The number of crossings in multigraphs with no empty lens

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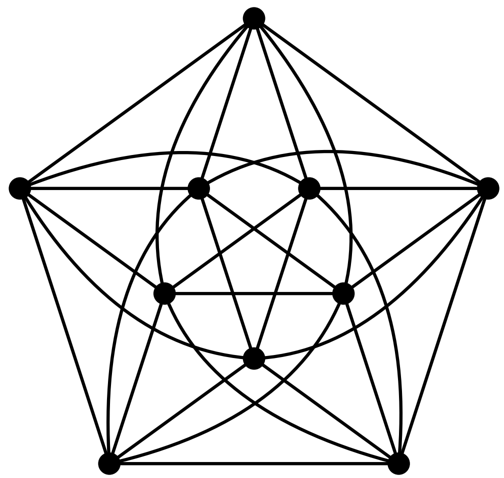
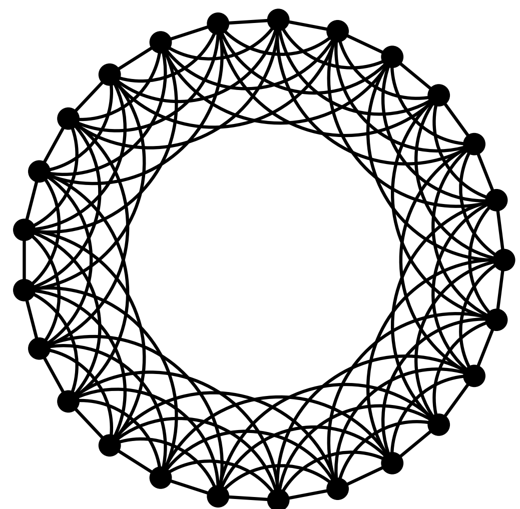
Crossing Lemma (Ajtai et al. 1982, Leighton 1983).

There exists $\alpha > 0$ s.t. for every n-vertex e-edge topological* graph G we have

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}, \quad \text{provided } e > 4n.$$



$L(K_n)$



C_n^k

Question: multigraphs?

*Note: G is drawn; there is no minimum in $\mathrm{cr}(G)$

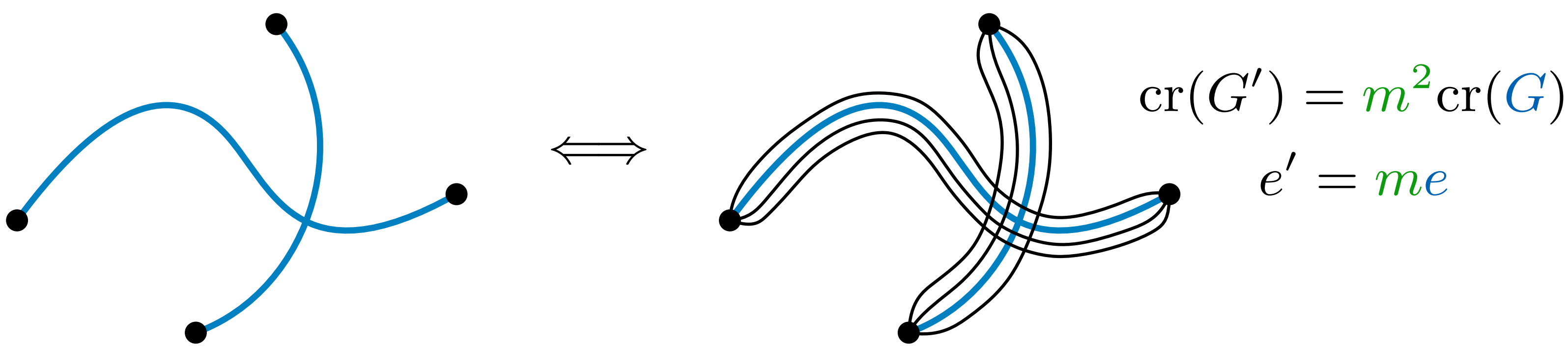
Crossing Lemma (Ajtai et al. 1982, Leighton 1983).

n vertices, e edges $\Longrightarrow$ $\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$, provided $e > 4n$.

Crossing Lemma for multigraphs (Székely 1997).

There exists $\alpha > 0$ s.t. for every n-vertex e-edge topological multigraph G with edge-multiplicity at most m we have

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{mn^2}, \quad \text{provided } e > 4mn.$$



Question: multigraphs with no bound on multiplicity?

Crossing Lemma (Ajtai et al. 1982, Leighton 1983).

n vertices, e edges $\implies$ $\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$, provided $e > 4n$.

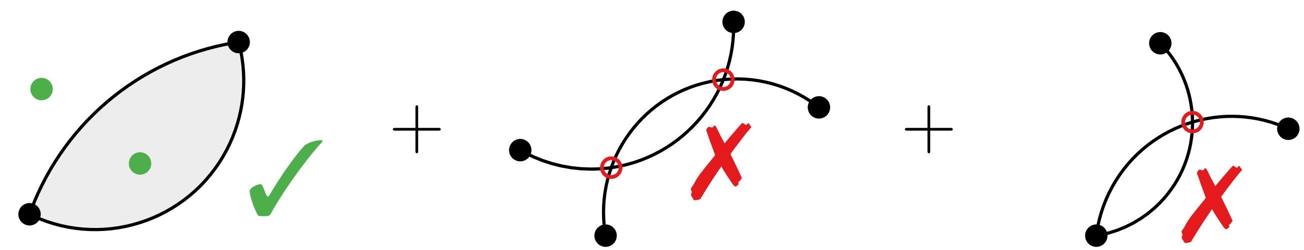
Crossing Lemma for multigraphs (Székely 1997).

n vertices, e edges, edge-multiplicity m $\implies$ $\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{mn^2}$, provided $e > 4mn$.

Crossing Lemma for branching multigraphs (Pach-Tóth 2018).

There exists $\alpha > 0$ s.t. for every n-vertex e-edge branching multigraph G we have

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}, \quad \text{provided } e > 4n.$$

branching = 

separated + single-crossing + locally starlike

Crossing Lemma (Ajtai et al. 1982, Leighton 1983).

n vertices, e edges $\implies$ $\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$, provided $e > 4n$.

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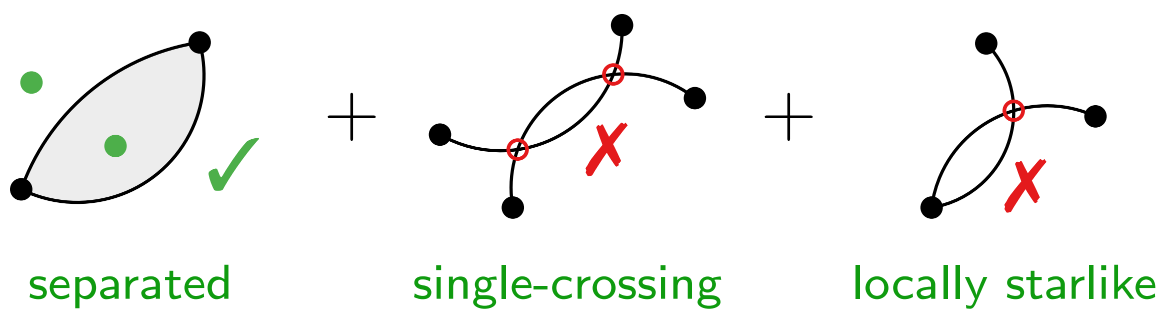
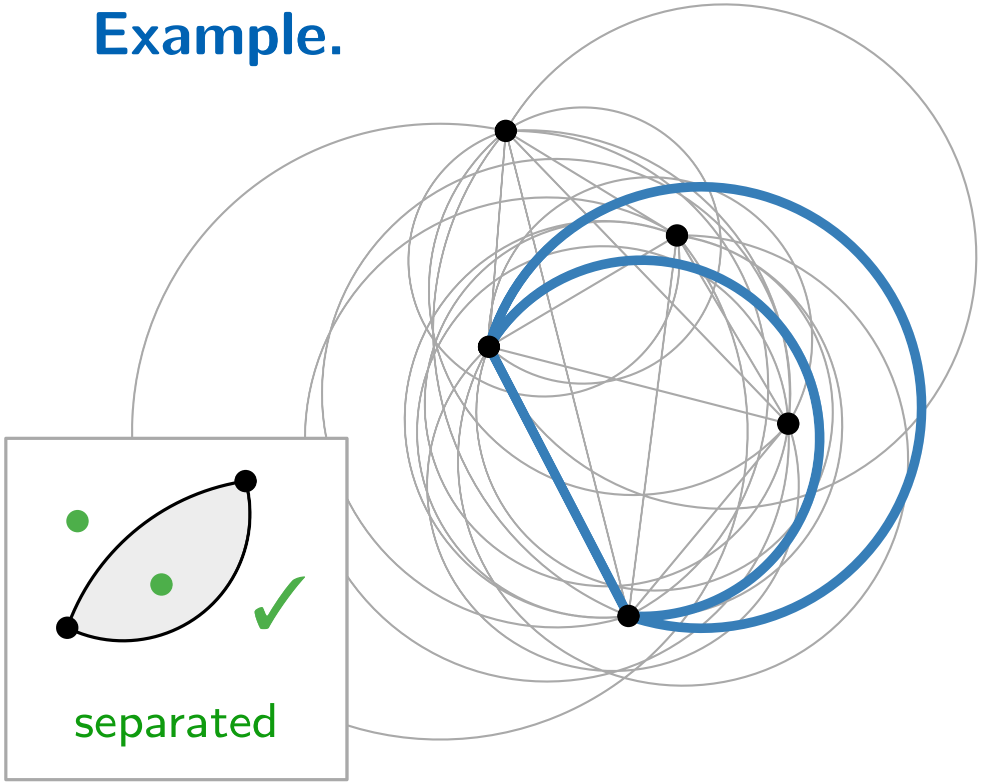
Crossing Lemma for branching multigraphs (Pach-Tóth 2018).

n vertices, e edges, branching $\implies$ $\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$, provided $e > 4n$.



Question: separated multigraphs?

Example.



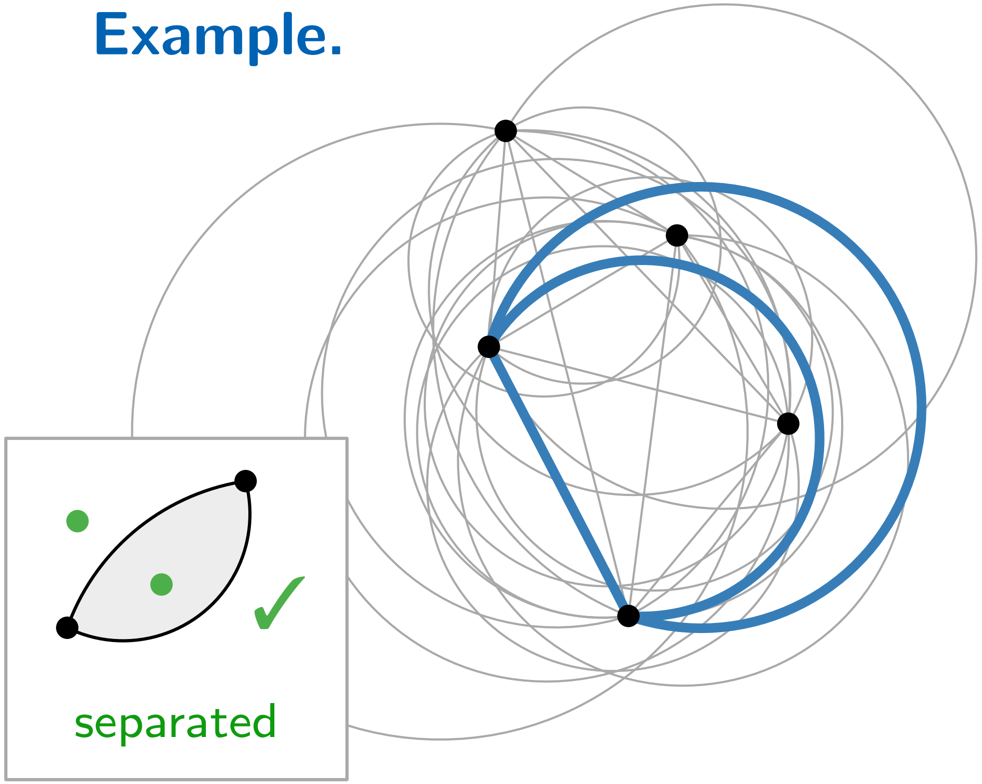
n vertices

$e \approx n^3$ edges

$\mathrm{cr}(G) \approx e^2$ crossings

$$n^6 \approx \mathrm{cr}(G) \not\gtrsim \alpha \cdot \frac{e^3}{n^2} \approx \alpha \cdot n^7 \quad ✗$$

Example.



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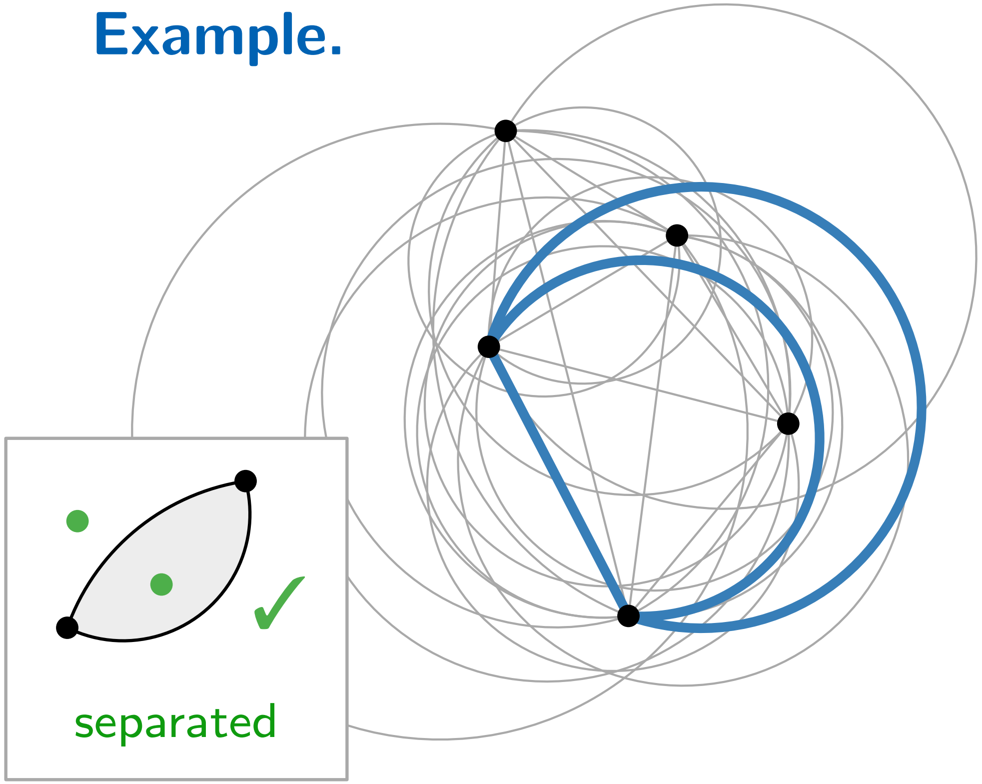
$$n^6 \approx \mathrm{cr}(G) \geq \alpha \cdot \frac{e^{2.5}}{n^{1.5}} \approx \alpha \cdot n^6$$ ✓

Example.



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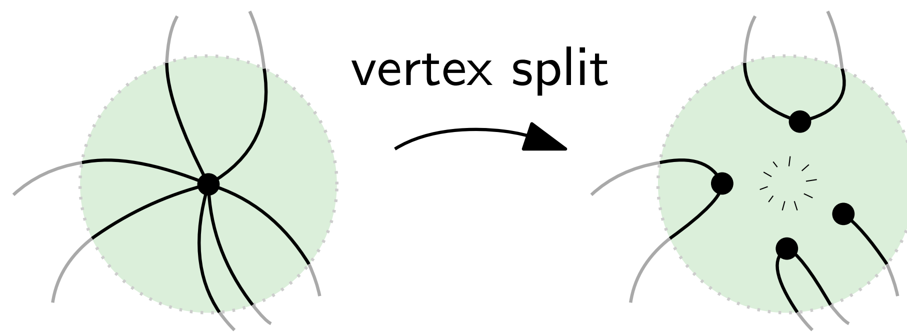
Theorem.
There exists $\alpha > 0$ s.t. for every n-vertex e-edge separated multigraph G we have

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^{2.5}}{n^{1.5}}, \quad \text{provided } e > 4n.$$

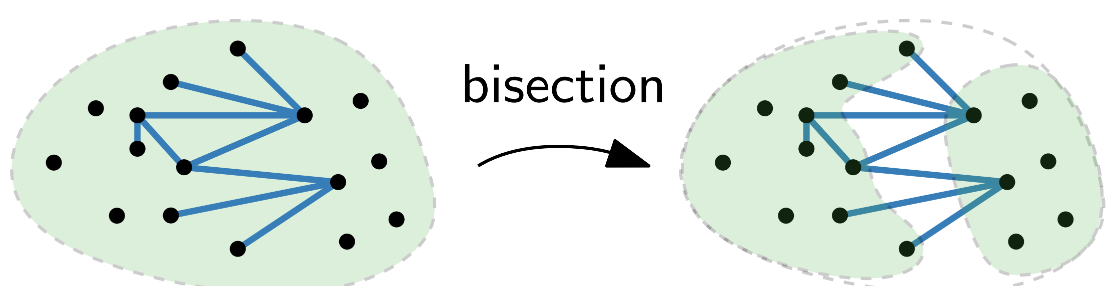
. . . follows from a very general crossing lemma . . .

General Drawing Styles

- D . . . predicate over the set of all topological multigraphs
- D monotone drawing style $\quad = \quad$ allows removing edges
- D split-compatible drawing style $\quad = \quad$ allows vertex splits



- D-bisection width $b_D(G) \quad = \quad$ min. # edges whose removal gives G_1, G_2 in drawing style D with $n(G_i) \geq \frac{1}{5}n(G)$ for $i = 1, 2$



A Generalized Crossing Lemma

Theorem.
Suppose D is a monotone and split-compatible drawing style, and that there are $k_1, k_2, k_3 > 0$ and $b > 1$ such that for every n-vertex, e-edge multigraph G in drawing style D we have:

(P1) $e \leq k_1 \cdot n$ if $\mathrm{cr}(G) = 0$

(P2) $b_D(G) \leq k_2 \cdot \sqrt{\mathrm{cr}(G) + \Delta(G) \cdot e + n}$

(P3) $e \leq k_3 \cdot n^b$

Then there exists $\alpha > 0$ s.t. for every n-vertex e-edge multigraph G in drawing style D we have

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}, \quad \text{provided } e > (k_1 + 1)n,$$

where $x(b) := 1/(b-1)$ and $\alpha = \alpha_b \cdot {k_2}^{-2} \cdot {k_3}^{-x(b)}$ for some positive constant α_b depending only on b.

Crossing Numbers $\longleftrightarrow$ Edge Densities

$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^{x(b)+2}}{n^{x(b)+1}}$$ where $x(b) := \dfrac{1}{b-1}$ and $e \leq k_3 \cdot n^b$

Crossing Numbers $\longleftrightarrow$ Edge Densities

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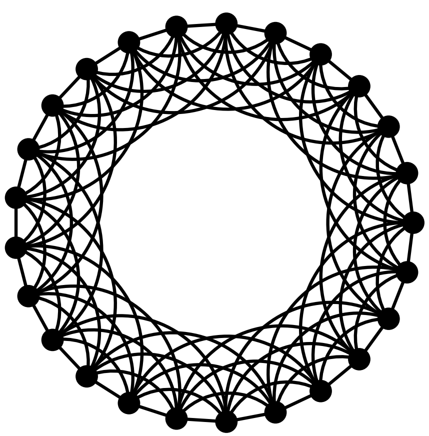
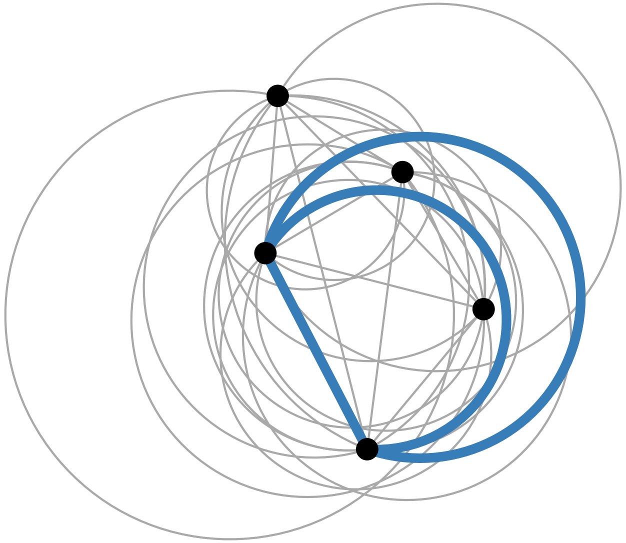
a few examples

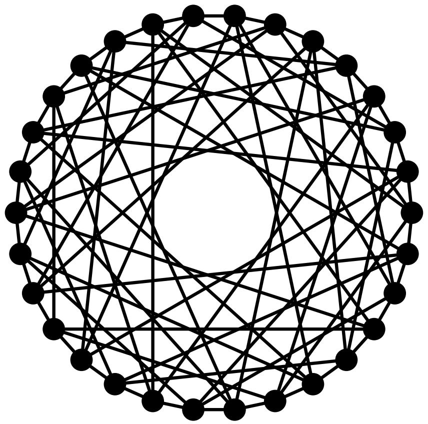
$$e = O(n^2)$$
$$x(b) = 1$$
$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$$

$$e = O(n^3)$$
$$x(b) = 0.5$$
$$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^{2.5}}{n^{1.5}}$$

$$e = O(n^{1+\frac{1}{r}})$$
$$x(b) = r$$
$$\mathrm{cr}(G) \geq \alpha_r \cdot \frac{e^{r+2}}{n^{r+1}}$$







Crossing Numbers $\longleftrightarrow$ Edge Densities

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a few examples

$e = O(n^2)$	$e = O(n^3)$	$e = O(n^{1+\frac{1}{r}})$
$x(b) = 1$	$x(b) = 0.5$	$x(b) = r$
$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^3}{n^2}$	$\mathrm{cr}(G) \geq \alpha \cdot \frac{e^{2.5}}{n^{1.5}}$	$\mathrm{cr}(G) \geq \alpha_r \cdot \frac{e^{r+2}}{n^{r+1}}$

a little algebra

$$\frac{e^{x(b)+2}}{n^{x(b)+1}} = \Theta\left(\frac{e^{x(b)+2}}{e^{\frac{1}{b}(x(b)+1)}}\right) = \Theta\left(\frac{e^{x(b)+2}}{e^{x(b)}}\right) = \Theta\left(e^2\right)$$

▷ Tightness* of the Generalized Crossing Lemma

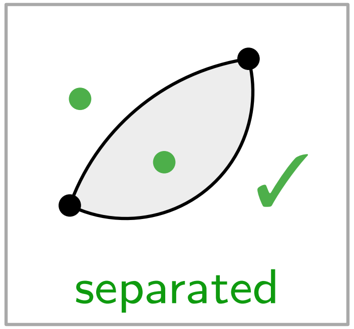
*provided: $e = \Theta(n^b)$ and pairs of edges cross constantly often

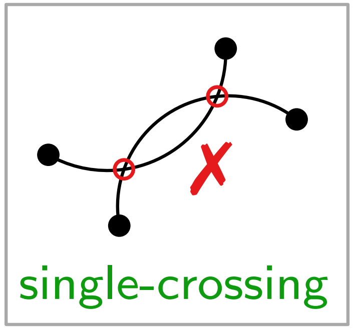
Checking (P1), (P2), and (P3)

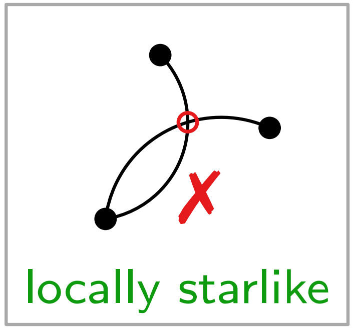
(P1) $e \leq k_1 \cdot n$ if $\mathrm{cr}(G) = 0$

(P2) $b_D(G) \leq k_2 \cdot \sqrt{\mathrm{cr}(G) + \Delta(G) \cdot e + n}$

(P3) $e \leq k_3 \cdot n^b$







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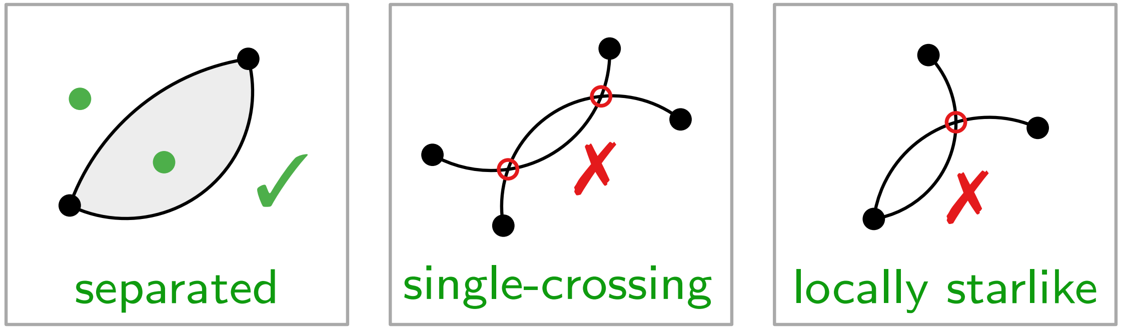

(P1) Separated $\Longrightarrow$ Euler's Formula: $e \leq 3n - 6$

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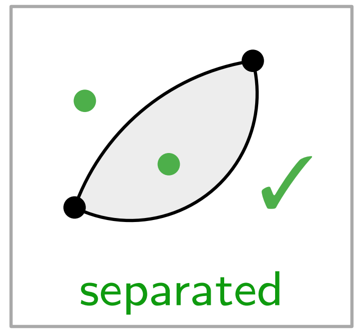
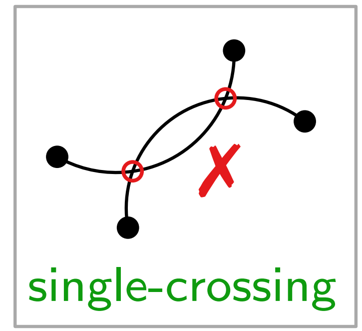
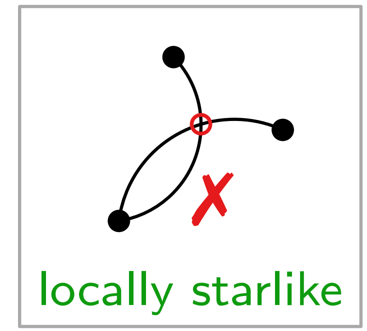
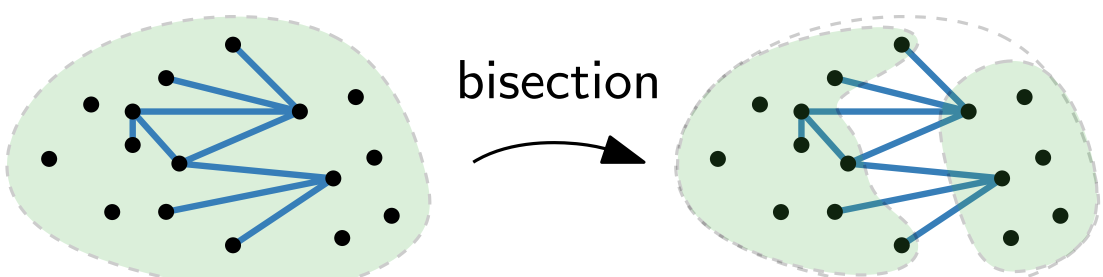






(P1) Separated $\Longrightarrow$ Euler's Formula: $e \leq 3n - 6$

(P2) Separated $\Longrightarrow$ Bisection Lemma of Pach and Tóth: $k_2 \leq 44$
(variant of Planar Separator Theorem)



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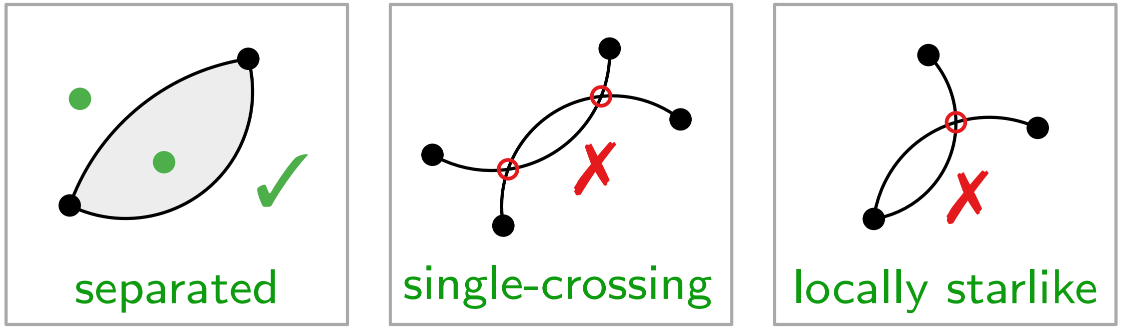

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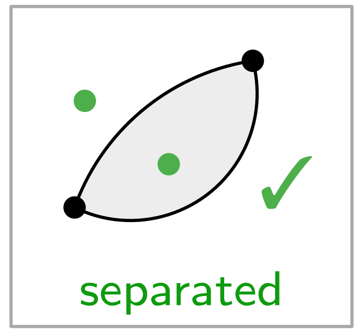

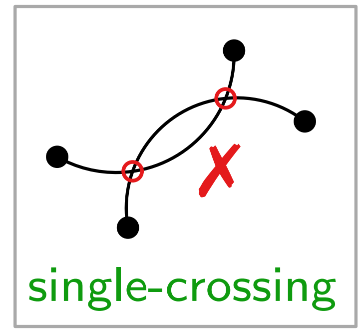

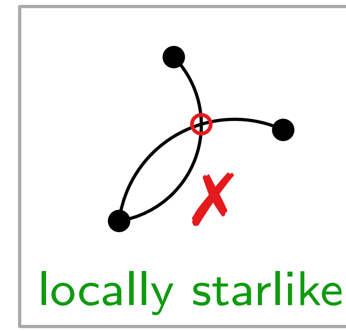

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Separated and locally starlike $\Longrightarrow$ $\Delta(G) < 2n$ $\Longrightarrow$ $e < n^2$

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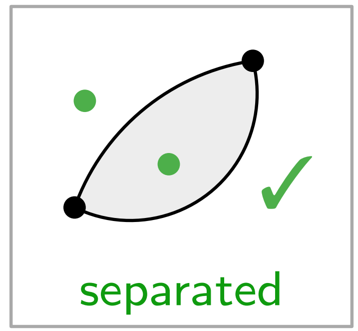

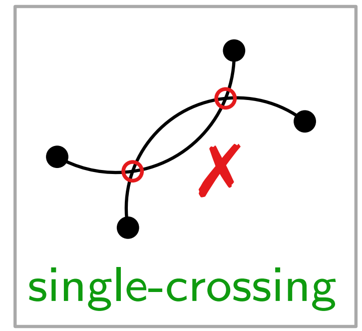

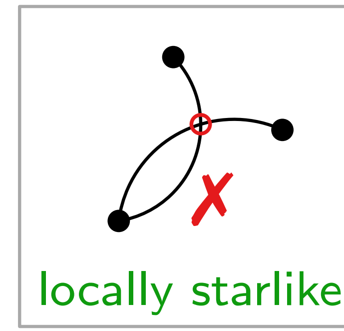

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Open Problem: Separated and single-crossing $\Longrightarrow$ $e = O(n^2)$ **?**

Summary

If drawing style D is monotone and split-compatible,
and for every n-vertex e-edge multigraph G in style D we have

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for constants $k_1, k_2, k_3 > 0$ and $b > 1$,

Then there is an absolute constant $\alpha > 0$ such that

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provided $e > (k_1 + 1)n$.

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Thank you for your attention!

(with special thanks to S. Felsner, V. Roselli, and P. Valtr)