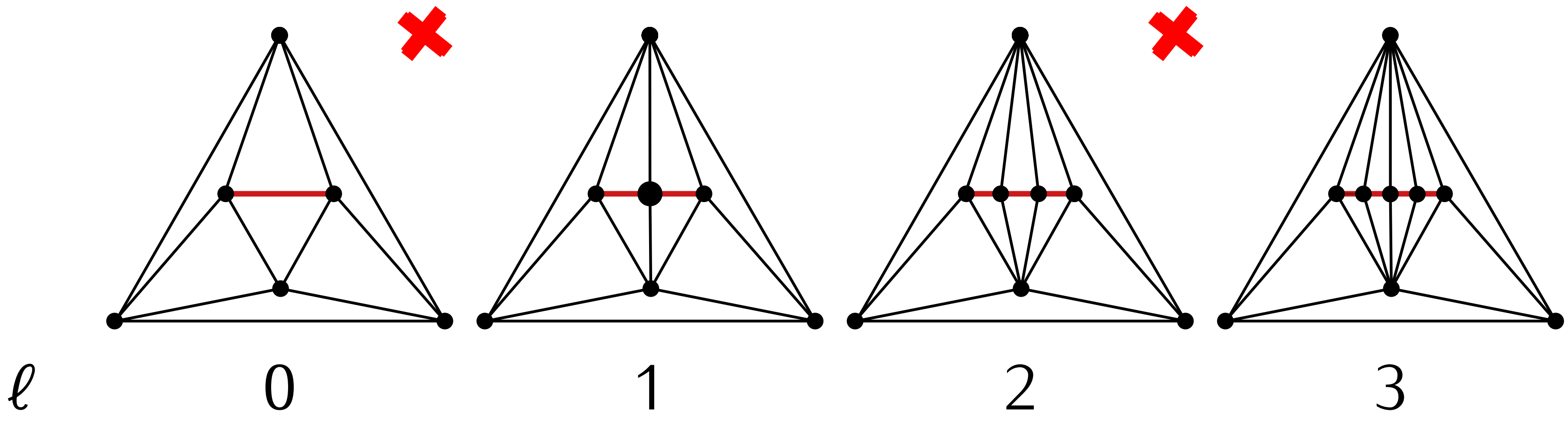
- even accordions are not area-universal (Eulerian)



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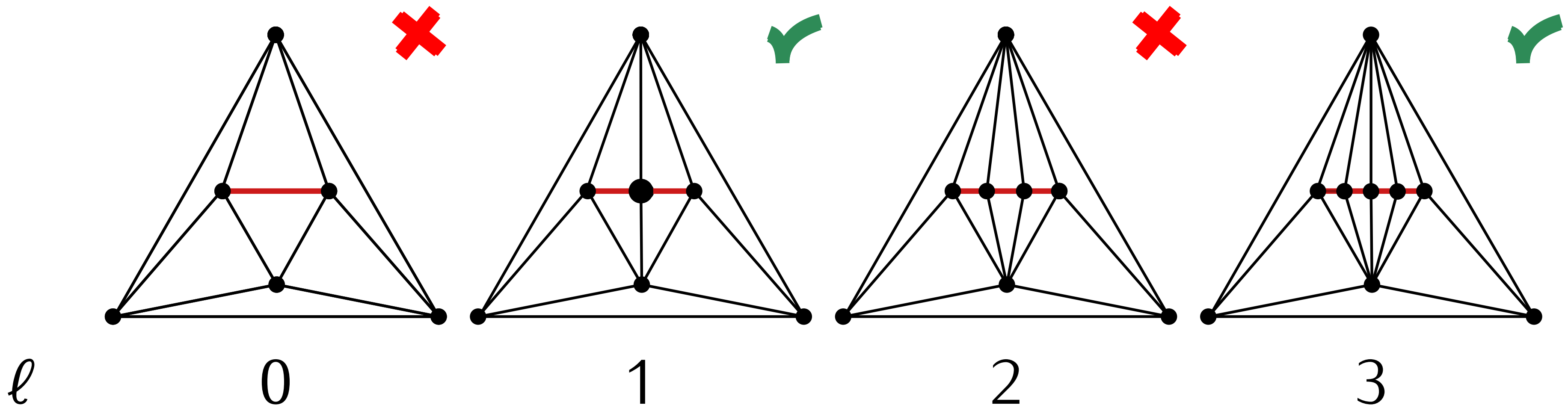

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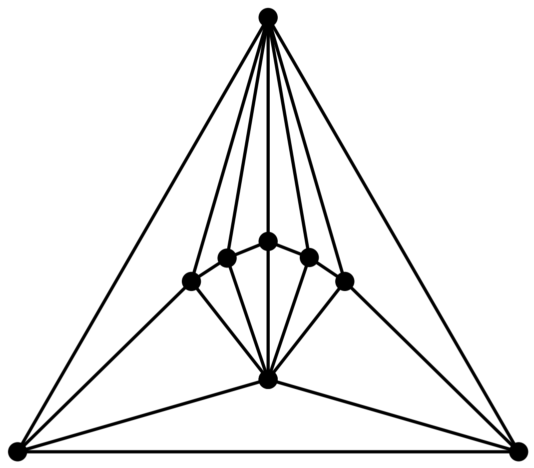

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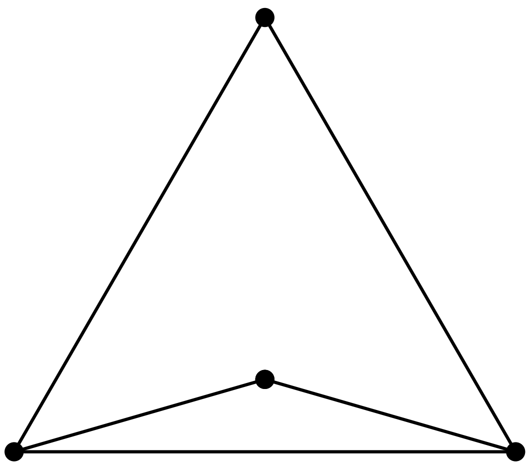

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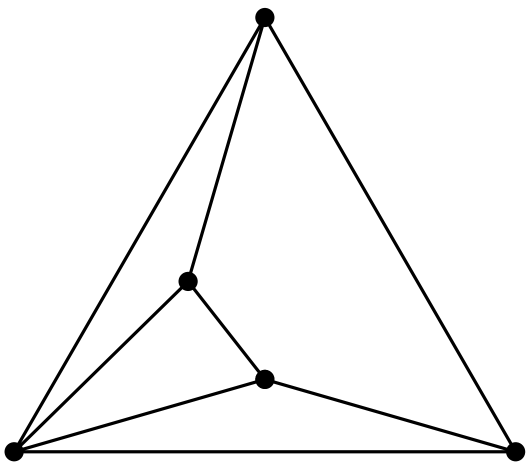

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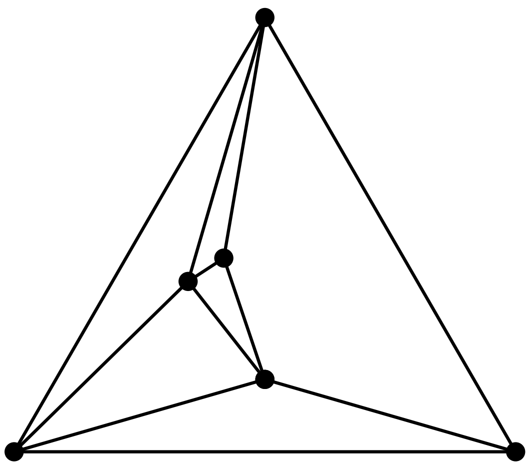

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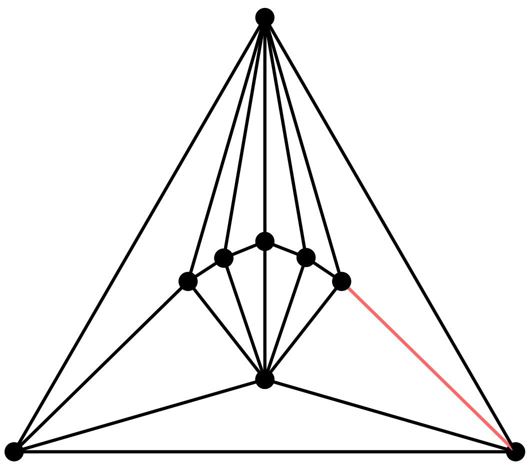

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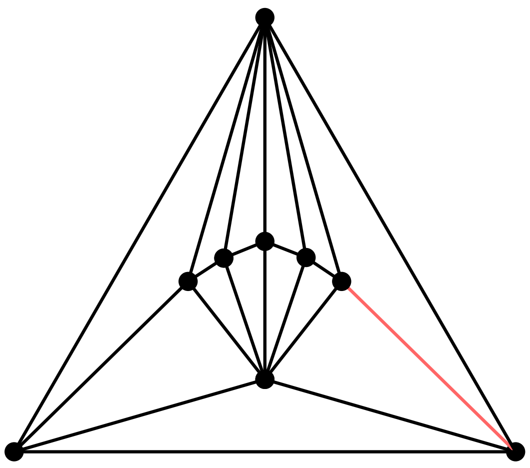
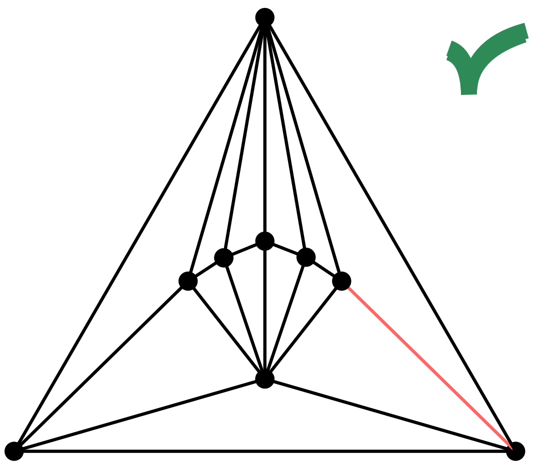


p-Order ✓

Application

T plane triangulation with p-order.
If $\mathfrak{f}$ is super nice (crr-free, odd max-degree) for generic $\mathcal{A}$,
then every $G \in [T]$ is area-universal.



p-Order ✓

ℓ odd $\implies$ $\mathfrak{f}$ super nice

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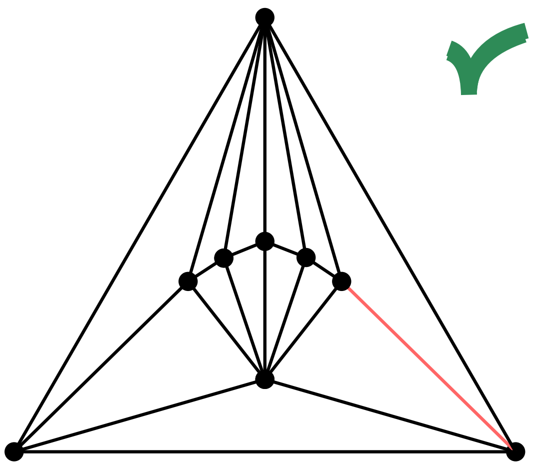

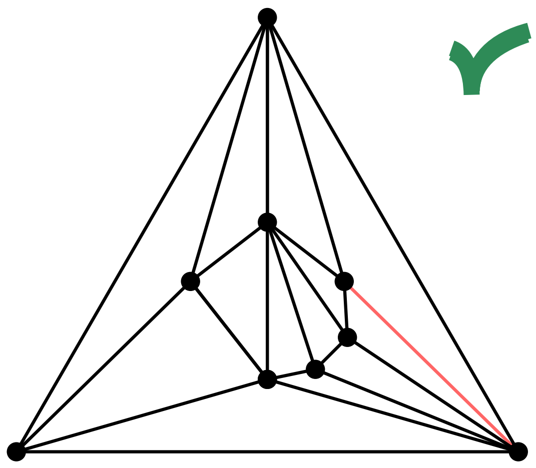

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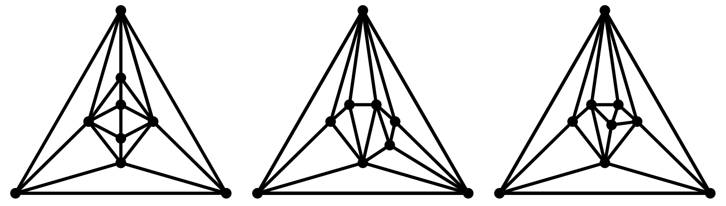
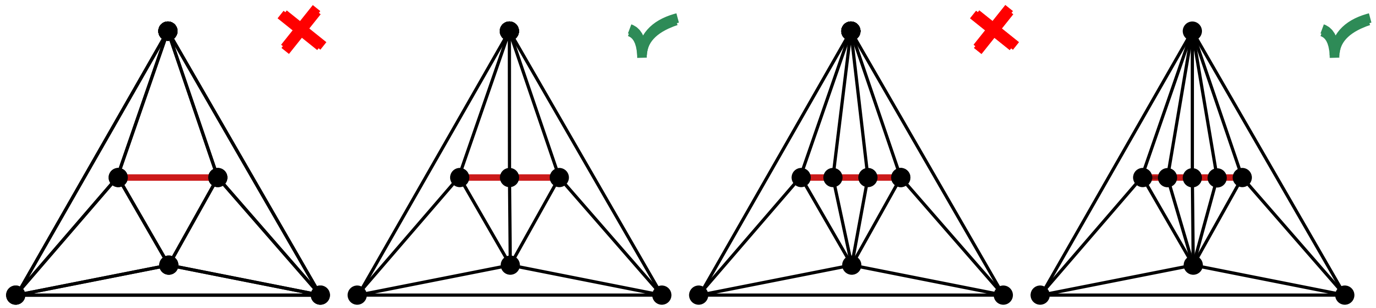


Summary & Open Problems

- Sufficient criterion for area-universality of triangulations with p-Order

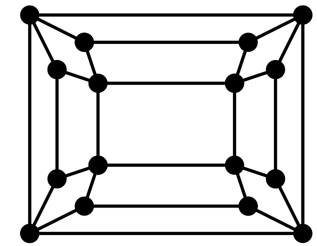


  - analysis of one area assignment
  - shows area-universality for all embeddings

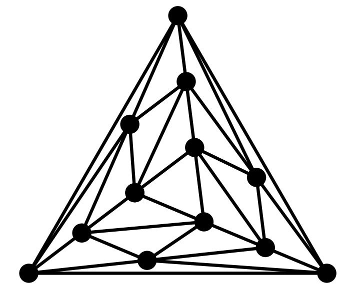
- Is area-universality a property of *plane or planar* graphs?
- ∃ characterization by *local* properties?

Open Problems II

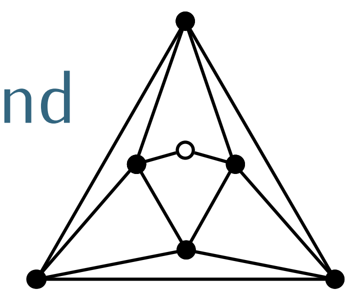
- Area-universal graph classes?
  - bipartite?



- Are 4-connected triangulations equiareal?



- Optimal bend drawings
  - How many bends are always sufficient and sometimes necessary?



- Computational complexity
  - $\forall\exists\mathbb{R}$-complete?