On contact graphs of paths on a grid

Z. Deniz[1], E. Galby[2], A. Munaro[3], B. Ries[2]

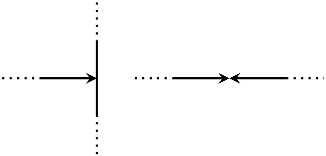
[1]Duzce University
[2]University of Fribourg
[3]University of West Virginia

Definition

$G = (V, E)$ is a *contact graph of paths on a grid* (CPG) if there exists a collection of interiorly disjoint paths on a grid in one-to-one correspondence with V such that two vertices are adjacent if and only if the corresponding paths have at least one grid-point in common.

Definition

$G = (V, E)$ is a *contact graph of paths on a grid* (CPG) if there exists a collection of **interiorly disjoint** paths on a grid in one-to-one correspondence with V such that two vertices are adjacent if and only if the corresponding paths have at least one grid-point in common.



(A) Allowed contacts.

(B) Forbidden contacts.

Figure: Types of contact between two paths.

Definition

$G = (V, E)$ is a *contact graph of paths on a grid* (CPG) if there exists a collection of interiorly disjoint paths on a grid in one-to-one correspondence with V such that two vertices are adjacent if and only if the corresponding paths have at least one grid-point in common.
If every path has at most k bends, then G is B_k-CPG.

Definition

$G = (V, E)$ is a *contact graph of paths on a grid* (*CPG*) if there exists a collection of interiorly disjoint paths on a grid in one-to-one correspondence with V such that two vertices are adjacent if and only if the corresponding paths have at least one grid-point in common.
If every path has at most k bends, then G is B_k-*CPG*.

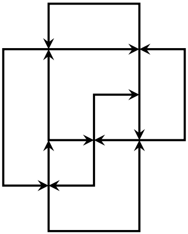


FIGURE: A 2-bend CPG representation of K_6.

Structural Properties

Lemma

A CPG graph is either 6-regular or has a vertex a degree at most 5.

Structural Properties

Lemma

A CPG graph is either 6-regular or has a vertex a degree at most 5.

- There exists an infinite family of 6-regular CPG graphs.

Structural Properties

Lemma

A CPG graph is either 6-regular or has a vertex a degree at most 5.

- There exists an infinite family of 6-regular CPG graphs.

Proposition

Every B_1-CPG graph has a vertex of degree at most 5.

Maximum Cliques

Theorem

CPG graphs are K_7-free.

Maximum Cliques

Theorem

CPG graphs are K_7-free.

Proposition

K_6 *is in* B_2*-CPG* $\setminus B_1$*-CPG.*

Maximum Cliques

Theorem

CPG graphs are K_7-free.

	maximum clique
B_0-CPG	≤ 4
B_1-CPG	≤ 5
B_k-CPG, $k \geq 2$	≤ 6

Recognition

Theorem

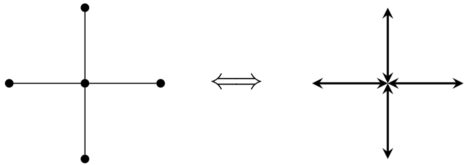
Recognition *is* NP*-complete for* B_0*-CPG graphs.*

Recognition

Theorem

Recognition *is* NP*-complete for* B_0*-CPG graphs.*

- G is rectilinear planar if and only if $L(G)$ is B_0-CPG.

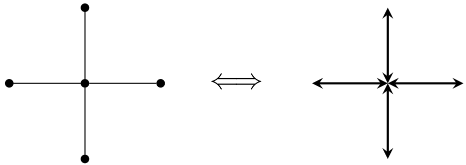


Recognition

Theorem

Recognition *is* NP*-complete for* B_0*-CPG graphs.*

- G is rectilinear planar if and only if $L(G)$ is B_0-CPG.

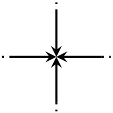


- Recognition is NP-complete for rectilinear planar graphs.

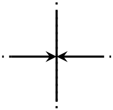
Planarity and CPG graphs

Lemma

If G is a CPG graph for which there exists a CPG representation containing no grid-point of type I or II.a, then G is planar. In particular, if G is a triangle-free CPG graph, then G is planar.



Type I



Type II.a

Planarity and CPG graphs

- It follows that CPG graphs are $K_{3,3}$-free.

Planarity and CPG graphs

- It follows that CPG graphs are $K_{3,3}$-free.
- However, for any $k \geq 0$, B_k-CPG $\not\subseteq$ Planar.

Figure: A B_0-CPG graph containing $K_{3,3}$ as a minor (contract e).

Planarity and CPG graphs

Lemma

If G is a planar CPG graph, then G has at most $4n - 2f + 4$ vertices of degree at most 3. In particular, if G is maximally planar, then G has at most 12 vertices of degree at most 3.

Planarity and CPG graphs

Lemma

If G is a planar CPG graph, then G has at most $4n - 2f + 4$ vertices of degree at most 3. In particular, if G is maximally planar, then G has at most 12 vertices of degree at most 3.

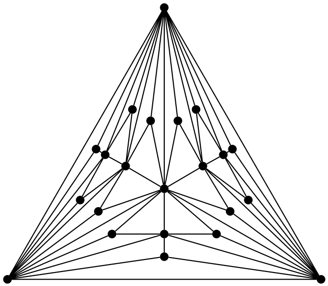


Figure: A non CPG maximally planar graph.

Vertex Coloring

Theorem

CPG graphs are 6-colorable.

Vertex Coloring

Theorem

CPG graphs are 6-colorable.

- Bound tight for B_k-CPG graphs with $k \geq 2$ since $K_6 \in B_2$-CPG.

Vertex Coloring

Theorem

CPG graphs are 6-colorable.

- Bound tight for B_k-CPG graphs with $k \geq 2$ since $K_6 \in B_2$-CPG.

Theorem

B_0-CPG graphs are 4-colorable.

Vertex Coloring

Theorem

CPG graphs are 6-colorable.

- Bound tight for B_k-CPG graphs with $k \geq 2$ since $K_6 \in B_2$-CPG.

Theorem

B_0-CPG graphs are 4-colorable.

- Bound tight since $K_4 \in B_0$-CPG.

Vertex Coloring

Theorem

CPG graphs are 6-colorable.

- Bound tight for B_k-CPG graphs with $k \geq 2$ since $K_6 \in B_2$-CPG.

Theorem

B_0-CPG graphs are 4-colorable.

- Bound tight since $K_4 \in B_0$-CPG.
- Open: tight bound for B_1-CPG graphs.

3-Colorability

- 3-colorability is NP-complete in CPG (Hliněný 1998).

3-COLORABILITY

- 3-COLORABILITY is NP-complete in CPG (Hliněný 1998).

THEOREM

3-COLORABILITY *is* NP*-complete in* B_0*-CPG.*

3-Colorability

- 3-colorability is NP-complete in CPG (Hliněný 1998).

Theorem

3-colorability *is* NP*-complete in* B_0*-CPG.*

- Reduce from 3-colorability restricted to planar graphs of maximum degree 4 (Garey et al. 1976).

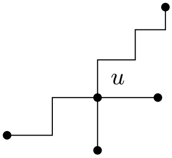
3-Colorability

- 3-colorability is NP-complete in CPG (Hliněný 1998).

Theorem

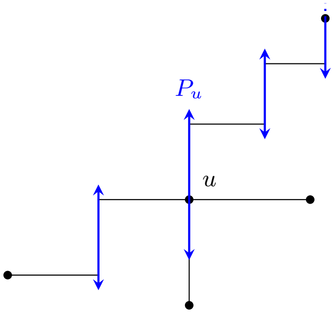
3-colorability *is* NP*-complete in* B_0*-CPG.*

- Given an instance $G = (V, E)$, we construct a B_0-CPG graph G' s.t. G is 3-colorable if and only if G' is 3-colorable, as follows. Consider a grid embedding of G (Tamassia and Tollis 1989).



3-Colorability

- Associate with each vertex $u \in V$ a vertical path P_u containing (x_u, y_u), and consider every interior vertical segment of an edge as a vertical path.



3-Colorability

- Associate with each vertex $u \in V$ a vertical path P_u containing (x_u, y_u), and consider every interior vertical segment of an edge as a vertical path.

- Add between two consecutive vertical paths a sequence of gadgets H and H', where H' is $H[\{b, c, 4, 5, 6, 7, 8, 9, 10\}]$.

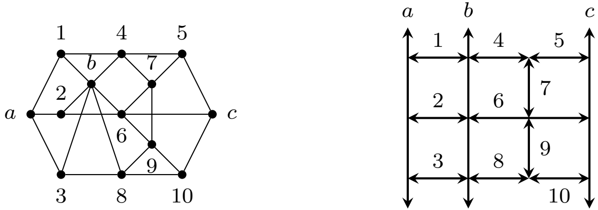


Figure: H (left) and a 0-bend CPG representation of H (right).

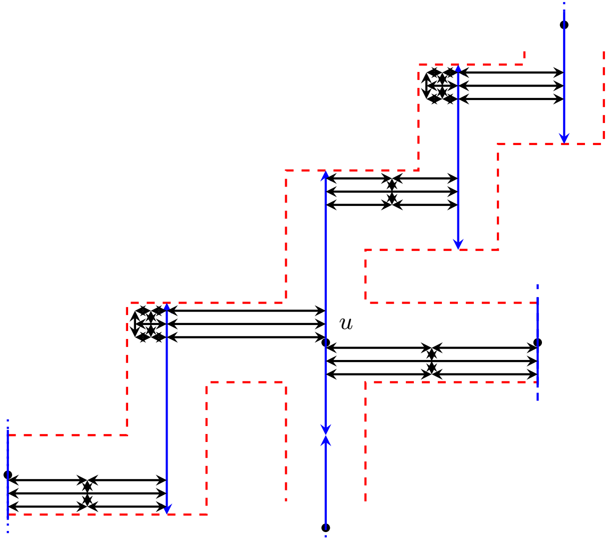


Figure: The transformations for u.

Open questions

- Can we characterize those planar graphs which are CPG?
- Is Recognition NP-complete for B_1-CPG graphs?
- Are B_1-CPG graphs 5-colorable?