The Queue-Number of Planar Posets

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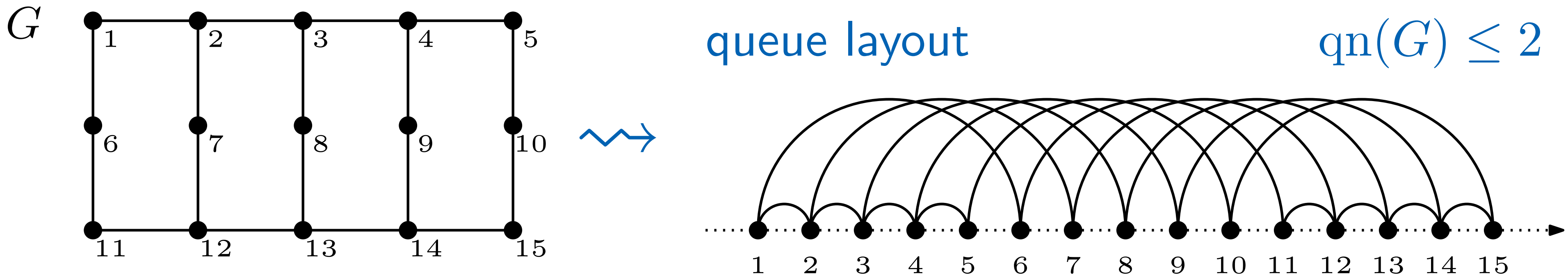
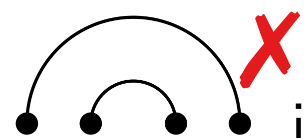
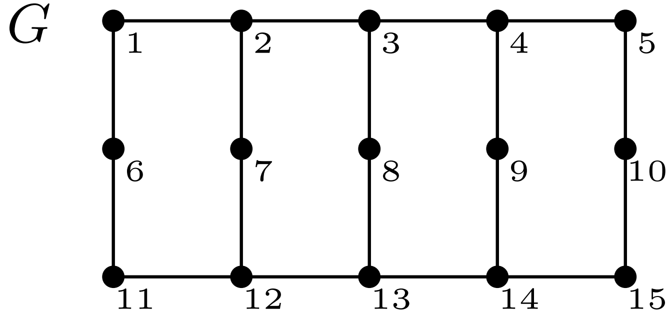
Barcelona



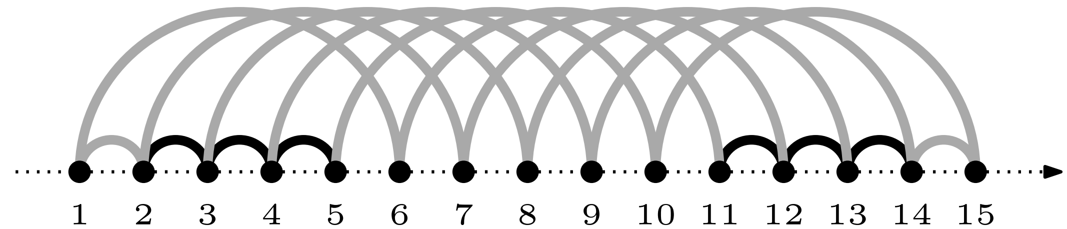
▷ **Queue-Number of a Graph** (Heath, Rosenberg 1992).

$\mathrm{qn}(G) =$ min k s.t. $\left\{ \begin{array}{l} \exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition} \end{array} \right\}$ with  in each part

no nesting



queue layout $\text{qn}(G) \leq 2$



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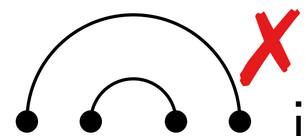

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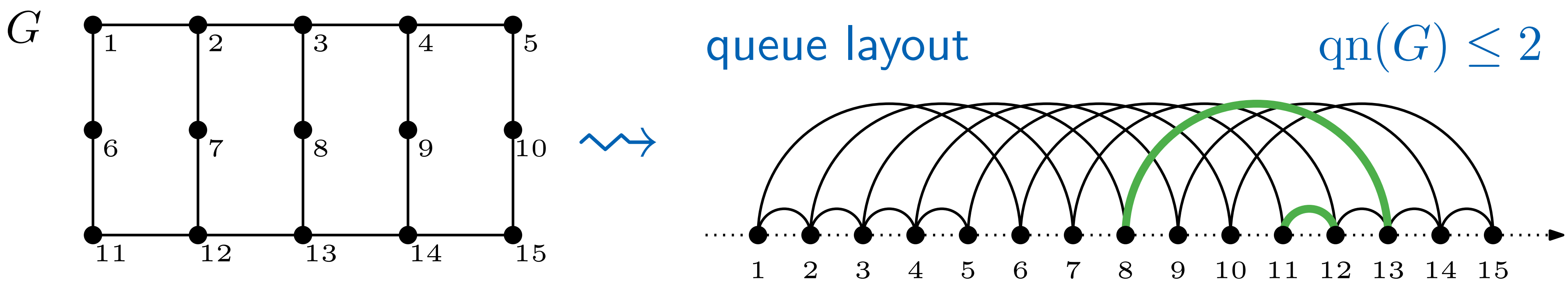

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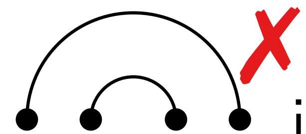
$\mathrm{qn}(G) =$ min k s.t. $\left\{ \begin{array}{l} \exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition} \end{array} \right\}$ with  in each part

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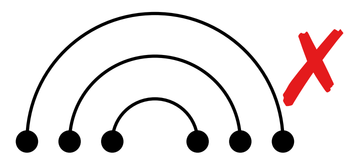
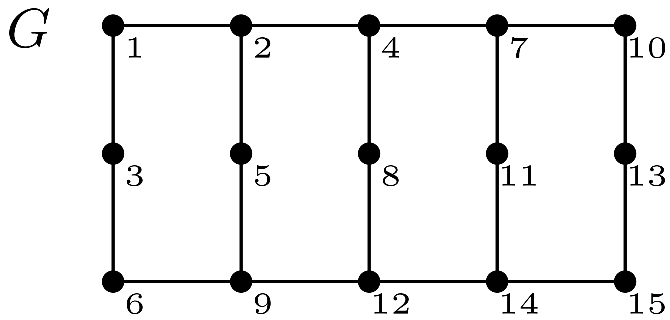
$\mathrm{qn}(G) =$ min k s.t. $\exists$ vertex ordering with 

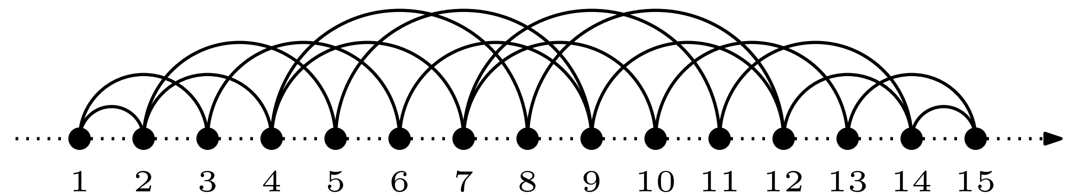
no k-nesting*

*also called k-rainbow



$\rightsquigarrow$

queue layout $\quad \mathrm{qn}(G) = 1$



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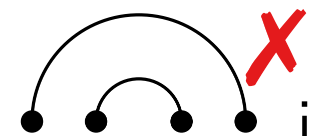
$\mathrm{qn}(G) =$ min k s.t. $\left\{ \begin{array}{l} \exists \text{ vertex ordering} \\ \exists\ k\text{-edge partition} \end{array} \right\}$ with  in each part

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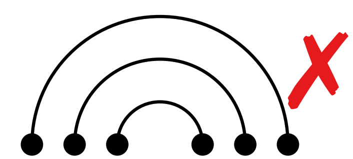
$\mathrm{qn}(G) =$ min k s.t. $\exists$ vertex ordering with 

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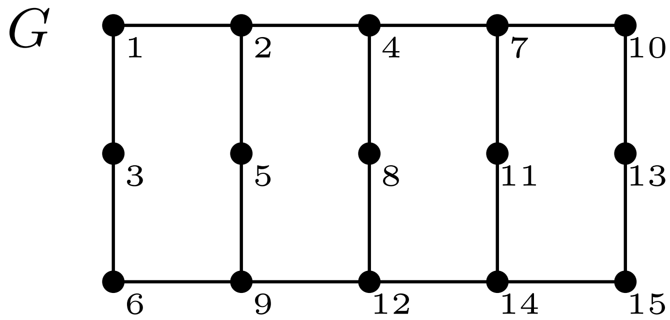

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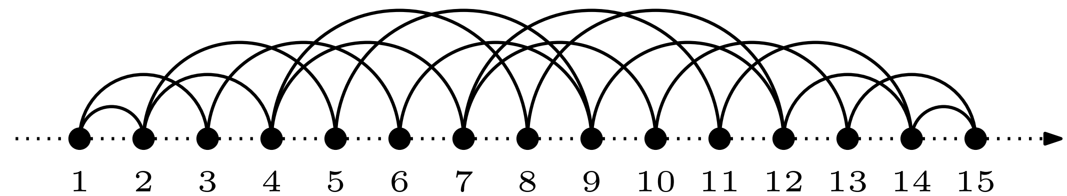

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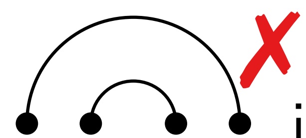
$\mathrm{qn}(G) =$ min k s.t. $\left\{ \begin{array}{l} \exists \text{ vertex ordering} \\ \exists\, k\text{-edge partition} \end{array} \right\}$ with  in each part

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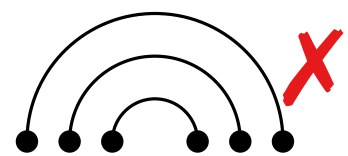
$\mathrm{qn}(G) =$ min k s.t. $\exists$ vertex ordering with 

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▷ **Queue-Number of a Poset** (Heath, Pemmaraju 1997).

$\mathrm{qn}(P) =$ min k s.t. $\exists$ linear extension with no k-nesting of covers

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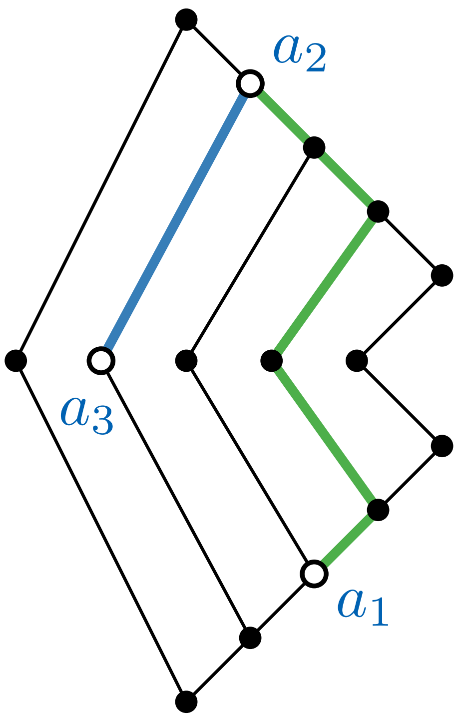
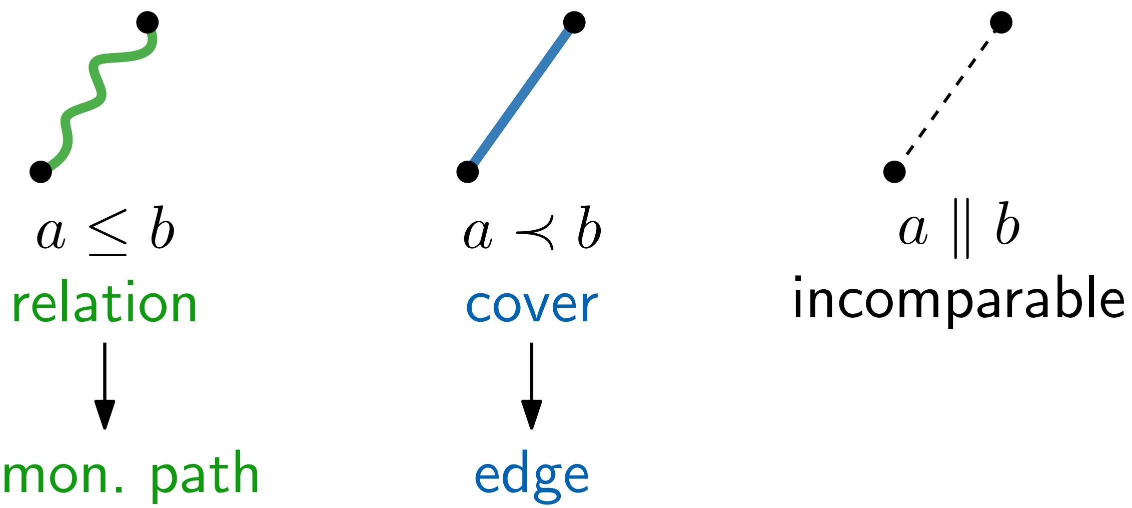


Hasse diagram

$P = (X, \leq)$ poset*

binary relation $\leq$ on finite set X

reflexive, antisymmetric, transitive



*short for partially ordered set

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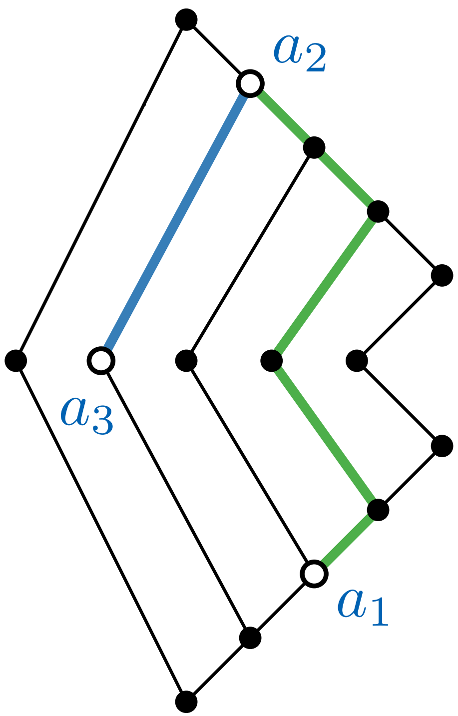

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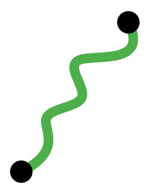
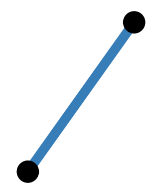


$a \leq b$

relation

↓

y-mon. path

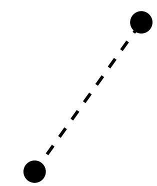


$a \prec b$

cover

↓

edge



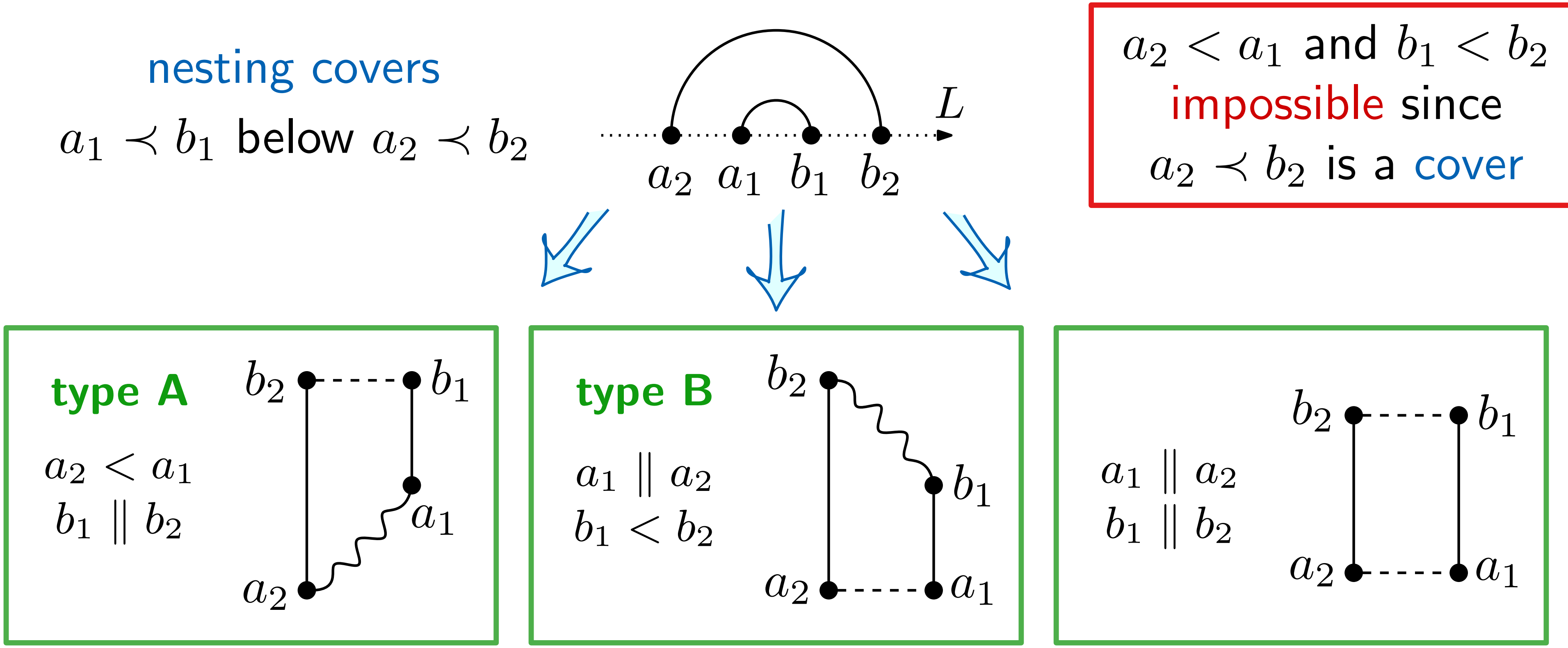
$a \parallel b$

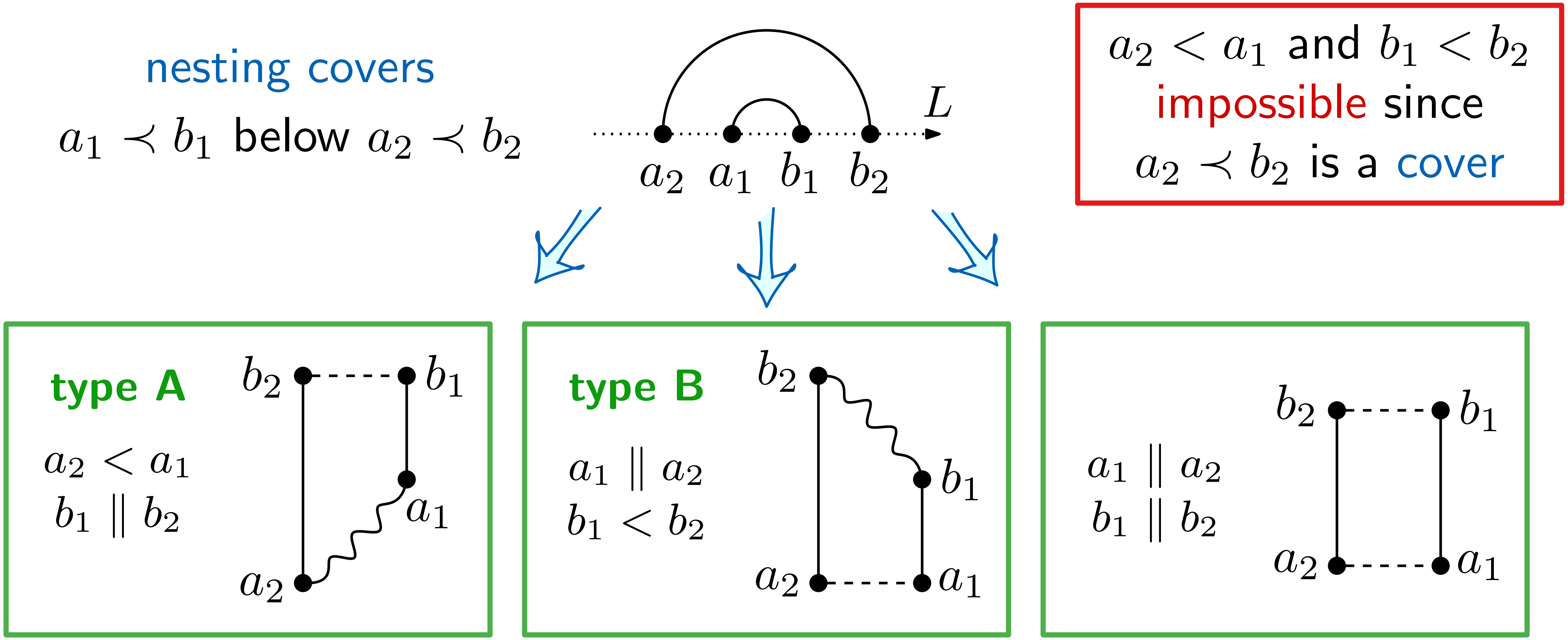
incomparable

L linear extension $\Leftrightarrow$ L is vertex ordering respecting P

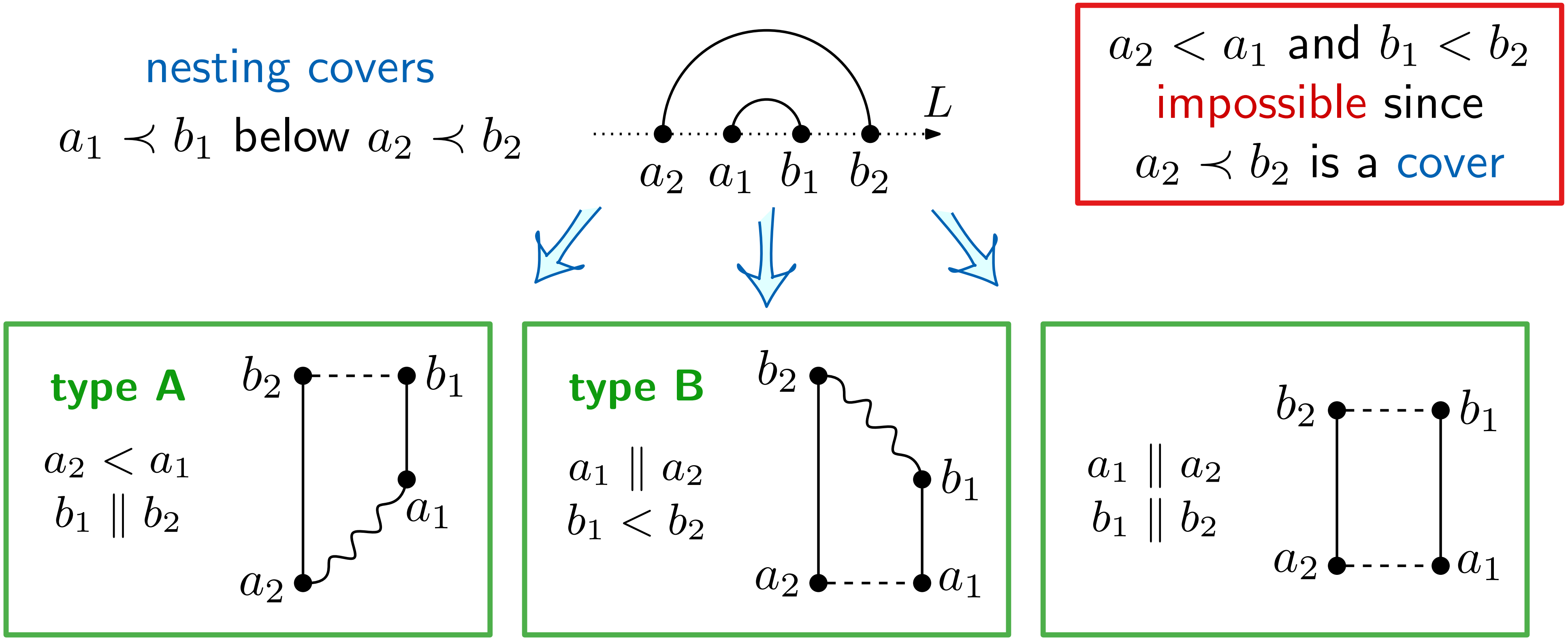
$a \prec b$ in P $\implies$ a before b in L

*short for partially ordered set


nesting covers
$a_1 \prec b_1$ below $a_2 \prec b_2$
L
a_2 a_1 b_1 b_2
$a_2 < a_1$ and $b_1 < b_2$
impossible since
$a_2 \prec b_2$ is a cover
type A
$a_2 < a_1$
$b_1 \parallel b_2$
b_2 b_1 a_1 a_2
type B
$a_1 \parallel a_2$
$b_1 < b_2$
b_2 b_1 a_2 a_1
$a_1 \parallel a_2$
$b_1 \parallel b_2$
b_2 b_1 a_2 a_1



Q1	$\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{width}(P)$ large?
Q2	P planar, $\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{width}(P)$ large?
Q3	$\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{height}(P)$ large?
Q4	P planar, $\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{height}(P)$ large?



Q1	$\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{width}(P)$ large	✓ **YES**
Q2	P planar, $\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{width}(P)$ large	✓ **YES**
Q3	$\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{height}(P)$ large	✗ **NO**
Q4	P planar, $\mathrm{qn}(P)$ large	$\Longrightarrow$	$\mathrm{height}(P)$ large	? **MAYBE**

Q1 $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{width}(P)$ large ✓**YES**

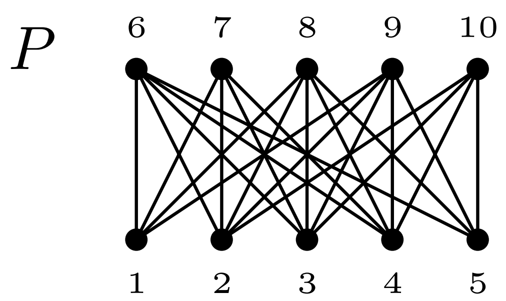
Theorem (Heath, Pemmaraju 1997).

$$w \leq \max\{\mathrm{qn}(P) \mid \mathrm{width}(P) = w\} \leq w^2$$

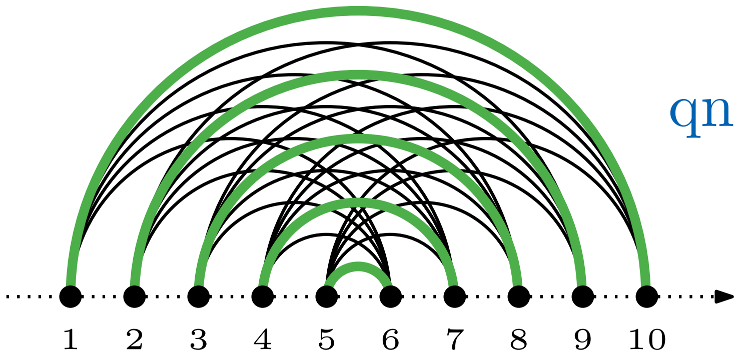
Conjecture (Heath, Pemmaraju 1997).

$$\max\{\mathrm{qn}(P) \mid \mathrm{width}(P) = w\} = w$$

lower bound



$\rightsquigarrow$



$\mathrm{qn}(P) = \mathrm{width}(P)$

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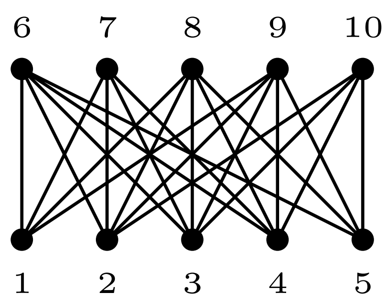
P



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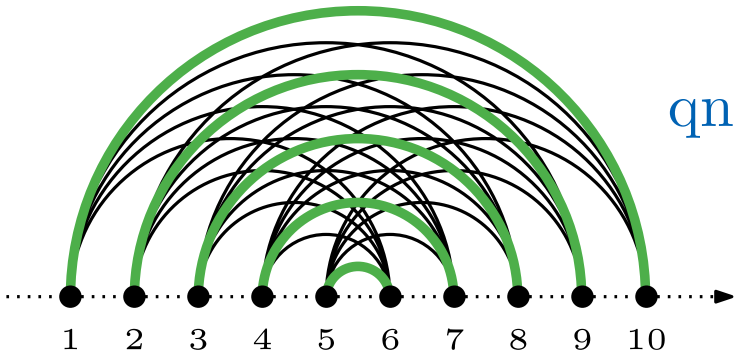


$\mathrm{qn}(P) = \mathrm{width}(P)$

Theorem.

$$\max\{\mathrm{qn}(P) \mid \mathrm{width}(P) = 2\} = 2$$

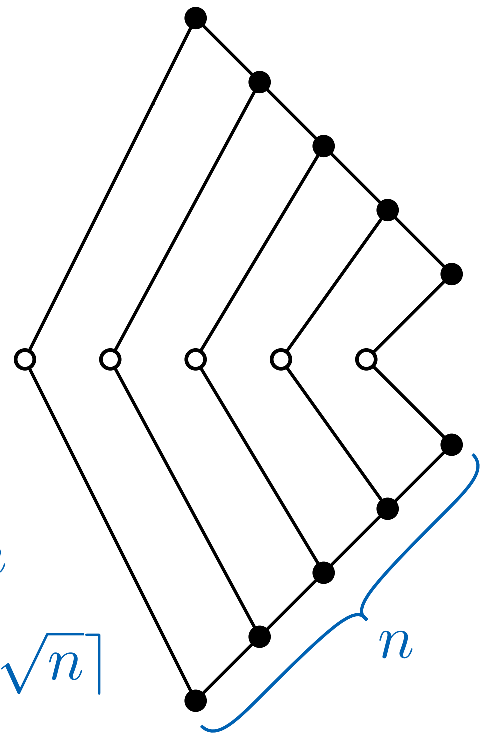
$$w \leq \max\{\mathrm{qn}(P) \mid \mathrm{width}(P) = w\} \leq w^2 - (w+1)$$

Q2 P planar, $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{width}(P)$ large ✓**YES**

Theorem (Heath, Pemmaraju 1997).

$$\sqrt{w} \le \max\{\mathrm{qn}(P) \mid P \text{ planar}, \mathrm{width}(P) = w\} \le 4w - 1$$

lower bound



$\mathrm{width}(P) = n$

$\mathrm{qn}(P) = \lceil \sqrt{n} \rceil$

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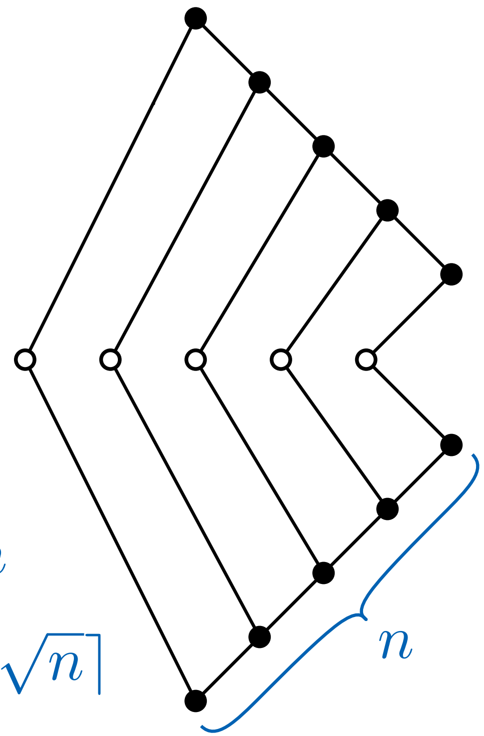

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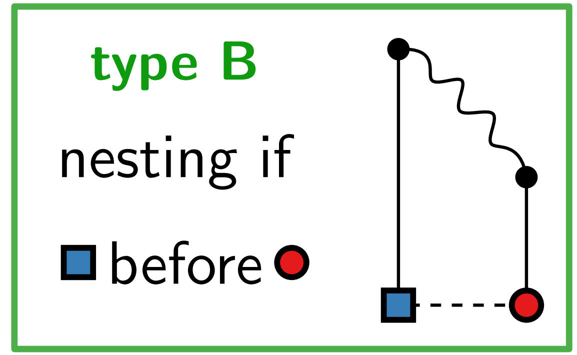

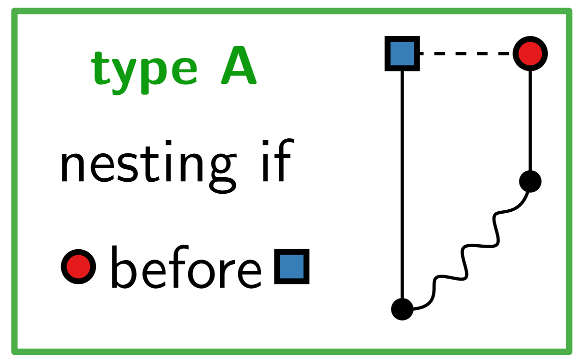

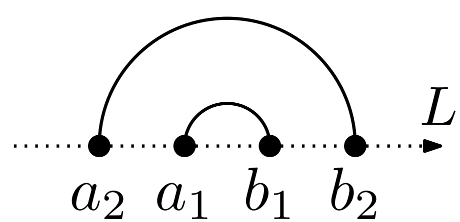

Q2 P planar, $\mathrm{qn}(P)$ large $\implies$ $\mathrm{width}(P)$ large ✓YES

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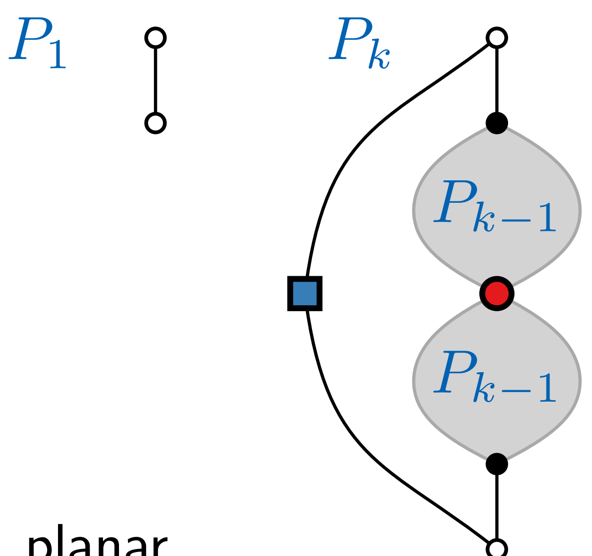
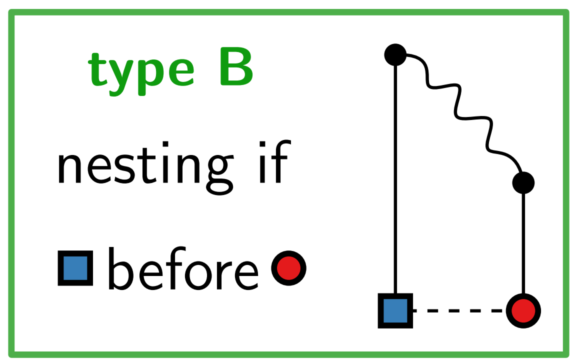
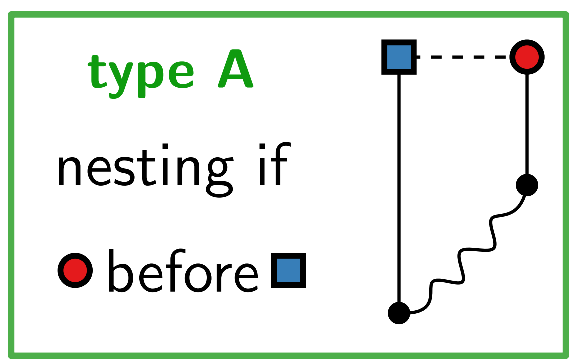
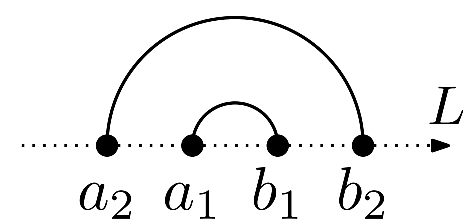


planar
with 0 and 1







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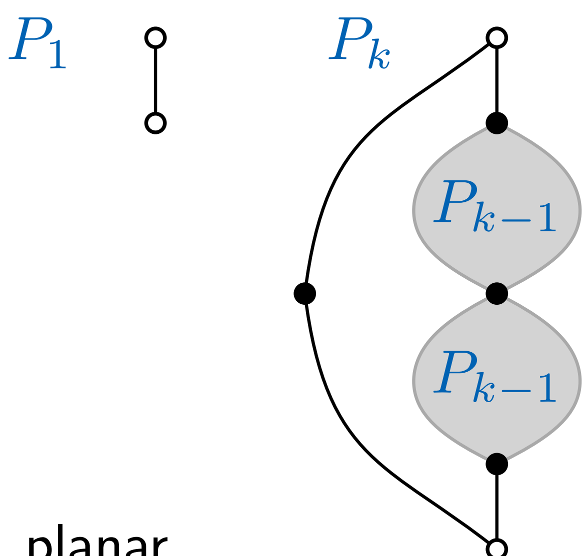
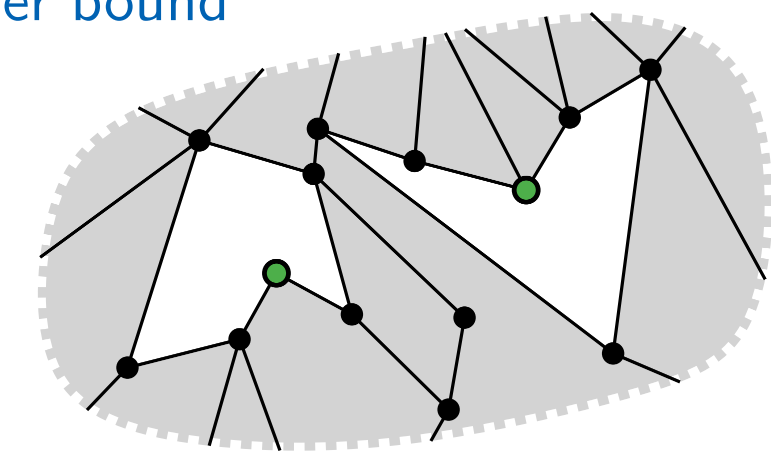


planar
with 0 and 1

upper bound



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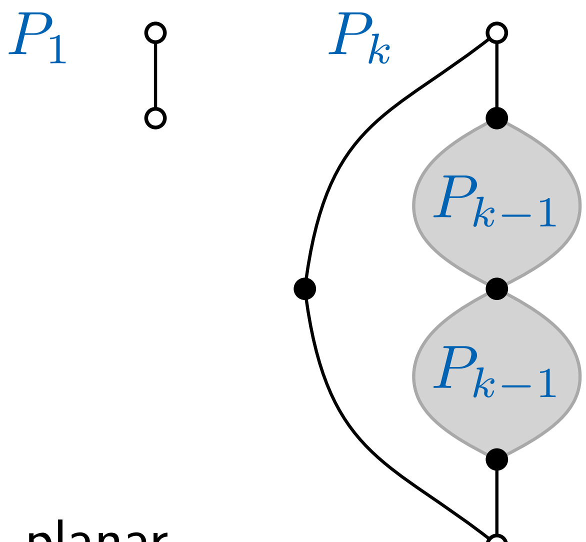

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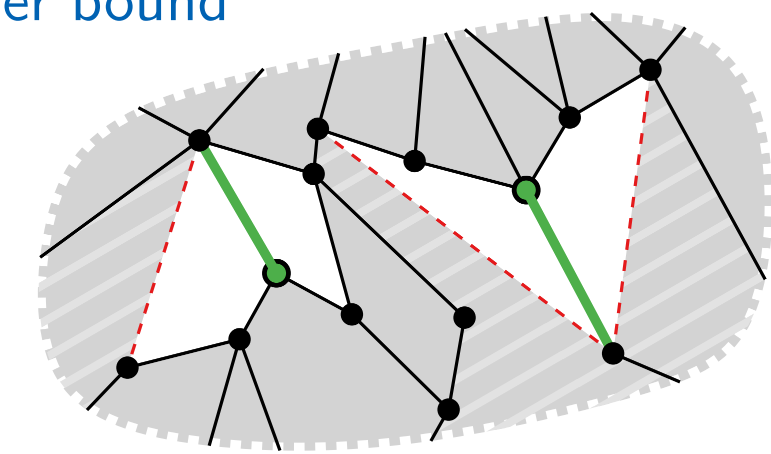
upper bound



$\leq 2w - 2$ new covers
≤ 2 lost covers each $\rightsquigarrow$ planar P' with 0 and 1

$\implies \mathrm{qn}(P') \leq w \implies \mathrm{qn}(P) \leq 3w - 2$

Q2 P planar, $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{width}(P)$ large ✓YES

Theorem (Heath, Pemmaraju 1997).

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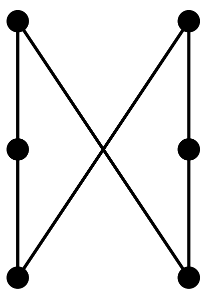
Theorem.

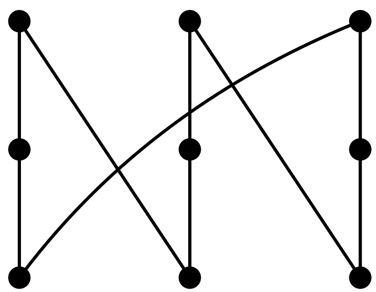
$$w \leq \max\{\mathrm{qn}(P) \mid P \text{ planar}, \mathrm{width}(P) = w\} \leq 3w - 2$$

Theorem.

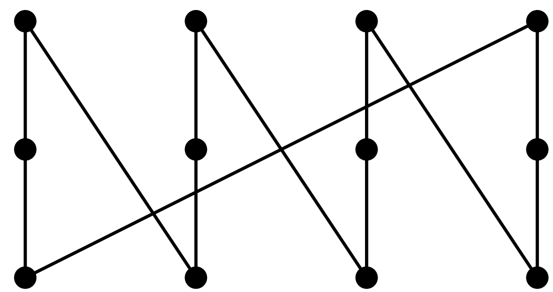
If P has no embedded subdivided crown, then $\mathrm{qn}(P) \leq \mathrm{width}(P)$.



subdivided 2-crown



subdivided 3-crown



subdivided 4-crown

embedded = all long diagonals are cover relations

Q2 P planar, $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{width}(P)$ large ✓**YES**

Theorem (Heath, Pemmaraju 1997).

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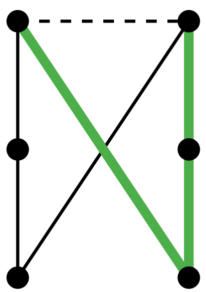
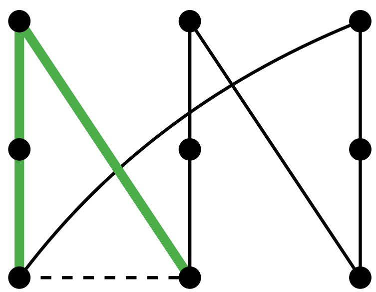


subdivided
2-crown

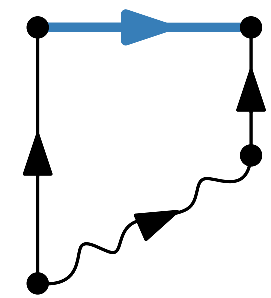


subdivided
3-crown

embedded = all long diagonals are cover relations

proofidea

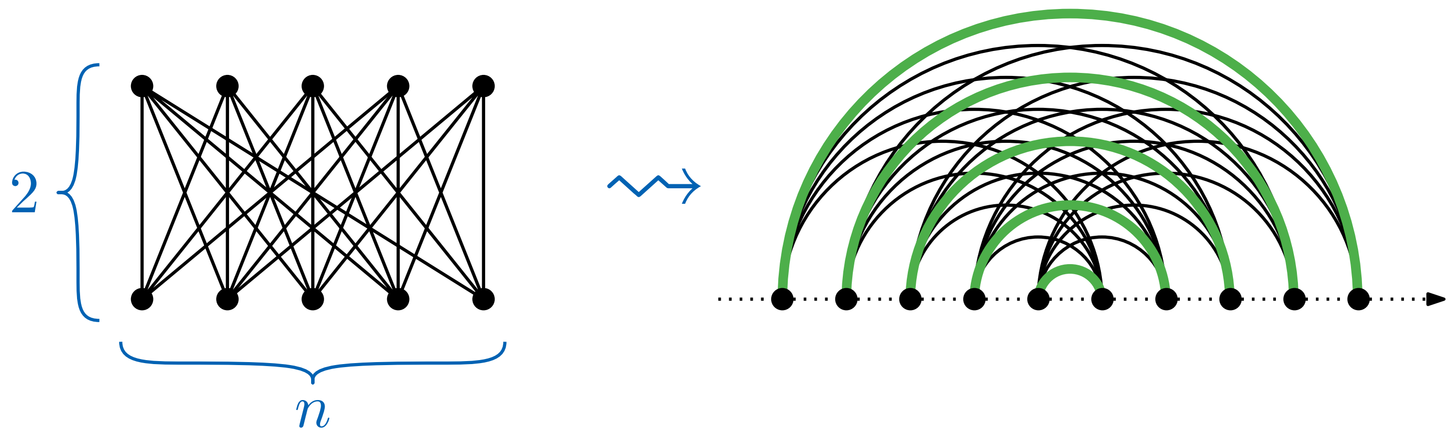
introduce new edges



⇝ show acyclicity

⇝ use topological ordering

Q3 $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{height}(P)$ large ✗ **NO**



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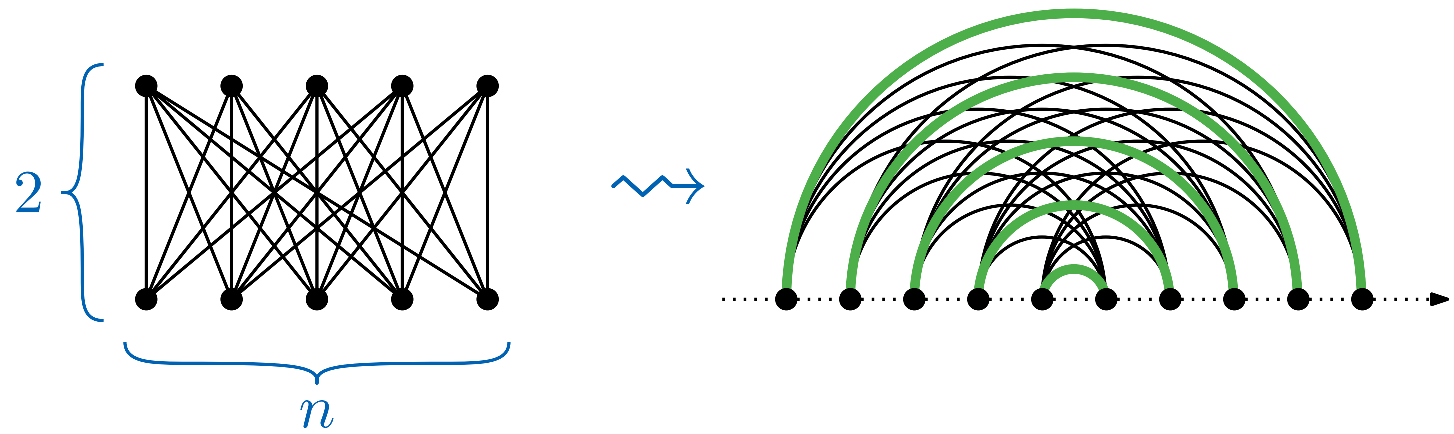

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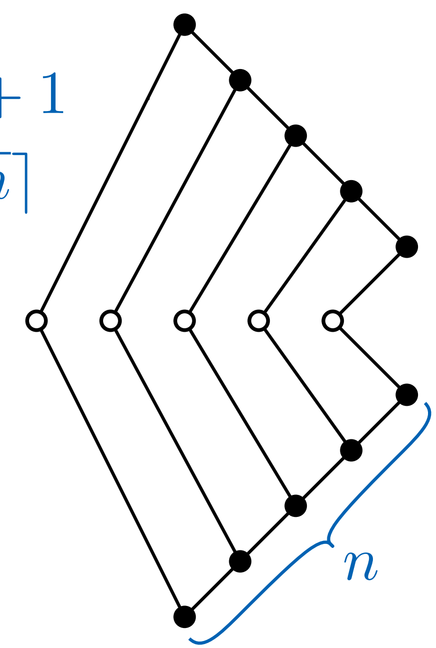
Q4 P planar, $\mathrm{qn}(P)$ large $\Longrightarrow$ $\mathrm{height}(P)$ large **? MAYBE**

lower bound

$$\mathrm{qn}(P) \geq \sqrt{\mathrm{height}(P)/2}$$

$\mathrm{height}(P) = 2n + 1$

$\mathrm{qn}(P) = \lceil \sqrt{n} \rceil$



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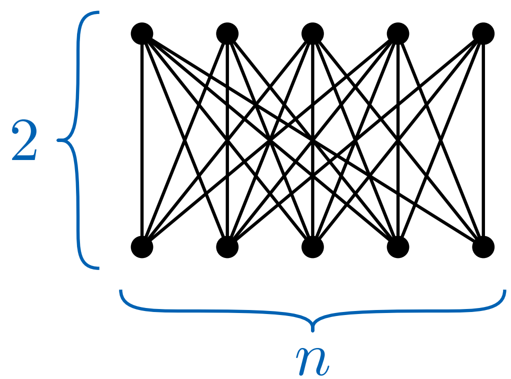
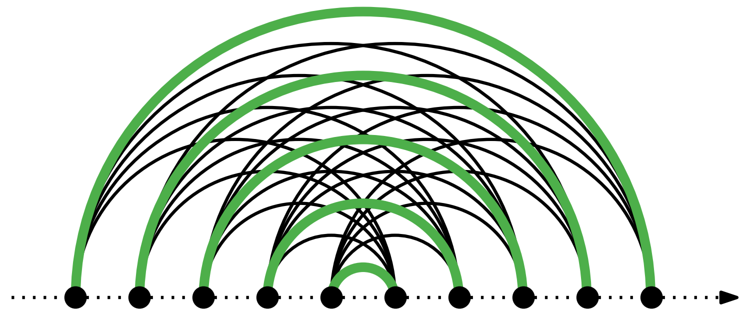


$\rightsquigarrow$



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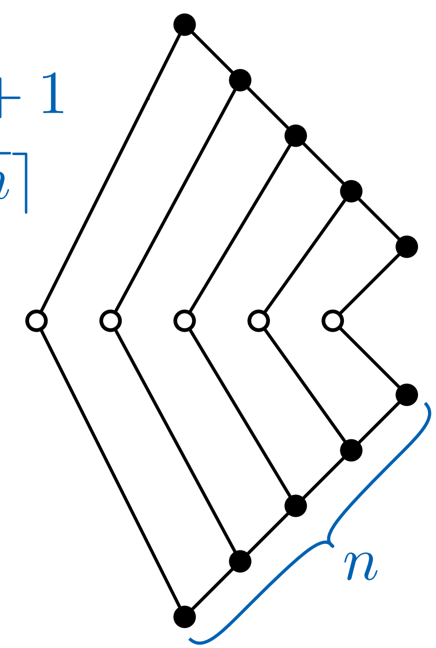

Conjecture (Heath, Pemmaraju 1997).

P planar $\Longrightarrow$ $\mathrm{qn}(P) \leq \mathrm{height}(P)$

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*hence, the conjecture is false

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  - $\forall\ G$ planar graph: $\mathrm{qn}(G) \leq C''$

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Take Home Slide

Conjecture (Heath, Pemmaraju 1997)**.**

For every poset P we have $\mathrm{qn}(P) \leq \mathrm{width}(P)$.

$\rightsquigarrow$ interesting and looks doable

Open Question.

For every planar poset P we have $\mathrm{qn}(P) \leq f(\mathrm{height}(P))$?

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Thank you for your attention!