

Recognition and Drawing of Stick Graphs

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Graph Drawing, 2018

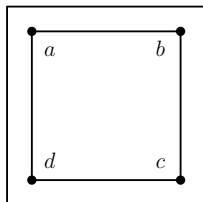
- 1 Introduction to Intersection Graphs
- 2 Stick Graphs
- 3 Related Work
- 4 Our Contribution
 - Studied Cases
 - Fixed As and Bs
 - Fixed As
 - Stick
- 5 Conclusion and Open Problems

Intersection Graphs

Context: Intersection Graphs

Vertices: Shapes in the plane

Edges: Intersection of shapes



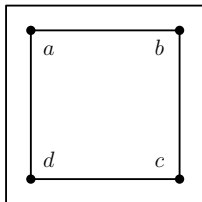
Graph

Intersection Graphs

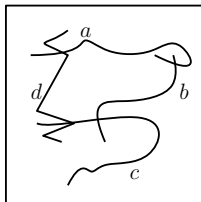
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Graph



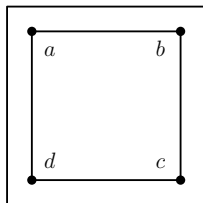
Strings

Intersection Graphs

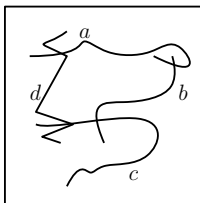
Context: Intersection Graphs

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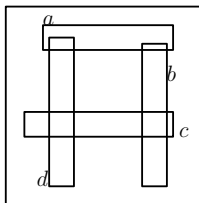
Edges: Intersection of shapes



Graph



Strings



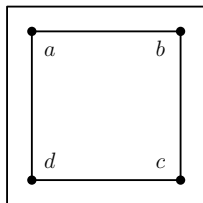
Boxes

Intersection Graphs

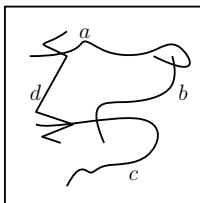
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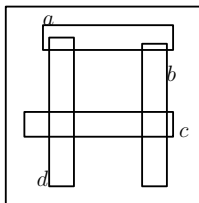
Edges: Intersection of shapes



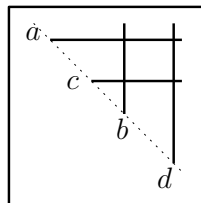
Graph



Strings



Boxes



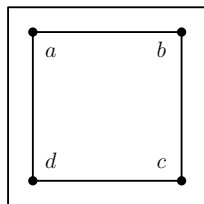
Segments

Intersection Graphs

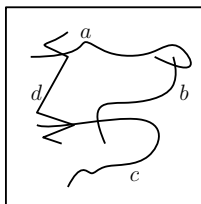
Context: Intersection Graphs

Vertices: Shapes in the plane

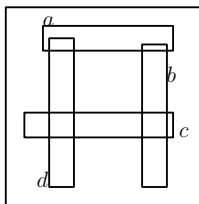
Edges: Intersection of shapes



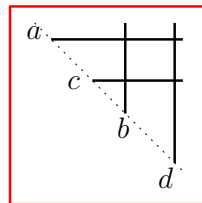
Graph



Strings



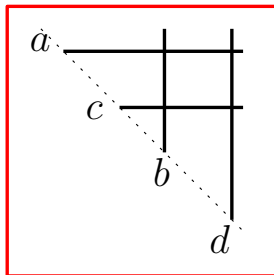
Boxes



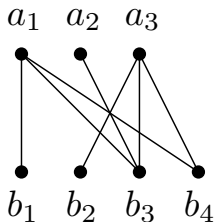
Stick

Stick Graphs

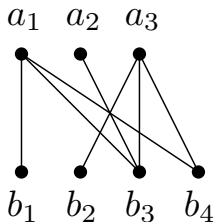
A **Stick Graph** is an intersection graph of axis-aligned segments such that the left end-points of the horizontal segments and the bottom end-points of the vertical segments lie on a “ground line”, a line with slope -1 .



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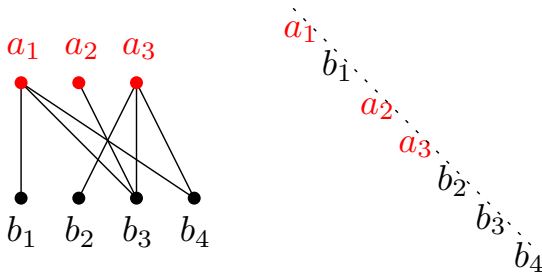


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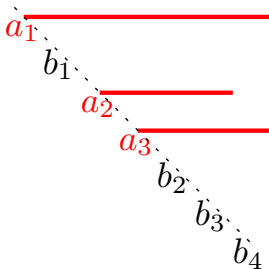
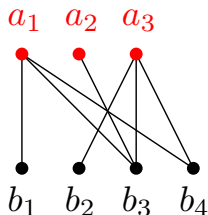


Stick Graphs

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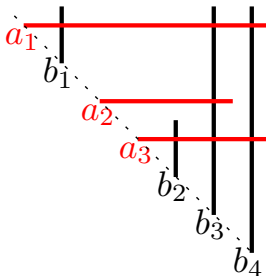
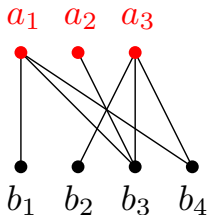


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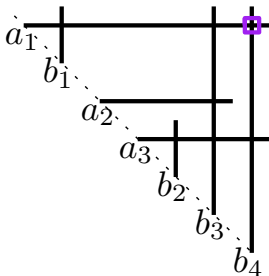
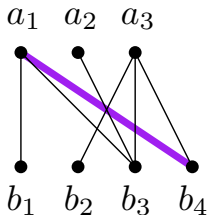
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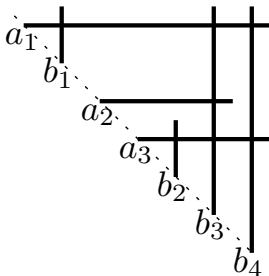
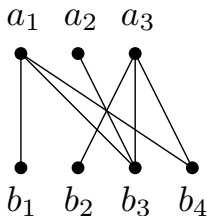


Stick Graphs

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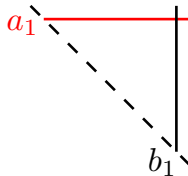
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Stick Graphs

Examples

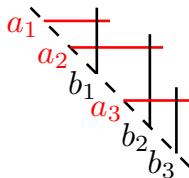
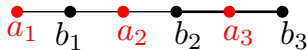
Edge



Stick Graphs

Examples

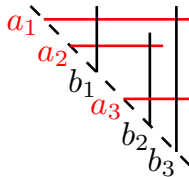
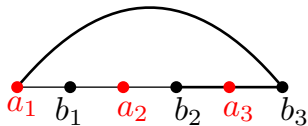
Path



Stick Graphs

Examples

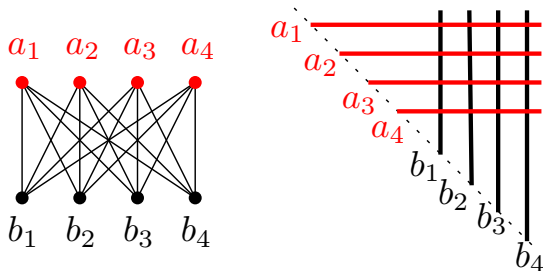
Cycle



Stick Graphs

Examples

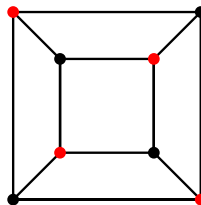
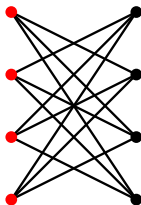
Complete bipartite graph



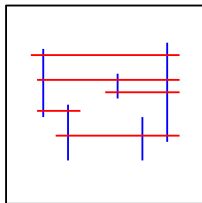
Stick Graphs

Examples

Not a Stick Graph



Grid Intersection Graphs Recognition

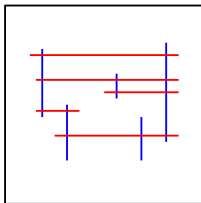


GIG

NP-Complete

(Kratochvil 1994)

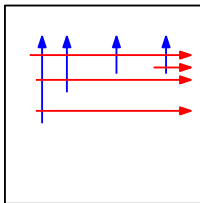
Grid Intersection Graphs Recognition



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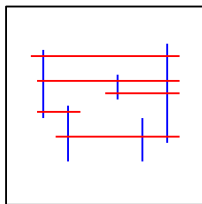


2DOR

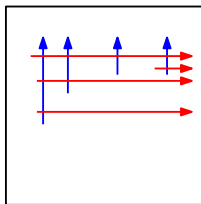
$O(nm)$ -time

(Cogis 1982)

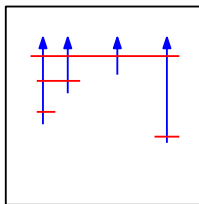
Grid Intersection Graphs Recognition



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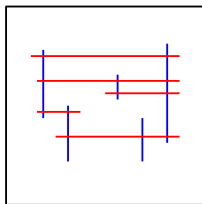


2DOR
 $O(nm)$ -time
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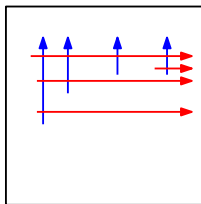


SegRay
Open
(Chaplick 2018)

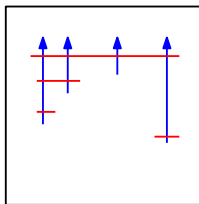
Grid Intersection Graphs Recognition



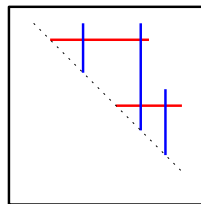
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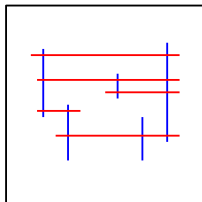


SegRay
Open
(Chaplick 2018)



Stick
Open
(Chaplick 2018)

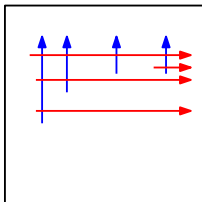
Grid Intersection Graphs Forbidden Matrices



GIG

γ_1

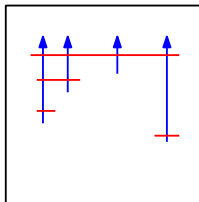
(Hartman 1991)



2DOR

γ_2, γ_3

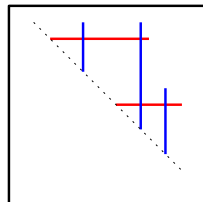
(Shrestha 2010)



SegRay

γ_4

(Chaplick 2014)



Stick

???

our contribution

$$\gamma_1 = \begin{bmatrix} * & 1 & * \\ 1 & 0 & 1 \\ * & 1 & * \end{bmatrix}$$

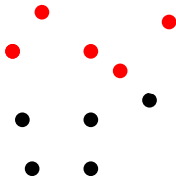
$$\gamma_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\gamma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\gamma_4 = \begin{bmatrix} 1 & 0 & 1 \\ * & 1 & * \end{bmatrix}$$

Stick Graph Introduction

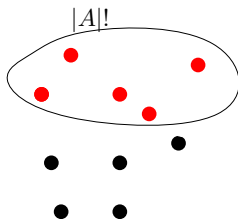
Naive algorithm



Complexity

Stick Graph Introduction

Naive algorithm

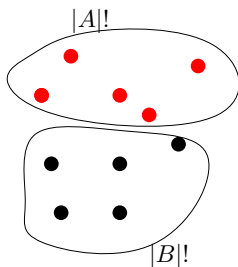


Complexity

- $|A|!$ ordering for A s vertices (σ_A)

Stick Graph Introduction

Naive algorithm

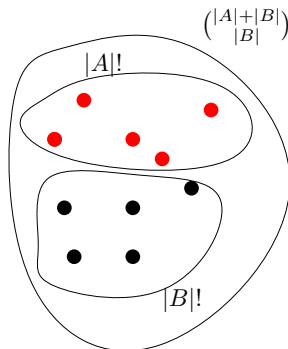


Complexity

- $|A|!$ ordering for A's vertices (σ_A)
- $|B|!$ ordering for B's vertices (σ_B)

Stick Graph Introduction

Naive algorithm

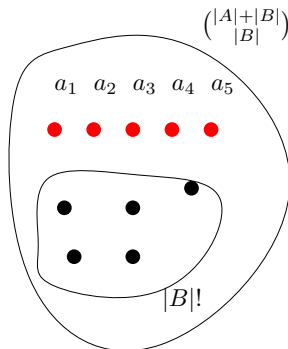


Complexity

- $|A|!$ ordering for As vertices (σ_A)
 - $|B|!$ ordering for Bs vertices (σ_B)
 - $\binom{|A|+|B|}{|B|}$ merging ordered As and Bs (σ)
- $|A|!|B|!\binom{|A|+|B|}{|B|}$

Stick Graph Introduction

Naive algorithm

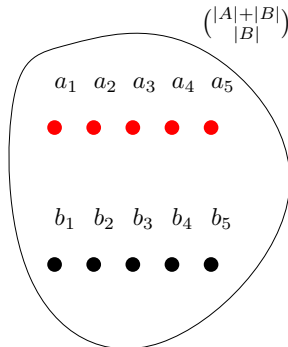


Complexity

- ~~$|A|!$ ordering for A s vertices~~
 - $|B|!$ ordering for B s vertices (σ_B)
 - $\binom{|A|+|B|}{|B|}$ merging ordered A s and B s (σ)
- $|B|! \binom{|A|+|B|}{|B|}$

Stick Graph Introduction

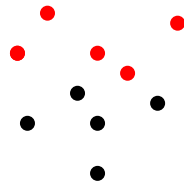
Naive algorithm



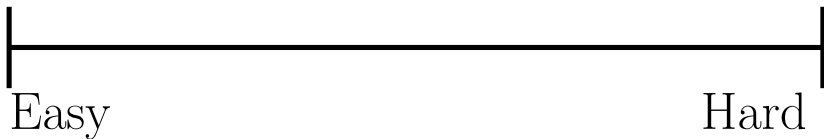
Complexity

- ~~$|A|!$ ordering for As vertices~~
- ~~$|B|!$ ordering for Bs vertices~~
- $\binom{|A|+|B|}{|B|}$ merging ordered As and Bs (σ)

→ $\binom{|A|+|B|}{|B|}$

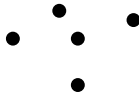


no order given
Stick



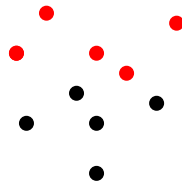
Studied Cases

a_1 a_2 a_3 a_4 a_5



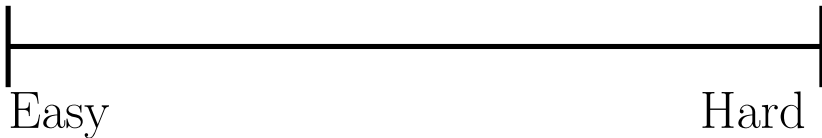
σ_A given

Fixed A_s



no order given

Stick



Studied Cases

a_1 a_2 a_3 a_4 a_5



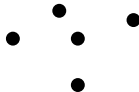
b_1 b_2 b_3 b_4 b_5



σ_A and σ_B given

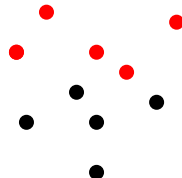
Fixed A s and B s

a_1 a_2 a_3 a_4 a_5



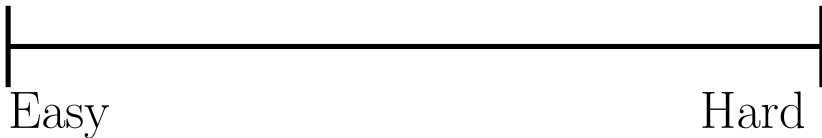
σ_A given

Fixed A s



no order given

Stick



Fixed As Bs

Statement

Problem: *Stick* REPRESENTATION WITH FIXED AS AND BS (*Stick*_{AB})

Input: A bipartite graph $G = (A \cup B, E)$, an ordering σ_A of the vertices in A , and an ordering σ_B of the vertices in B .

Question: Does G admit a *Stick* representation such that the i th horizontal segment on the ground line corresponds to the i th vertex of σ_A and the j th vertical segment on the line corresponds to the j th vertex of σ_B ?

a_1 a_2 a_3 a_4 a_5



b_1 b_2 b_3 b_4 b_5



σ_A and σ_B given

Fixed As and Bs

Fixed A s B s

Ordering constraints

Input: M ; σ_A ; σ_B ;

Constraints for the ordering on the ground line

Fixed As Bs

Ordering constraints

Input: $M; \sigma_A; \sigma_B;$

Constraints for the ordering on the ground line

- $(a_{i-1} \prec a_i)$ according to σ_A and $(b_{i-1} \prec b_i)$ according to σ_B

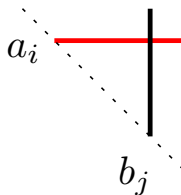
Fixed As Bs

Ordering constraints

Input: M ; σ_A ; σ_B ;

Constraints for the ordering on the ground line

- $(a_{i-1} \prec a_i)$ according to σ_A and $(b_{i-1} \prec b_i)$ according to σ_B
- $(a_i \prec b_j)$ for each $m_{i,j} = 1$



matrix entry

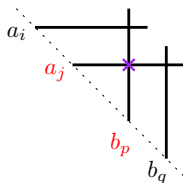
Fixed As Bs

Ordering constraints

Input: M ; σ_A ; σ_B ;

Constraints for the ordering on the ground line

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- $(a_i \prec b_j)$ for each $m_{i,j} = 1$
- $(b_p \prec a_j)$ for each configuration $\begin{matrix} a_i & b_p & b_q \\ a_j & \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \end{matrix}$



$(a_j \prec b_p)$
wrong ordering

Fixed As Bs

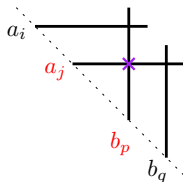
Ordering constraints

Input: M ; σ_A ; σ_B ;

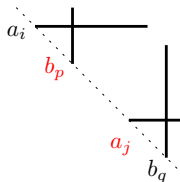
Constraints for the ordering on the ground line

- $(a_{i-1} \prec a_i)$ according to σ_A and $(b_{i-1} \prec b_i)$ according to σ_B
- $(a_i \prec b_j)$ for each $m_{i,j} = 1$
- $(b_p \prec a_j)$ for each configuration

$$\begin{matrix} a_i & b_p & b_q \\ a_j & \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \end{matrix}$$



$(a_j \prec b_p)$
wrong ordering

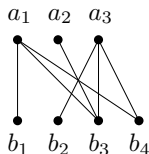


$(b_p \prec a_j)$
correct ordering

Fixed As Bs

Testing algorithm

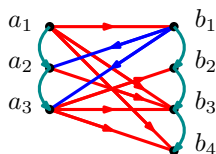
Transform the constraints into edges to build a testing algorithm.



G

$$M = \begin{matrix} & b_1 & b_2 & b_3 & b_4 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

M



H

Given a matrix representation M

Create a graph H

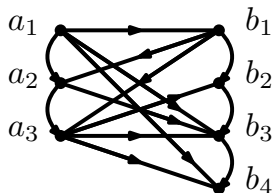
Add edges following these constraints (edges indicate precedence):

- (a_{i-1}, a_i) according to σ_A
- (b_{i-1}, b_i) according to σ_B
- (a_i, b_j) for each $m_{ij} = 1$
- (b_p, a_j) for each configuration

$$\begin{matrix} a_i \\ a_j \end{matrix} \begin{bmatrix} b_p & b_q \\ 1 & * \\ 0 & 1 \end{bmatrix}$$

Fixed As Bs

Testing algorithm

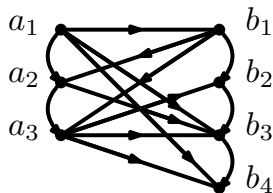


Lemma

A given graph admits a Stick representation respecting σ_A and σ_B if and only if H is acyclic, i.e., the constraints are consistent.

Fixed As Bs

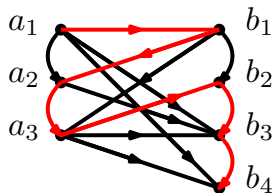
Testing algorithm



If the topological ordering of vertices of H gives the total ordering of vertices.

Fixed As Bs

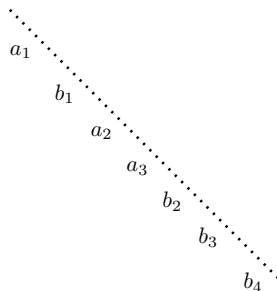
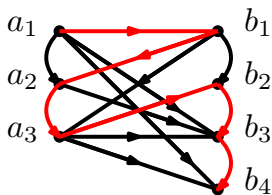
Testing algorithm



If the **topological ordering** of vertices of H gives the total ordering of vertices.

Fixed As Bs

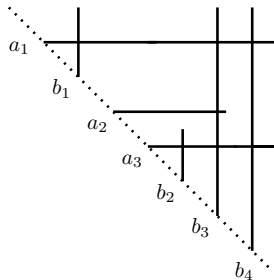
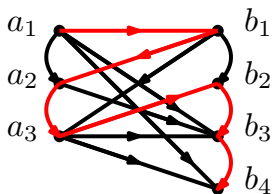
Testing algorithm



If the **topological ordering** of vertices of H gives the total ordering of vertices.

Fixed As Bs

Testing algorithm



If the **topological ordering** of vertices of H gives the total ordering of vertices.

Lemma

A given graph admits a Stick representation respecting σ_A and σ_B if and only if H is acyclic, i.e., the constraints are consistent.

Testing complexity

Theorem

There is an $O(|A||B|)$ -time algorithm to decide the Stick_{AB} problem, and construct a Stick representation if one exists.

Lemma

A given graph admits a Stick representation respecting σ_A and σ_B if and only if H is acyclic, i.e., the constraints are consistent.

Characterization

Theorem

An instance of $Stick_{AB}$ with graph $G = (A \cup B, E)$ has a solution if and only if G 's ordered adjacency matrix M has no ordered submatrix of the following form:

$$\begin{array}{c} \begin{array}{cc} & \begin{array}{ccc} b_p & b_q & b_r \end{array} \\ \begin{array}{c} a_i \\ a_j \\ a_k \end{array} & \begin{bmatrix} * & 1 & * \\ * & 0 & 1 \\ 1 & * & * \end{bmatrix} \end{array}, \quad \begin{array}{c} \begin{array}{cc} & \begin{array}{cc} b_p & b_q \end{array} \\ \begin{array}{c} a_i \\ a_j \\ a_k \end{array} & \begin{bmatrix} 1 & * \\ 0 & 1 \\ 1 & * \end{bmatrix} \end{array}, \quad \begin{array}{c} \begin{array}{ccc} & b_p & b_q & b_r \\ \begin{array}{c} a_i \\ a_j \end{array} & \begin{bmatrix} * & 1 & * \\ 1 & 0 & 1 \end{bmatrix} \end{array} \end{array}.$$

Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{c} \\ a_i \\ a_j \\ a_k \end{array} \begin{array}{ccc} b_p & b_q & b_r \\ \left[\begin{array}{ccc} * & 1 & * \\ * & 0 & 1 \\ 1 & * & * \end{array} \right] \end{array}$$

$$\begin{array}{c} \\ a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$

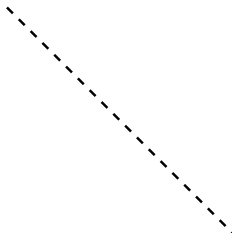
$$\begin{array}{c} \\ a_i \\ a_j \end{array} \begin{array}{ccc} b_p & b_q & b_r \\ \left[\begin{array}{ccc} * & 1 & * \\ 1 & 0 & 1 \end{array} \right] \end{array}$$

Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{cc} & b_p & b_q \\ \begin{array}{c} a_i \\ a_j \\ a_k \end{array} & \begin{bmatrix} 1 & * \\ 0 & 1 \\ 1 & * \end{bmatrix} \end{array}$$

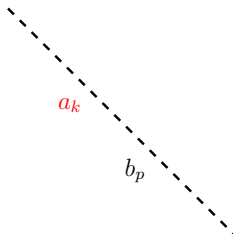


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{cc} & b_p & b_q \\ \begin{array}{c} a_i \\ a_j \\ a_k \end{array} & \begin{bmatrix} 1 & * \\ 0 & 1 \\ 1 & * \end{bmatrix} \end{array}$$

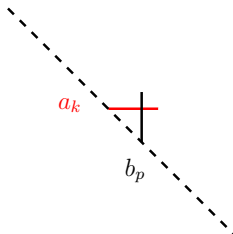


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{cc} & b_p & b_q \\ \begin{array}{c} a_i \\ a_j \\ a_k \end{array} & \begin{bmatrix} 1 & * \\ 0 & 1 \\ 1 & * \end{bmatrix} \end{array}$$

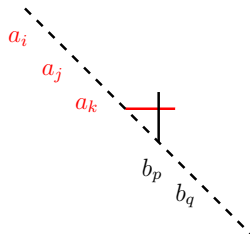


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$

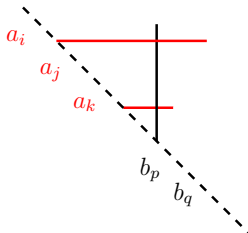


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$

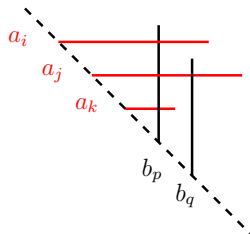


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$

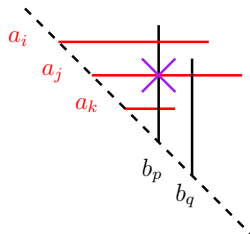


Fixed As Bs

Forbidden matrix

$$* = \{0, 1\}$$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$



Fixed As Bs

Characterization

Theorem

An instance of $Stick_{AB}$ with graph $G = (A \cup B, E)$ has a solution if and only if G 's ordered adjacency matrix M has no ordered submatrix of the following form:

$$\begin{array}{c} \begin{array}{ccc} & b_p & b_q & b_r \\ a_i & \begin{bmatrix} * & 1 & * \end{bmatrix} \\ a_j & \begin{bmatrix} * & 0 & 1 \end{bmatrix} \\ a_k & \begin{bmatrix} 1 & * & * \end{bmatrix} \end{array} , & \begin{array}{cc} & b_p & b_q \\ a_i & \begin{bmatrix} 1 & * \end{bmatrix} \\ a_j & \begin{bmatrix} 0 & 1 \end{bmatrix} \\ a_k & \begin{bmatrix} 1 & * \end{bmatrix} \end{array} , & \begin{array}{ccc} & b_p & b_q & b_r \\ a_i & \begin{bmatrix} * & 1 & * \end{bmatrix} \\ a_j & \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \end{array} \end{array}$$

Fixed As Free Bs

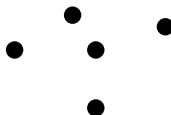
Statement

Problem: *Stick* REPRESENTATION WITH FIXED AS ($Stick_A$)

Input: A bipartite graph $G = (A \cup B, E)$, and a vertex-ordering σ_A of A .

Question: Does G admit a *Stick* representation such that the i th horizontal segment on the ground line corresponds to the i th vertex of σ_A ?

a_1 a_2 a_3 a_4 a_5



σ_A given

Fixed As

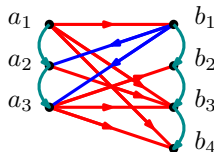
Fixed As Free Bs

Ingredients

Previous H graph model doesn't work since (b_{i-1}, b_i) is missing

$$\begin{matrix} & b_1 & b_2 & b_3 & b_4 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

M



H

$E(H)$:

- (a_{i-1}, a_i) according to σ_A
- ~~(b_{i-1}, b_i) according to σ_B~~
- (a_i, b_j) for each $m_{i,j} = 1$
- (b_p, a_j) for each configuration

$$\begin{matrix} a_i & \begin{bmatrix} b_p & b_q \\ 1 & * \\ 0 & 1 \end{bmatrix} \\ a_j & \end{matrix}$$

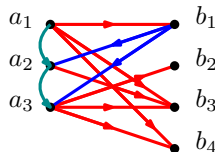
Fixed As Free Bs

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Previous H graph model doesn't work since (b_{i-1}, b_i) is missing

$$\begin{matrix} & b_1 & b_2 & b_3 & b_4 \\ \begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{matrix}$$

M



H

$E(H)$:

- (a_{i-1}, a_i) according to σ_A
- ~~(b_{i-1}, b_i) according to σ_B~~
- (a_i, b_j) for each $m_{i,j} = 1$
- (b_p, a_j) for each configuration

$$\begin{matrix} a_i & \begin{bmatrix} b_p & b_q \\ 1 & * \\ 0 & 1 \end{bmatrix} \\ a_j & \end{matrix}$$

Fixed As and Free Bs

Ingredients

In Fixed As and Free Bs case we can rearrange the columns of forbidden matrices.

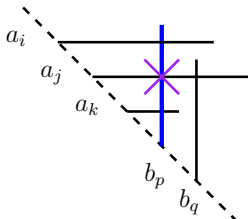
Fixed As and Free Bs

Ingredients

In Fixed As and Free Bs case we can rearrange the columns of forbidden matrices.

$* = 0, 1$

$$\begin{array}{l} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$



Fixed As and Free Bs

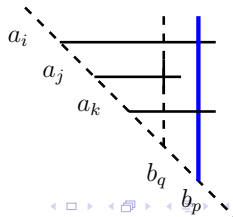
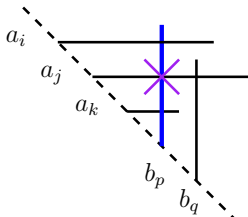
Ingredients

In Fixed As and Free Bs case we can rearrange the columns of forbidden matrices.

$* = 0, 1$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$$

$$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_q & b_p \\ \left[\begin{array}{cc} * & 1 \\ 1 & 0 \\ * & 1 \end{array} \right] \end{array}$$



Ingredients:

- As ordering based on σ_A
- Ordering A-B based on entries 1 on M ($m_{i,j} = 1$ as shown before)
- Arrangement of Bs to avoid forbidden matrices

Idea:

Express the ordering between any pair of vertices as variables:

v, w of G add $p_{v \prec w}$ and $p_{w \prec v}$

Since only one can be true we can use a truth *clause*:

$$(\neg p_{v \prec w} \vee \neg p_{w \prec v}) \wedge (p_{v \prec w} \vee p_{w \prec v})$$

→ 2-SAT formula

Fixed As and Free Bs

2-SAT

- **Variables:** $\forall (v, w) \in V(G)$ add $p_{v \prec w}$ and $p_{w \prec v}$
- **Clauses:** $(\neg p_{v \prec w} \vee \neg p_{w \prec v}) \wedge (p_{v \prec w} \vee p_{w \prec v})$
- **As truth:** $\forall (a_{i-1}, a_i) \in V(A)$ add $p_{a_{i-1} \prec a_i} = \text{true}$ based on σ_A
- **Adj truth:** $\forall (a_i, b_j) \in V(G)$ add $p_{a_i \prec b_j} : m_{i,j} = 1$ based on σ_A
- **Additional clauses:** order columns to avoid the three submatrices

$$\bullet \quad \begin{array}{c} a_i \\ a_j \\ a_k \end{array} \quad \begin{array}{ccc} b_p & b_q & b_r \\ \begin{bmatrix} * & 1 & * \\ * & 0 & 1 \\ 1 & * & * \end{bmatrix} \end{array} \quad \text{add } (\neg p_{b_q \prec b_r} \vee p_{b_q \prec b_p}) \text{ and } (\neg p_{b_p \prec b_q} \vee p_{b_r \prec b_q})$$

$$\bullet \quad \begin{array}{c} a_i \\ a_j \\ a_k \end{array} \quad \begin{array}{cc} b_p & b_q \\ \begin{bmatrix} 1 & * \\ 0 & 1 \\ 1 & * \end{bmatrix} \end{array} \quad \text{add } p_{b_q \prec b_p} = \text{true}$$

$$\bullet \quad \begin{array}{c} a_i \\ a_j \end{array} \quad \begin{array}{ccc} b_p & b_q & b_r \\ \begin{bmatrix} * & 1 & * \\ 1 & 0 & 1 \end{bmatrix} \end{array} \quad \text{add } (\neg p_{b_q \prec b_r} \vee p_{b_q \prec b_p}) \text{ and } (\neg p_{b_p \prec b_q} \vee p_{b_r \prec b_q})$$

The time complexity of the algorithm is dominated by the time to construct the 2-SAT formula, which is $O(|A|^3|B|^3)$.

Theorem

There is an algorithm with run-time $O(|A|^3|B|^3)$ to decide the $Stick_A$ problem, and construct a $Stick$ representation if one exists.

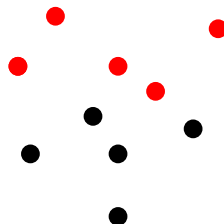
Stick

Statement

Problem: *Stick* REPRESENTATION

Input: A bipartite graph $G = (A \cup B, E)$.

Question: Does G admit a *Stick* representation such that the vertices in A and B correspond to horizontal and vertical segments, respectively?



no order given

Stick

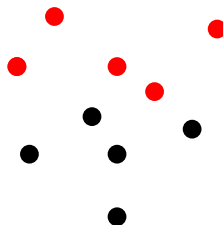
Stick

Statement

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Input: A bipartite graph $G = (A \cup B, E)$.

Question: Does G admit a *Stick* representation such that the vertices in A and B correspond to horizontal and vertical segments, respectively?



no order given

Stick

Recognition in polynomial time still open.

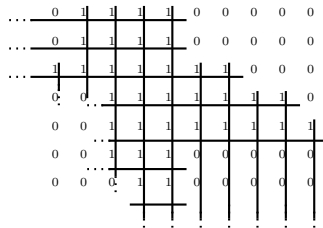
Positive result 1

Let $G = (A \cup B, E)$ be a bipartite graph and let M be its adjacency matrix, where the rows and columns correspond to A s and B s, respectively. If M has the simultaneous consecutive ones property, then G admits a *Stick* representation, which can be computed in $O(|A||B|)$ time.

0	1	1	1	1	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0
1	1	1	1	1	1	1	0	0	0
0	0	1	1	1	1	1	1	1	0
0	0	1	1	1	1	1	1	1	1
0	0	1	1	1	0	0	0	0	0
0	0	0	1	1	0	0	0	0	0

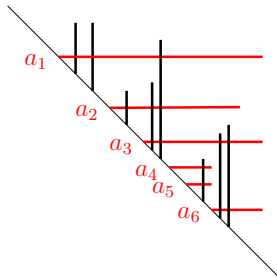
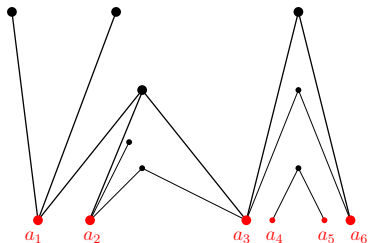
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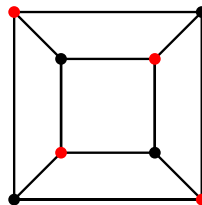
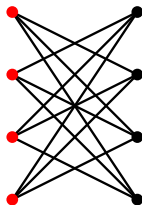
Positive result 2

Let $G = (A \cup B, E)$ be an n -vertex bipartite graph that admits a *one-sided planar drawing*. Then G is a *Stick* graph, and its *Stick* representation can be computed in $O(n^2)$ time.



Negative result 1

Let H be the graph obtained by deleting a perfect matching from a complete bipartite graph $K_{4,4}$. Any graph $G = (A \cup B, E)$ containing H as an induced subgraph does not admit a *Stick* representation. Since H is a planar graph, not all planar bipartite graphs are *Stick* graphs.



Conclusion and Open Problems

Stick Graphs

Characterization (forbidden submatrices for Stick representation)

$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{ccc} b_p & b_q & b_r \\ \left[\begin{array}{ccc} * & 1 & * \\ * & 0 & 1 \\ 1 & * & * \end{array} \right] \end{array}$	$\begin{array}{c} a_i \\ a_j \\ a_k \end{array} \begin{array}{cc} b_p & b_q \\ \left[\begin{array}{cc} 1 & * \\ 0 & 1 \\ 1 & * \end{array} \right] \end{array}$	$\begin{array}{c} a_i \\ a_j \end{array} \begin{array}{ccc} b_p & b_q & b_r \\ \left[\begin{array}{ccc} * & 1 & * \\ 1 & 0 & 1 \end{array} \right] \end{array}$
--	--	--

Setting	Test
Fixed As and Bs	$O(A B)$ -time
Fixed As	$O(A ^3 B ^3)$ -time
Stick	Open

Open Problems

- What is the complexity of recognizing the *Stick* graphs?
- Can we improve the time complexity of the recognition algorithm for graphs with fixed As?

Conclusion and Open Problems

Stick Graphs

Characterization (forbidden submatrices for Stick representation)

	b_p	b_q	b_r
a_i	$*$	1	$*$
a_j	$*$	0	1
a_k	1	$*$	$*$

	b_p	b_q
a_i	1	$*$
a_j	0	1
a_k	1	$*$

	b_p	b_q	b_r
a_i	$*$	1	$*$
a_j	1	0	1

Open Problems

- What is the complexity of recognizing the *Stick* graphs?
- Can we improve the time complexity of the recognition algorithm for graphs with fixed As ?

Thank you!

Felice De Luca

felicedeluca@email.arizona.edu