On RAC Drawings of Graphs with one Bend per Edge

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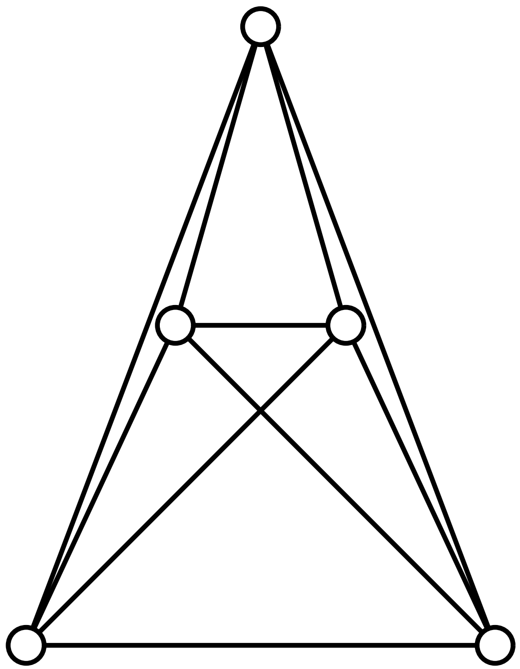
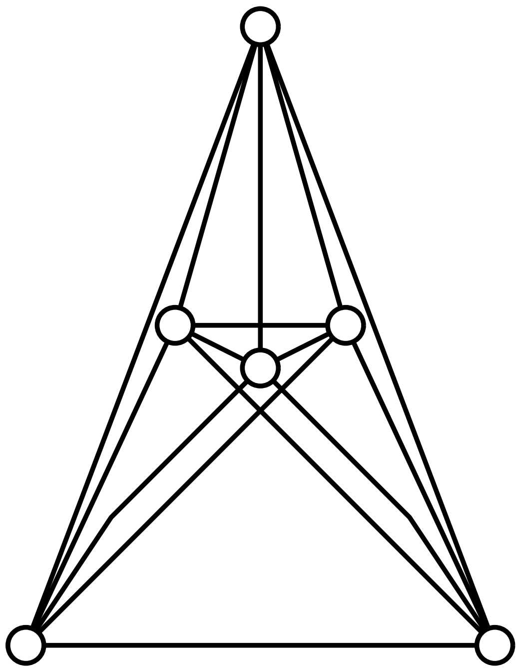
k-bend RAC Drawings

- k-bend: edges drawn as polylines with at most k bends



0-bend RAC Drawing of K_5



1-bend RAC Drawing of K_6

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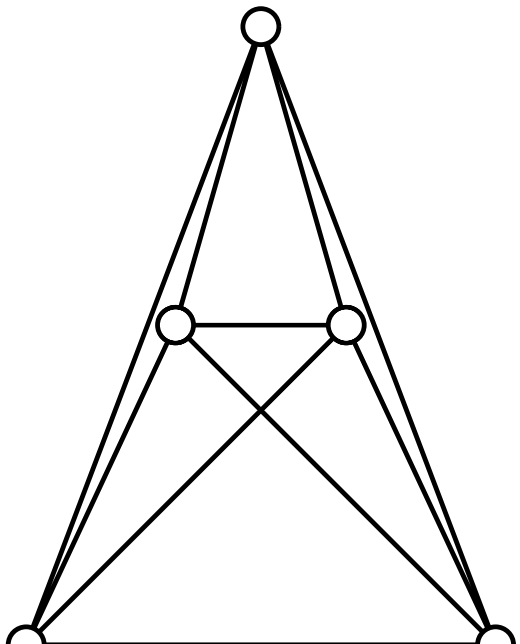
- **R**ight **A**ngle **C**rossing: all crossings at 90°



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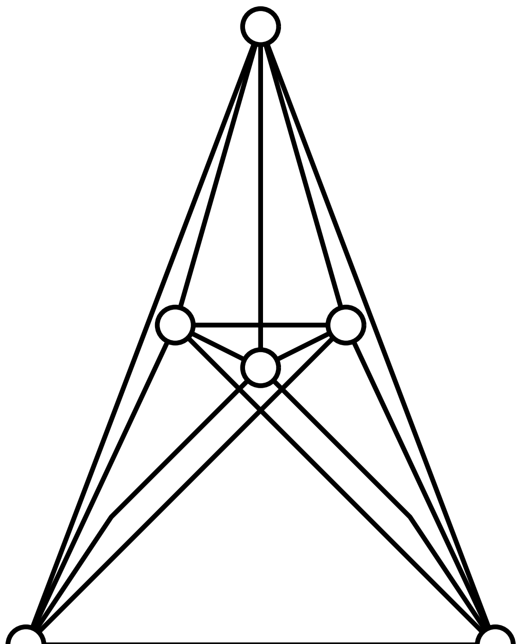

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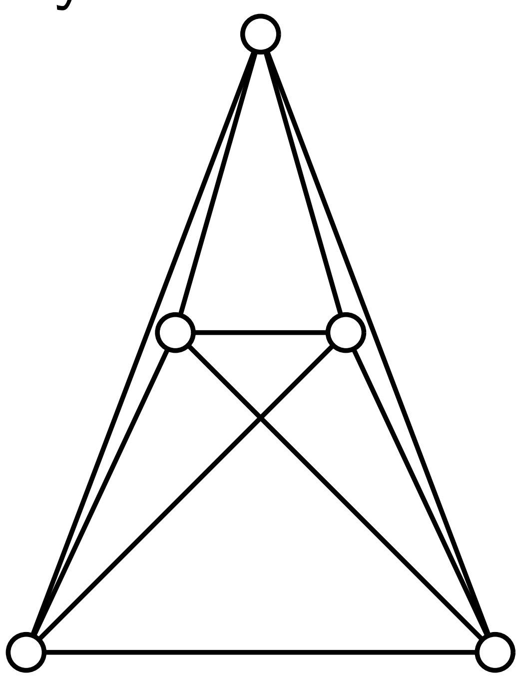
- k-bend: edges drawn as polylines with at most k bends
- **R**ight **A**ngle **C**rossing: all crossings at 90°
- Motivation: few bends and large crossing angles increase readability [Purchase'00, Purchase et al.'02, Huang'07, Huang et al.'14]



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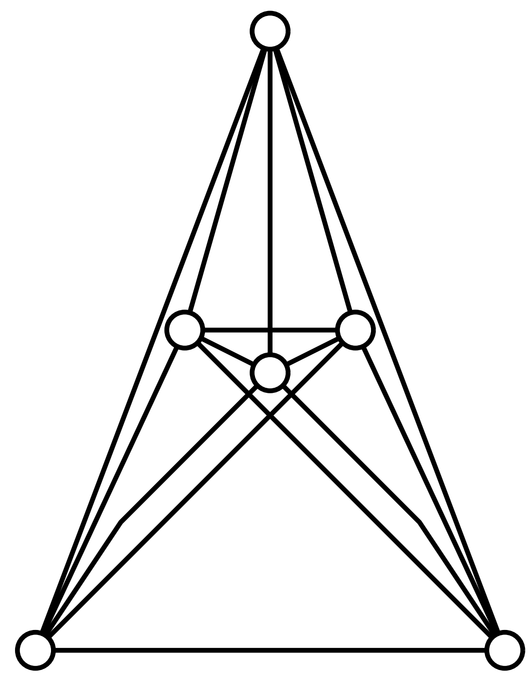

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  - At most $4n - 10$ edges (tight) [Didimo et al.'11]

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  - Studies on variants with restricted vertex position [Di Giacomo et al.'14] [Hong, Nagamochi'15]

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- **1-bend RAC:**
  - At most $6.5n - 13$ edges [Arikushi et al.'12]
  - 1-bend RAC graphs with $4.5n - O(\sqrt{n})$ edges
- **2-bend RAC:**
  - At most $74.2n$ edges [Arikushi et al.'12]
  - 2-bend RAC graphs with $7.83n - O(\sqrt{n})$ edges
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Our Contribution

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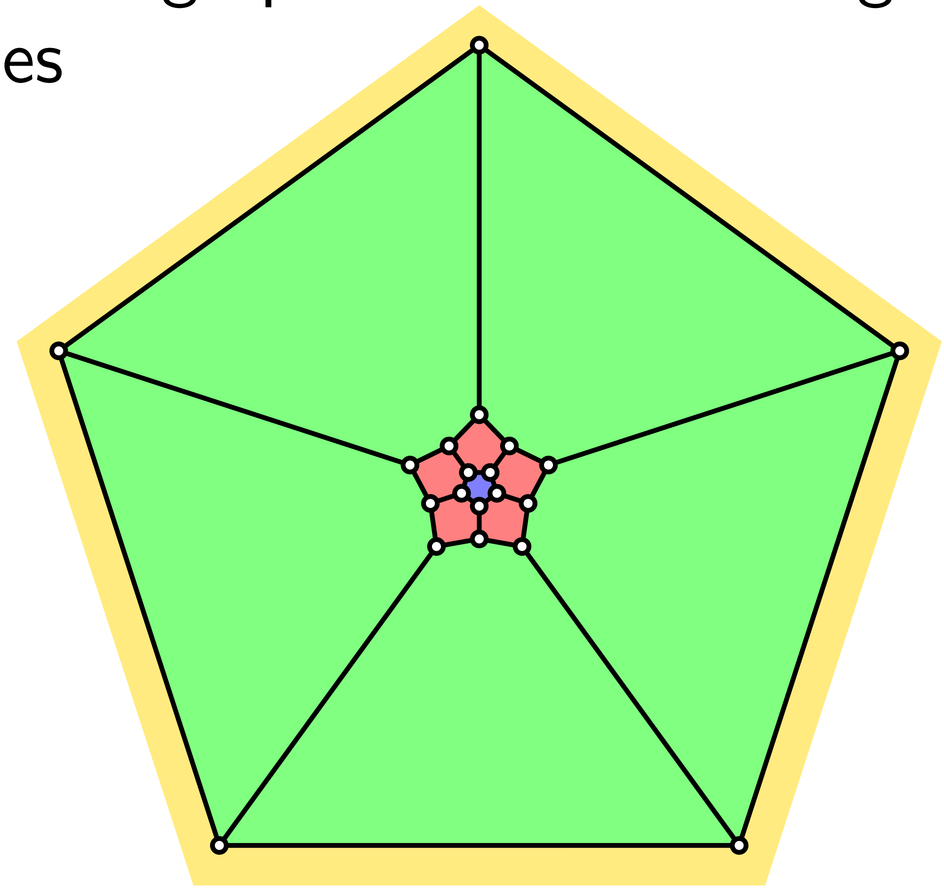
- There are infinitely many 1-bend RAC graphs with $5n - 10$ edges
- This reduces the gap from $2n$ to $0.5n$

The Lower Bound

- The dodecahedral graph admits a drawing with 4 types of face geometries



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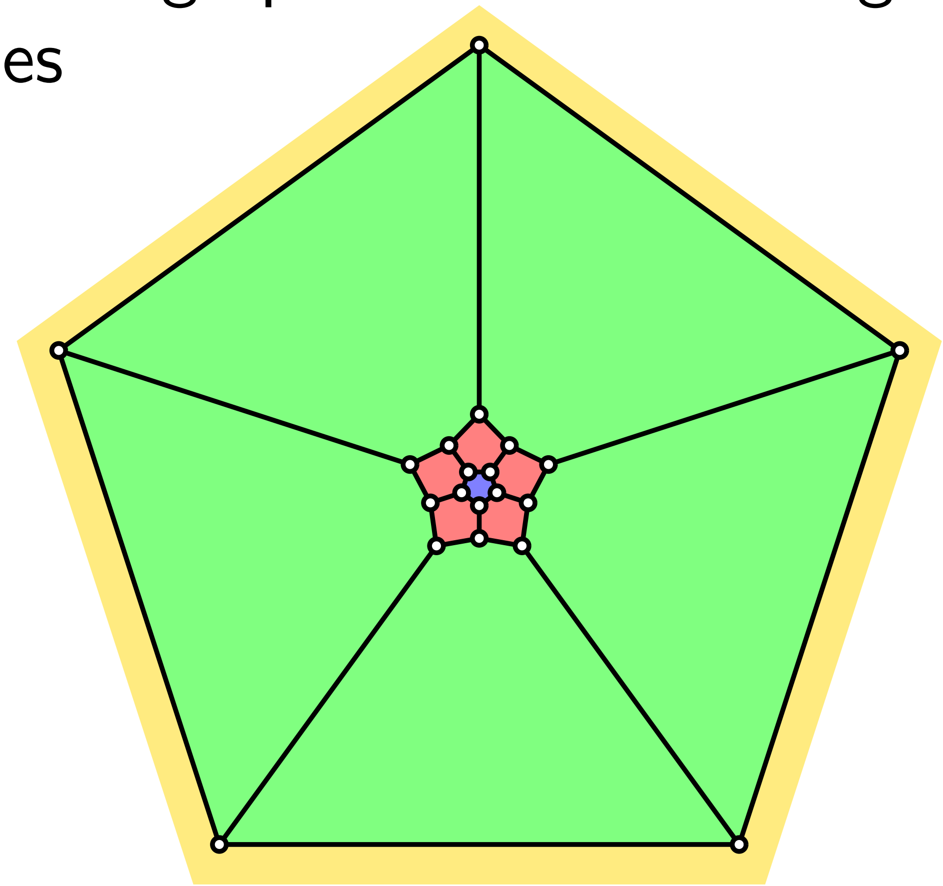

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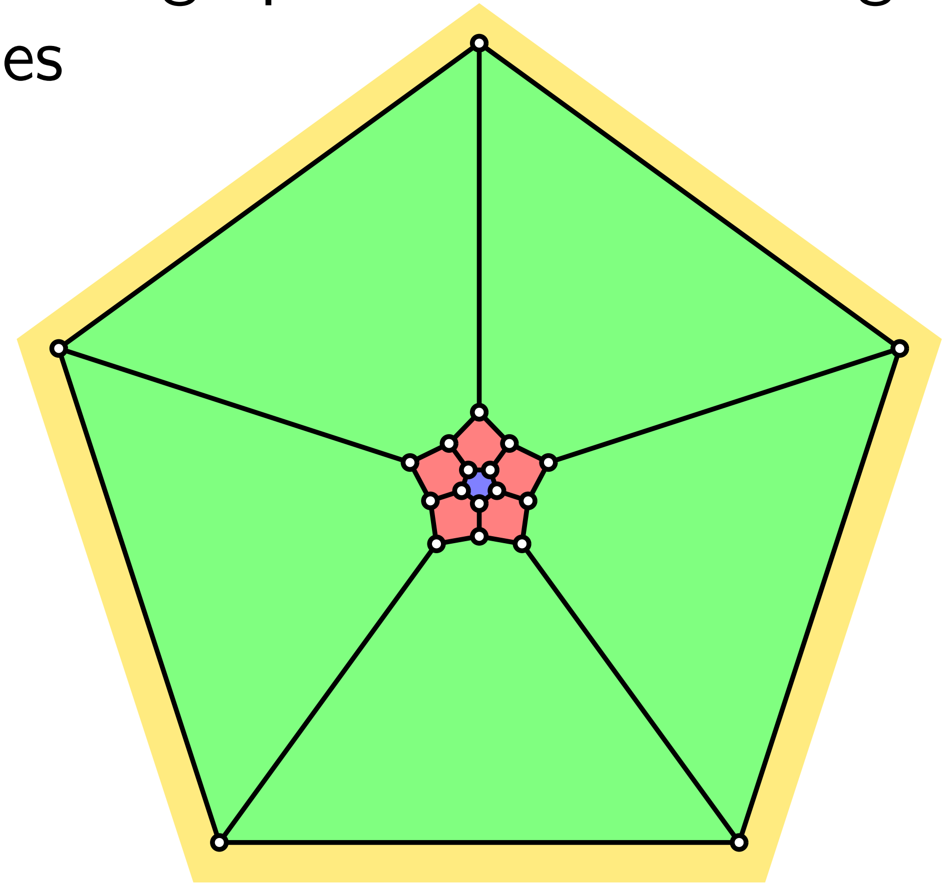

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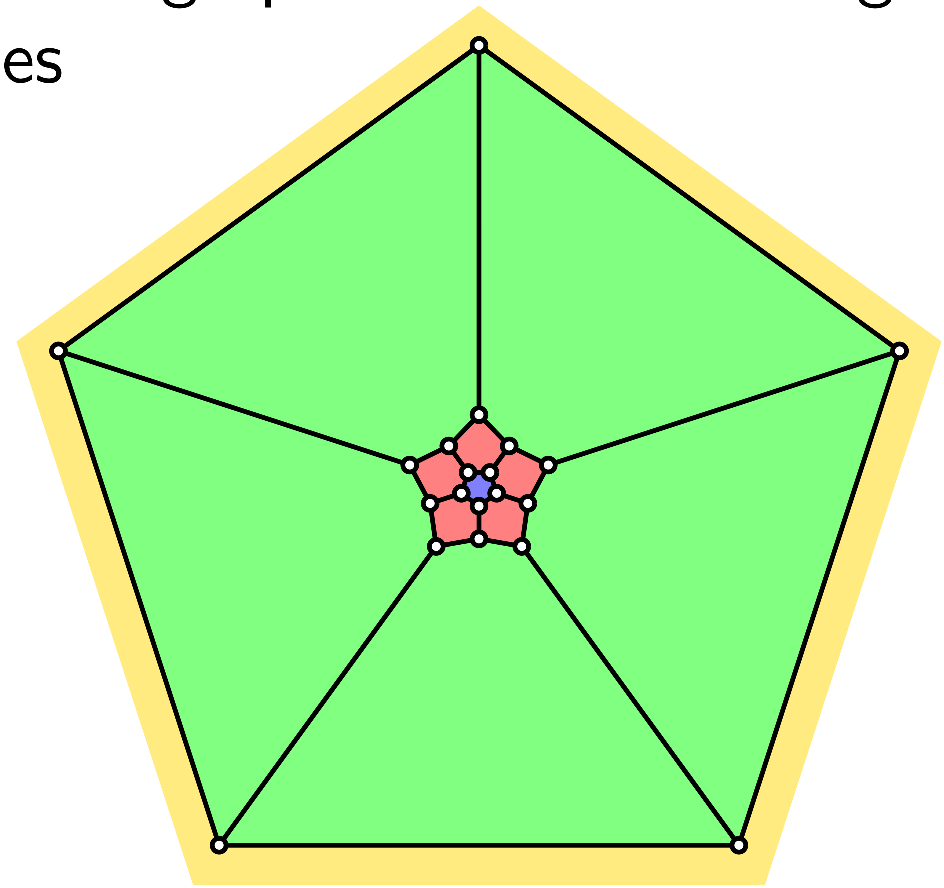

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  $\implies$ we can glue copies together
- By adding 5 edges in each face, we achieve $5n - 10$ edges

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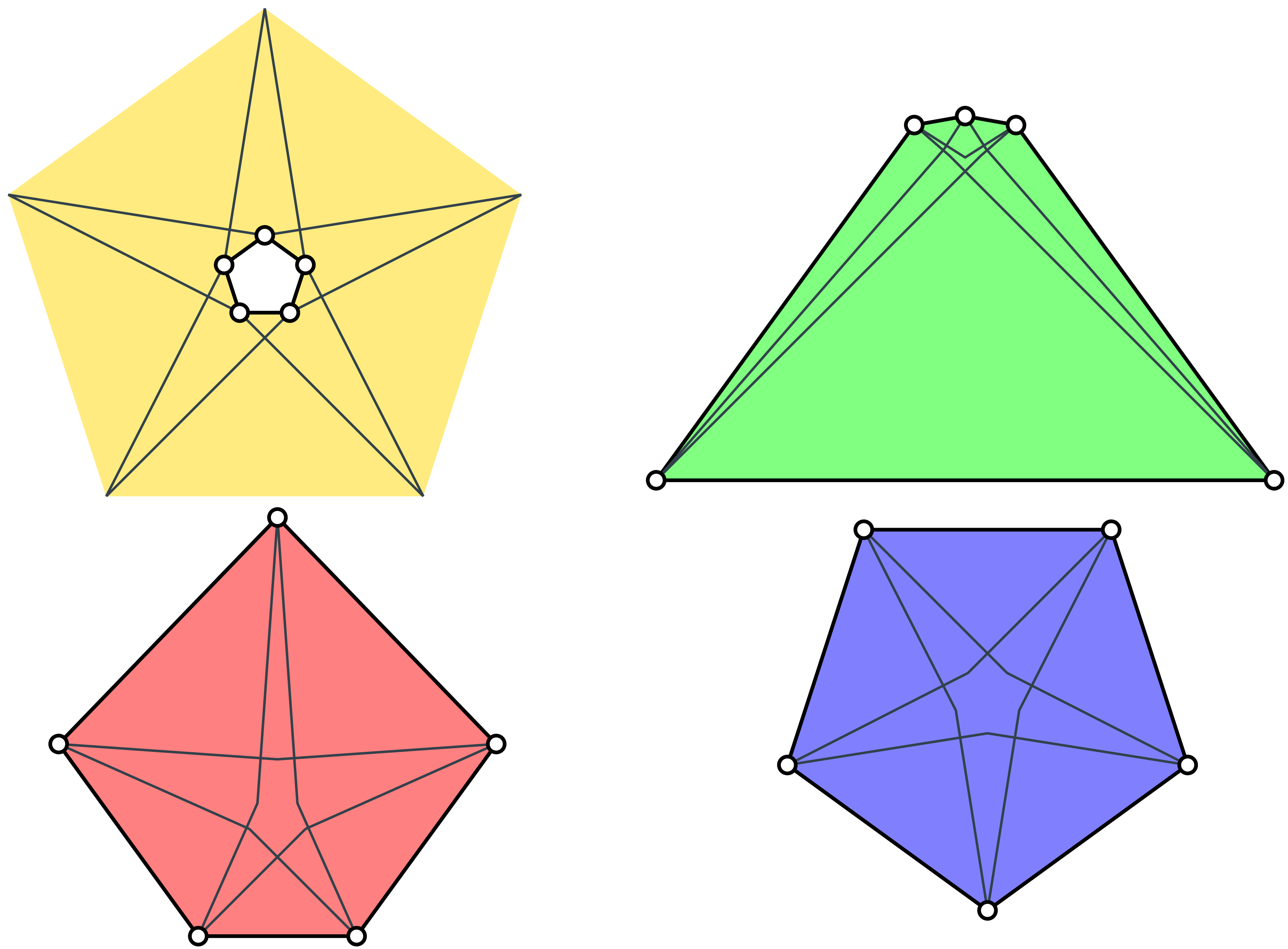
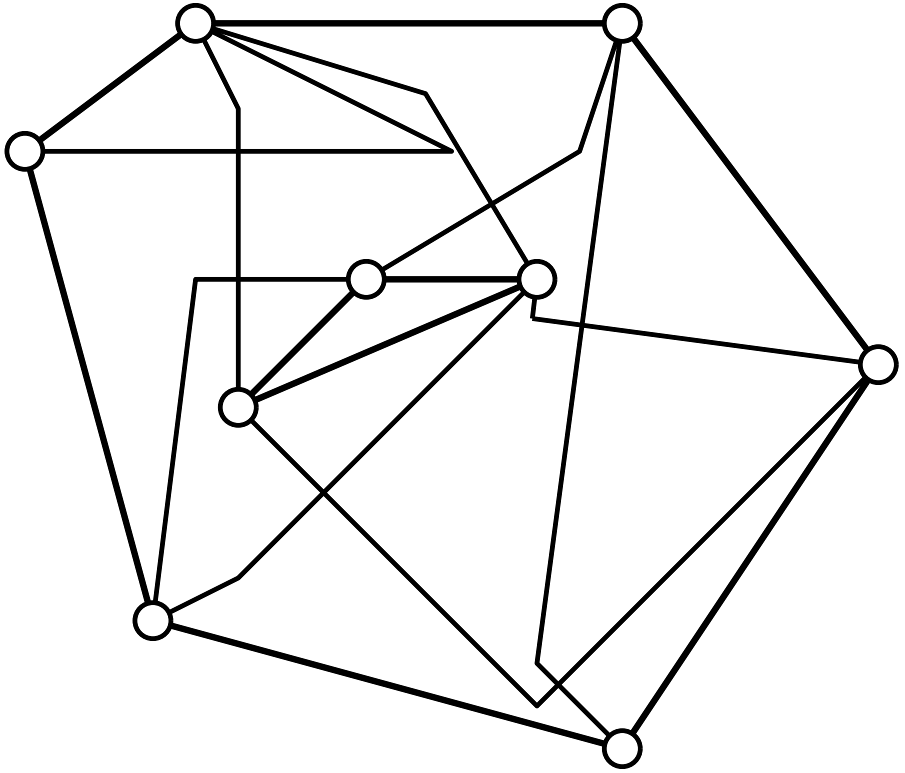


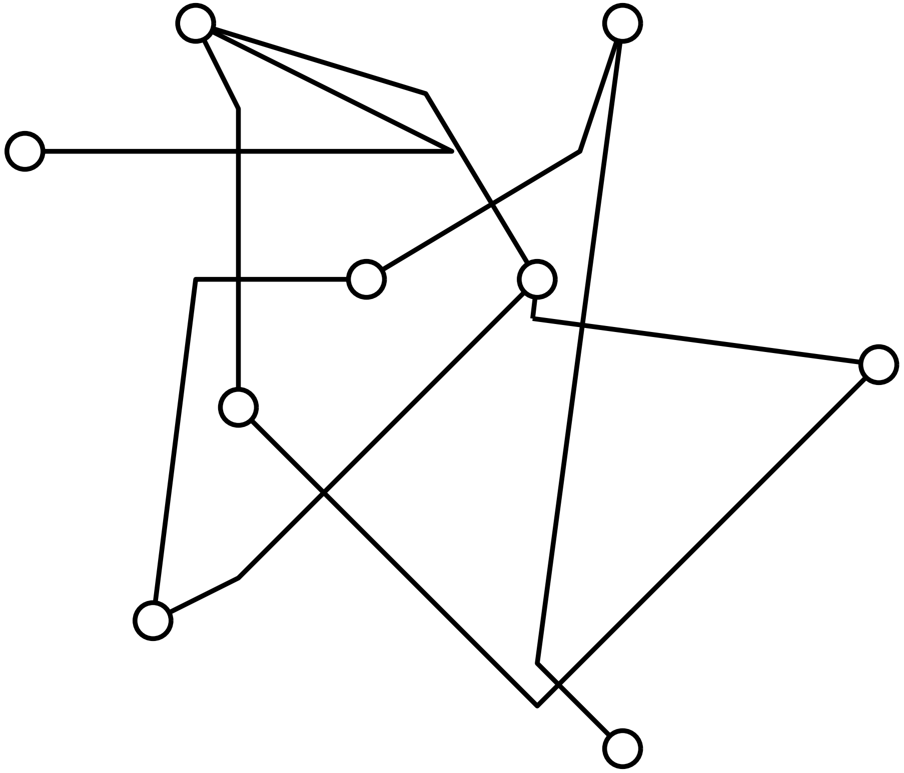
Arikushi et al. 2012

- Upper bound on the intersected edges



Arikushi et al. 2012

- Upper bound on the intersected edges
  - Remove the planar edges



Arikushi et al. 2012

- Upper bound on the intersected edges
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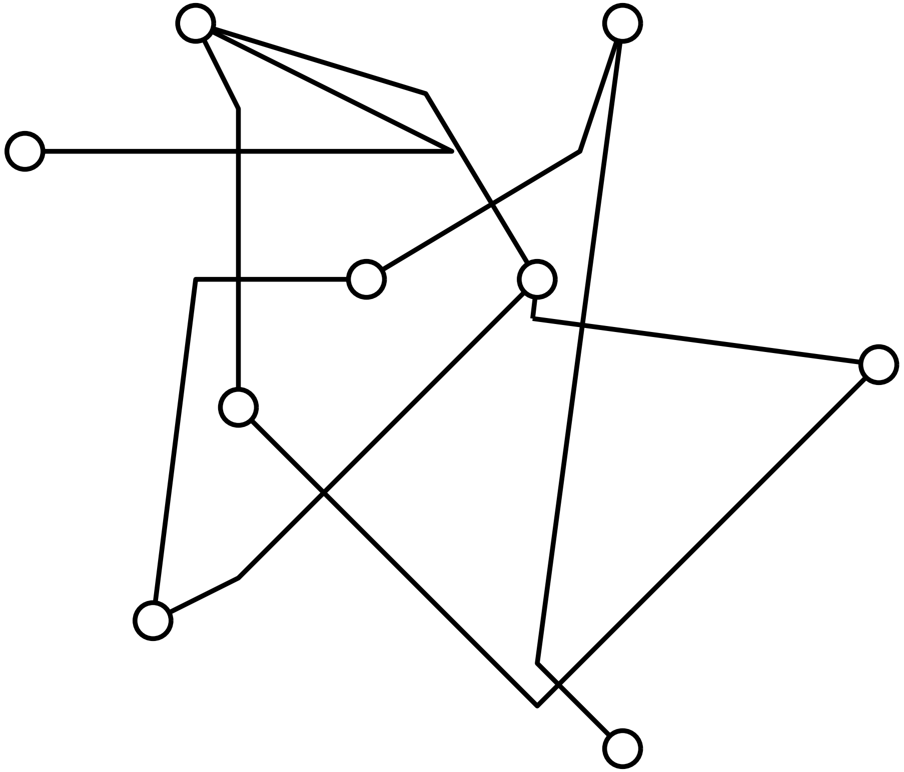

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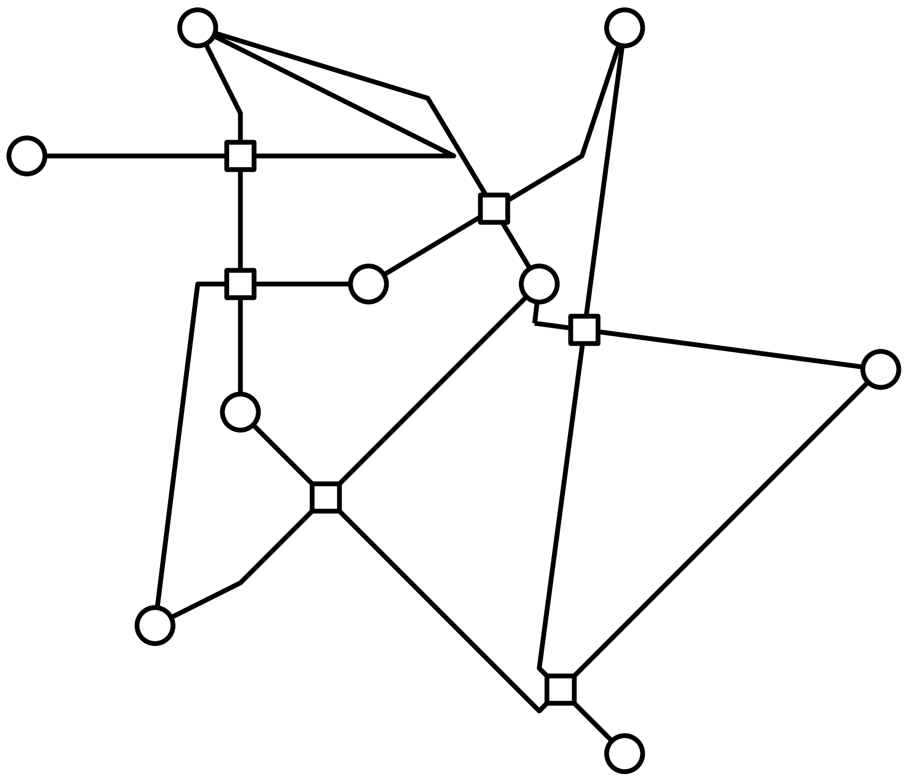

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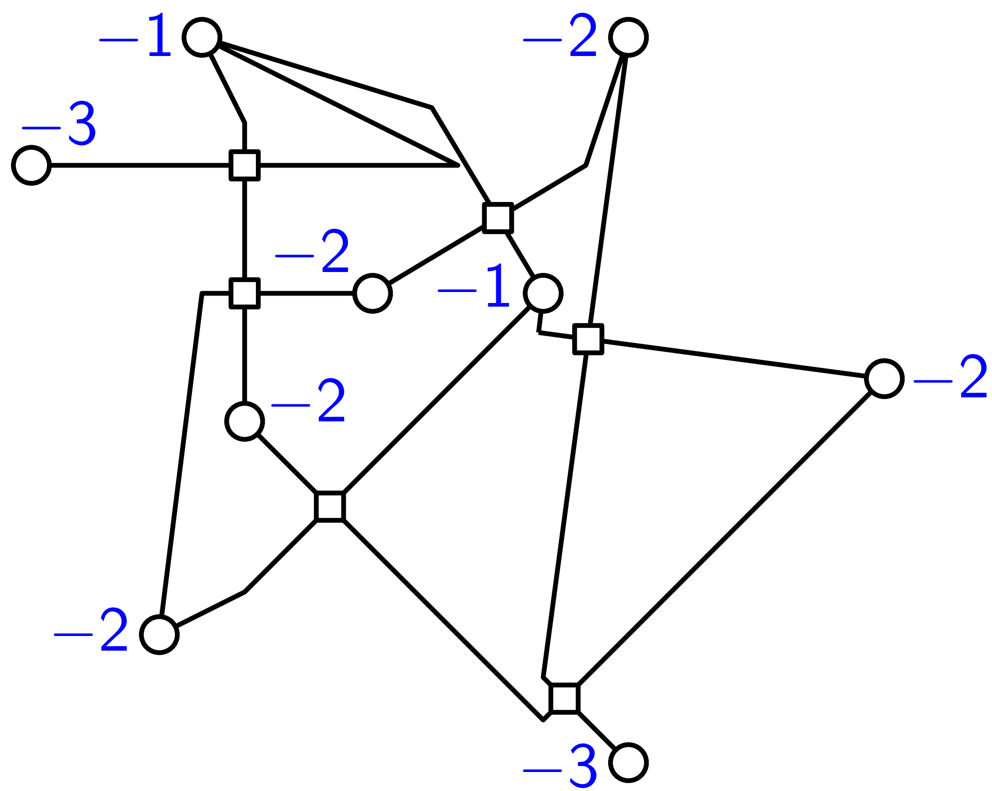

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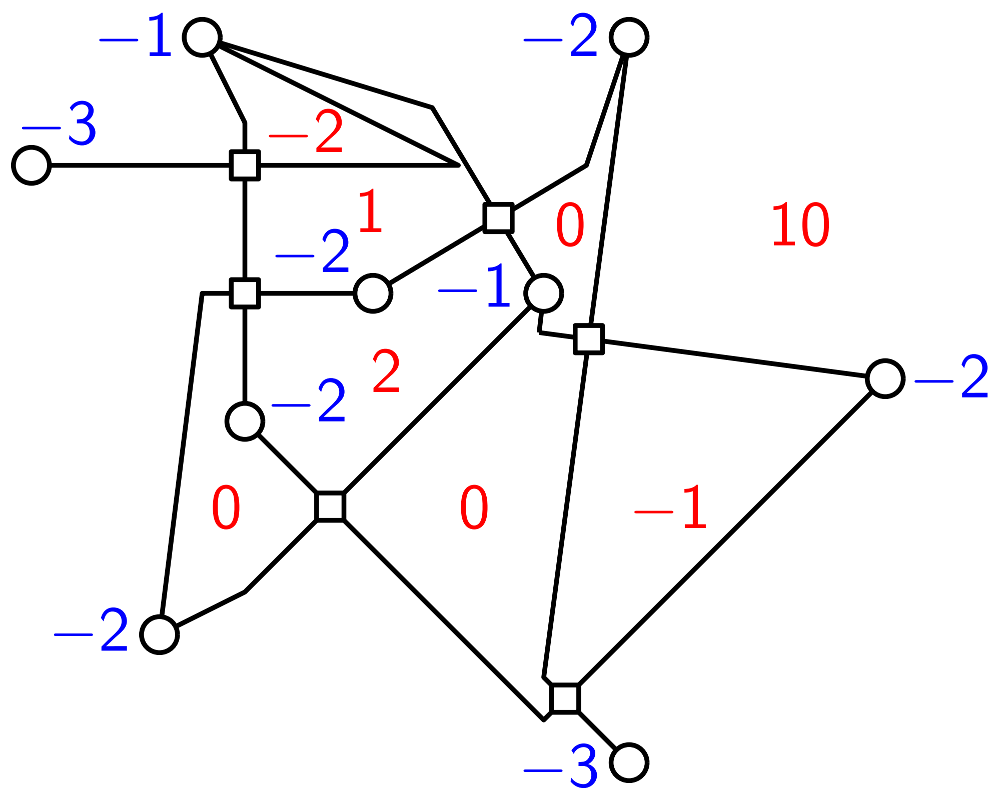

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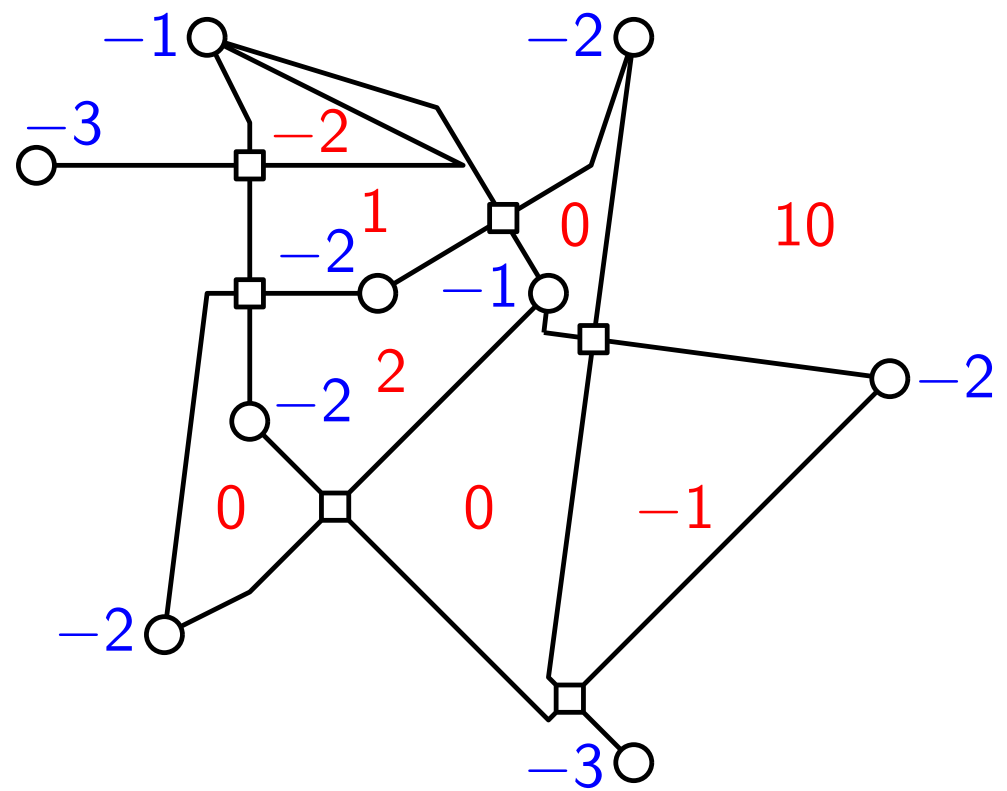
    $\sum ch(v) + \sum ch(f) = -8$



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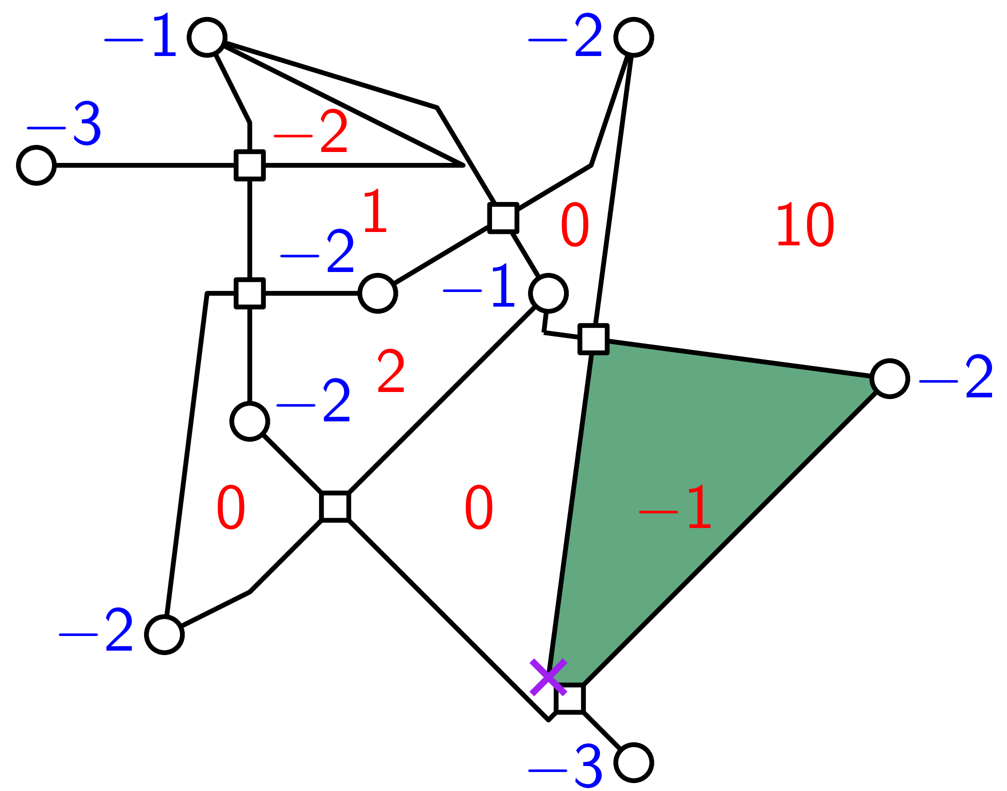

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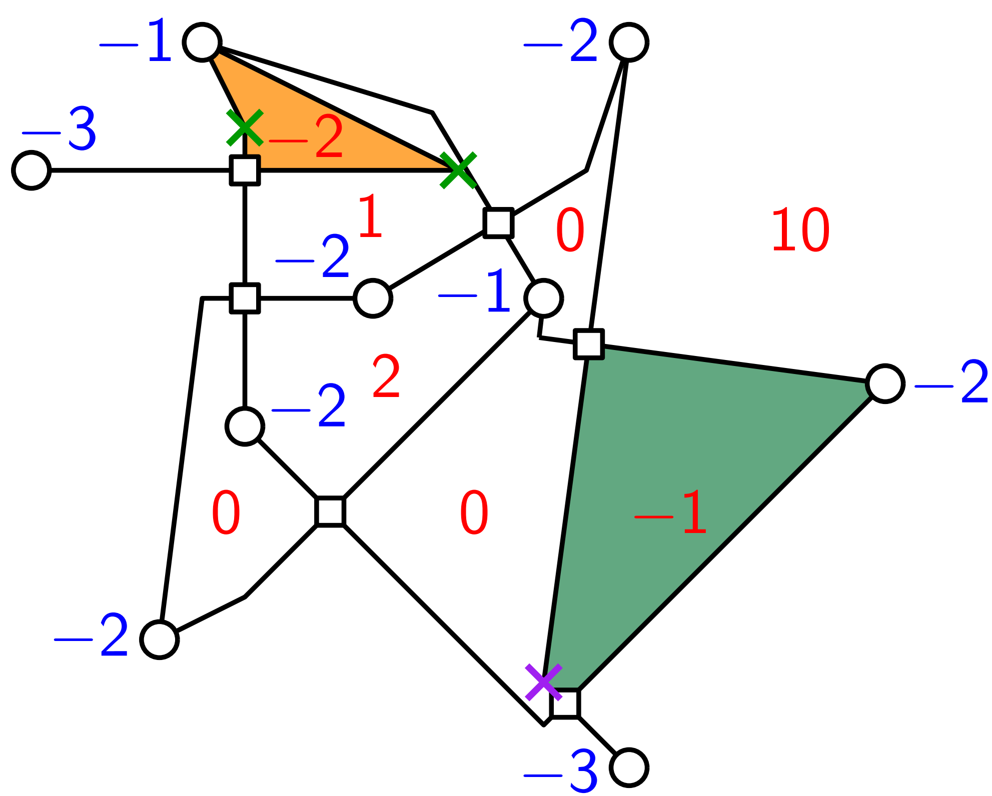
  - Lenses have charge -2 and are incident to two bends, one of which is convex



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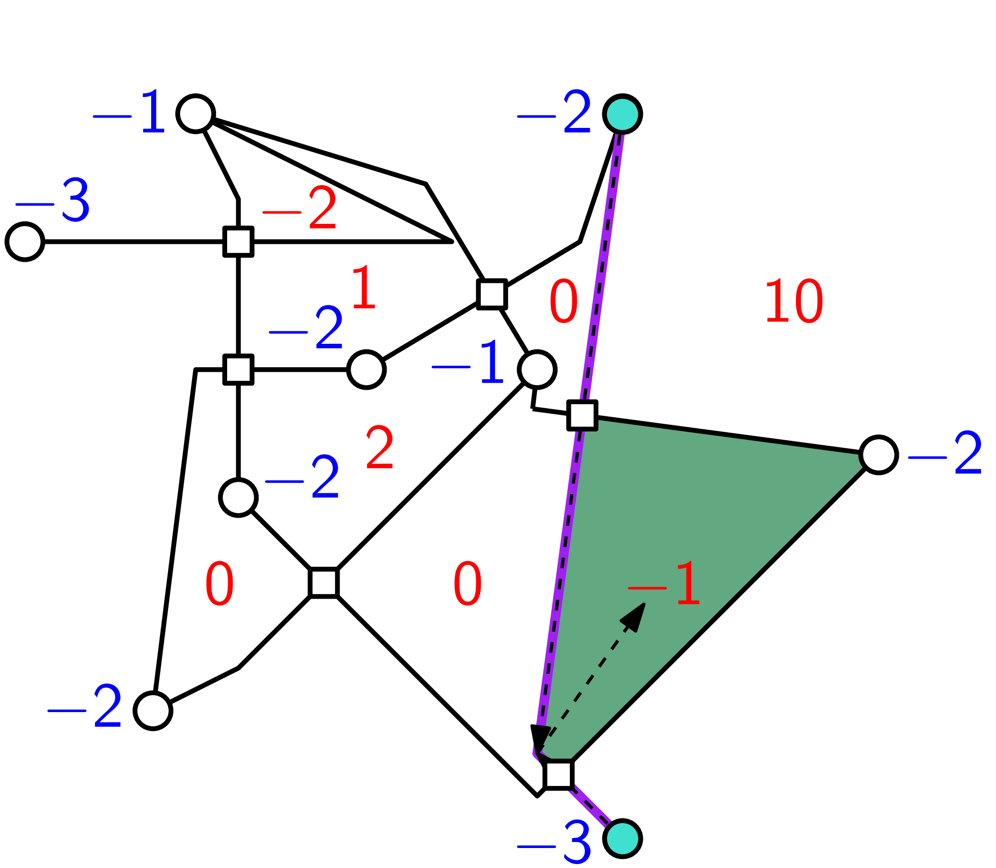
Arikushi et al. 2012

- Discharging phase 1
  - For each edge, move 1/2 charge from each endpoint to the face incident to its convex bend



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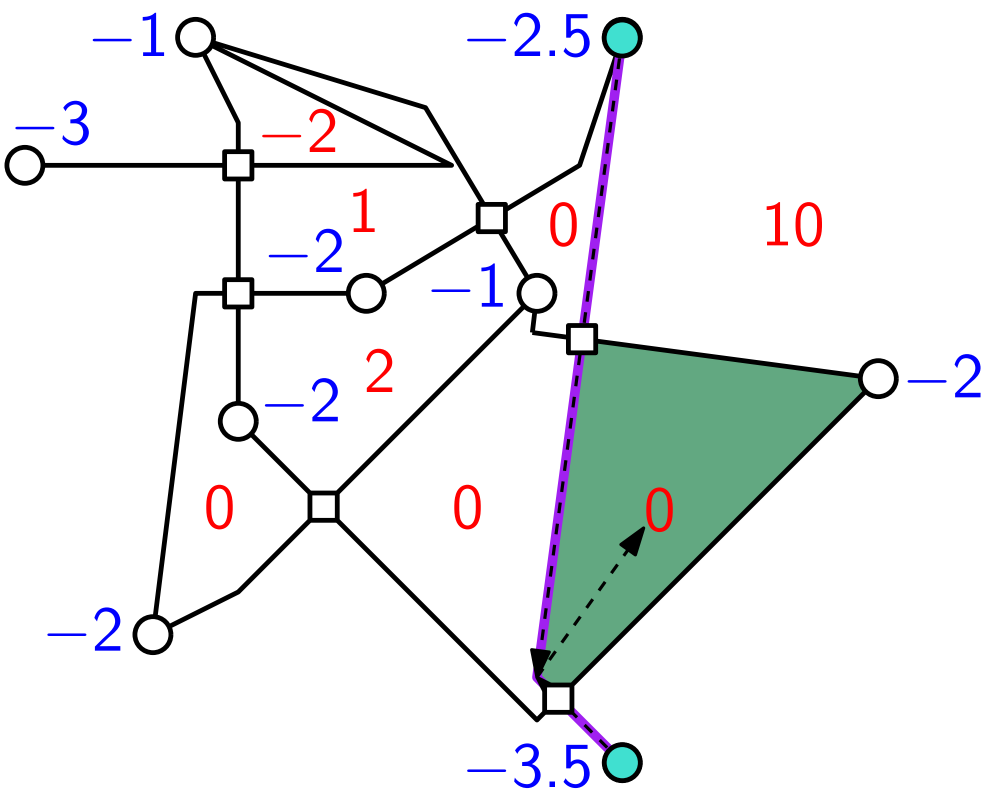

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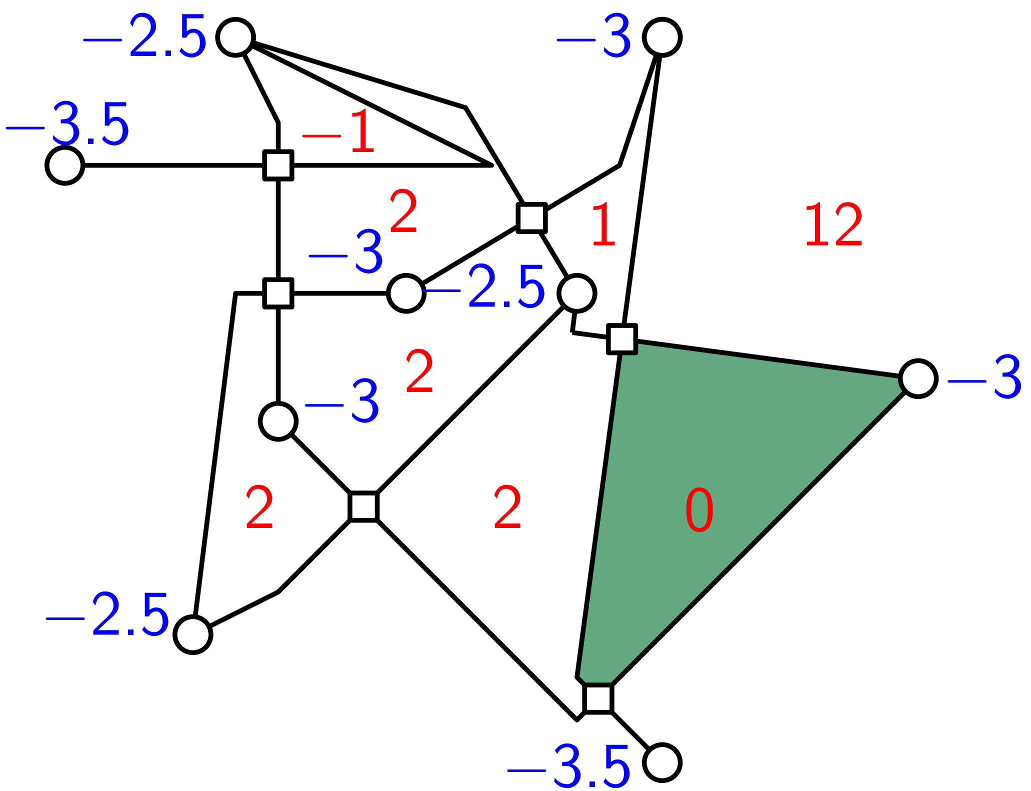

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Arikushi et al. 2012

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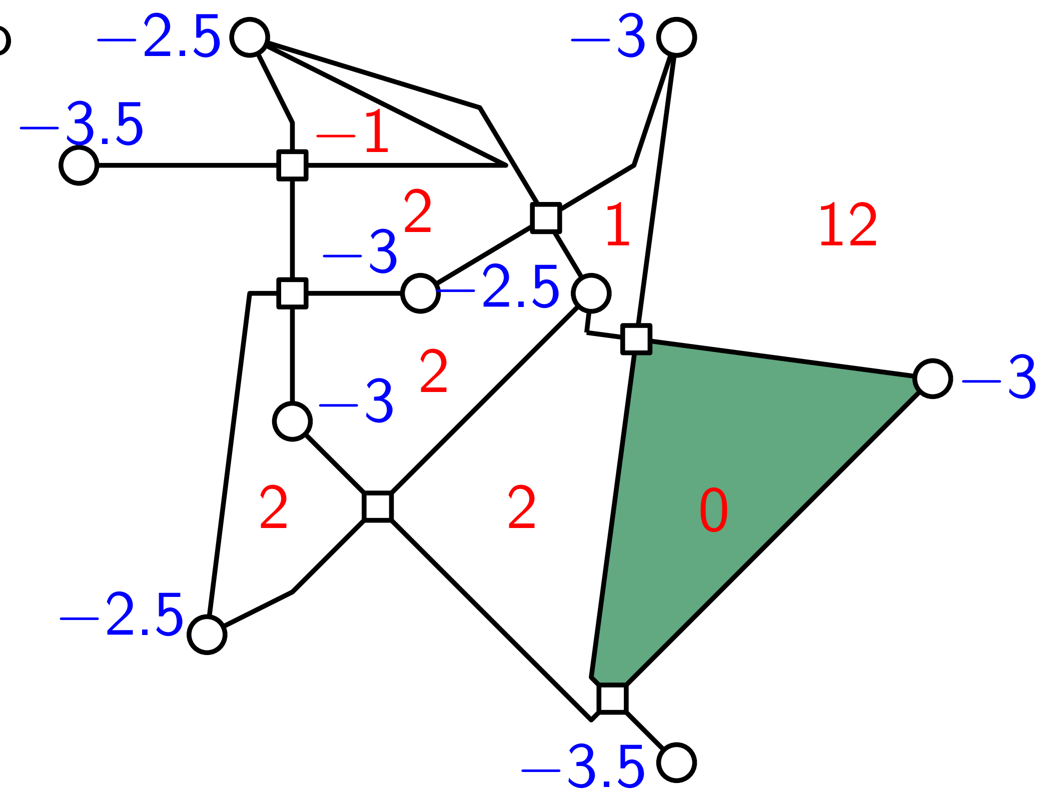

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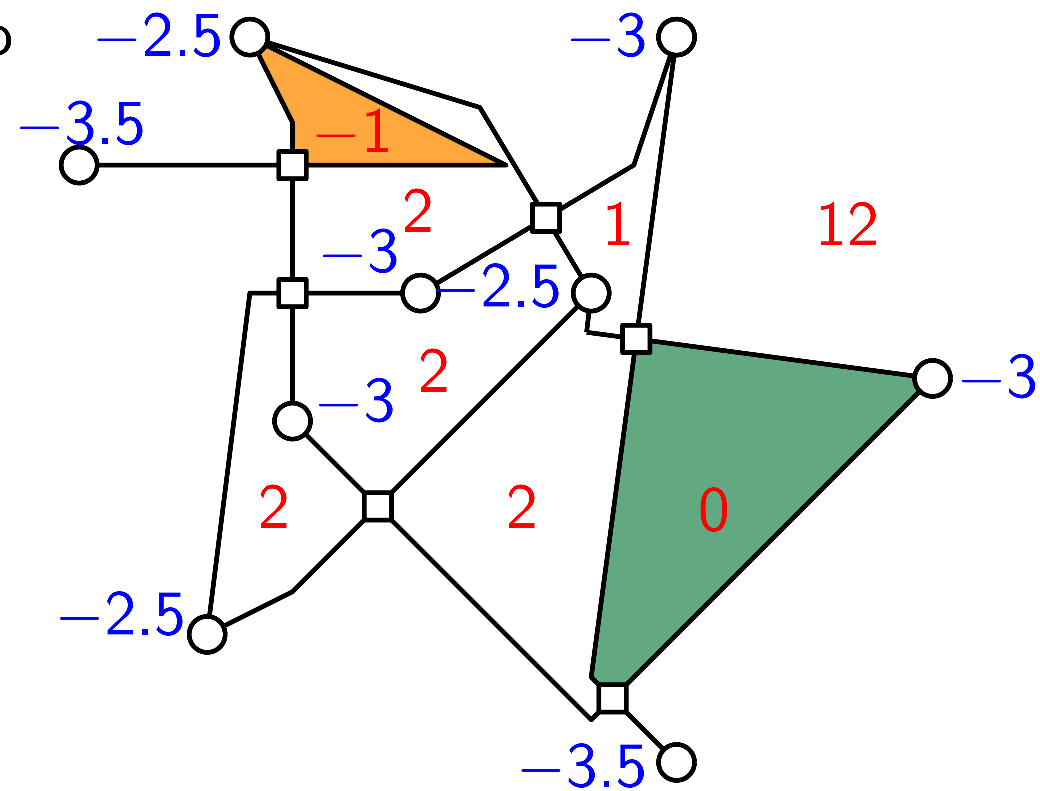

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Arikushi et al. 2012

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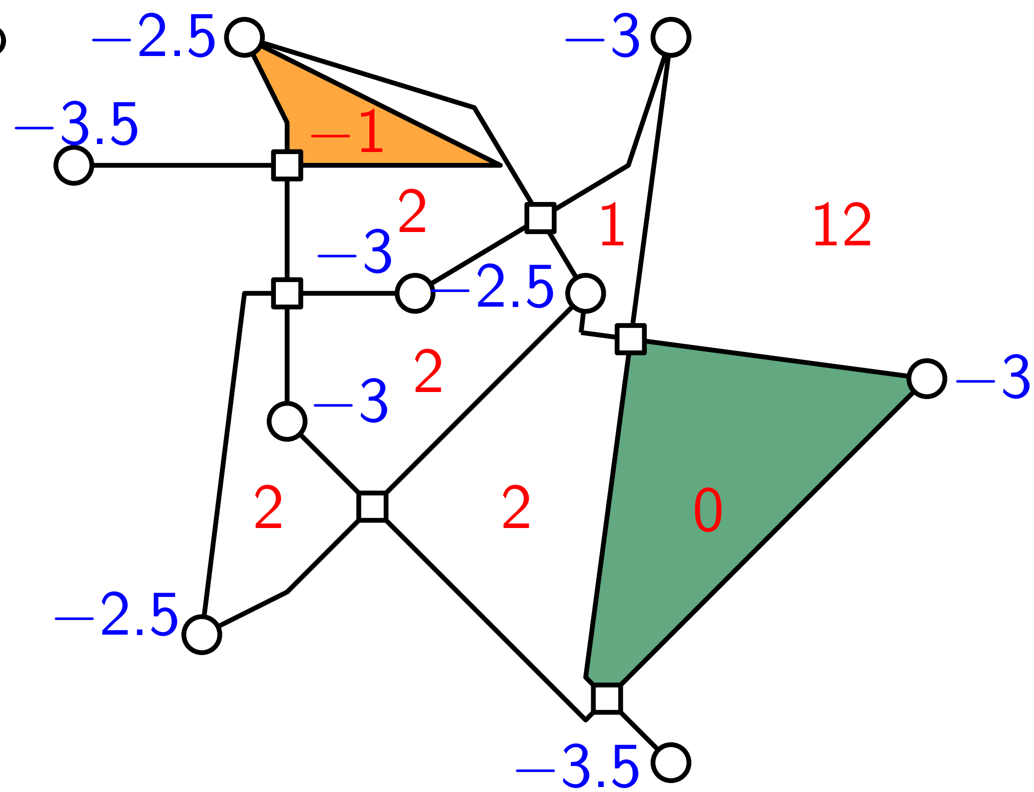
  - $ch'(v) \geq 1/2deg(v) - 4$



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Arikushi et al. 2012

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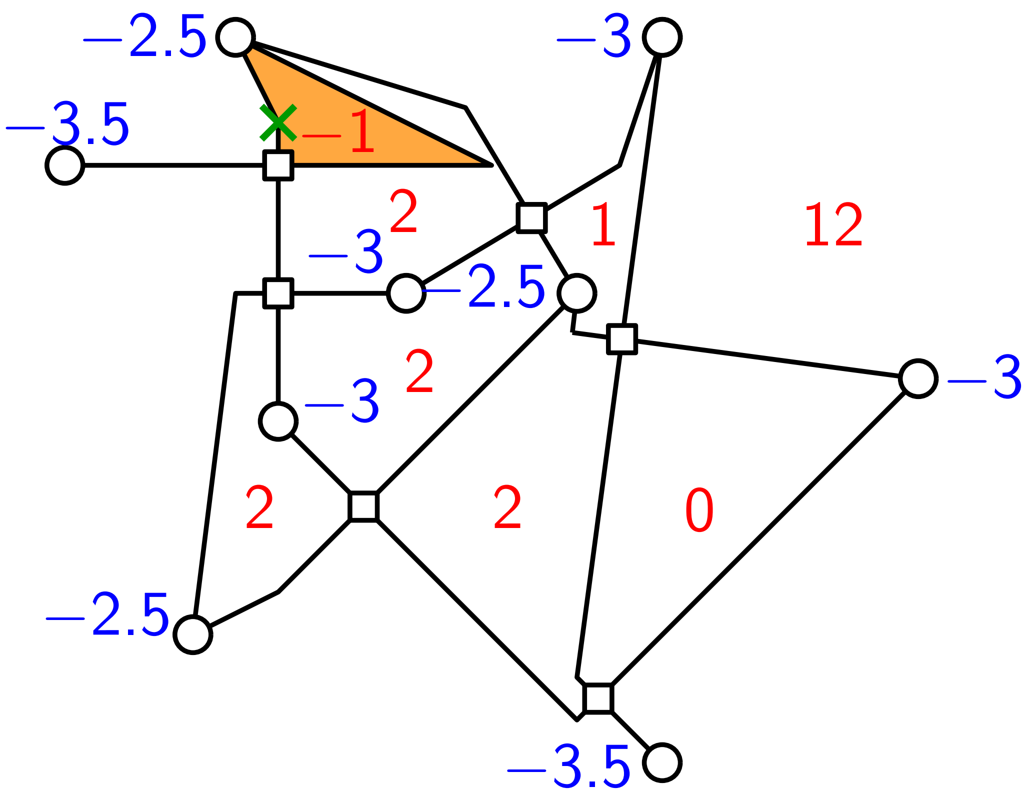
  - Injection from lenses with reflex bends to convex bends at faces of size at least 4



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Arikushi et al. 2012

- Discharging phase 2
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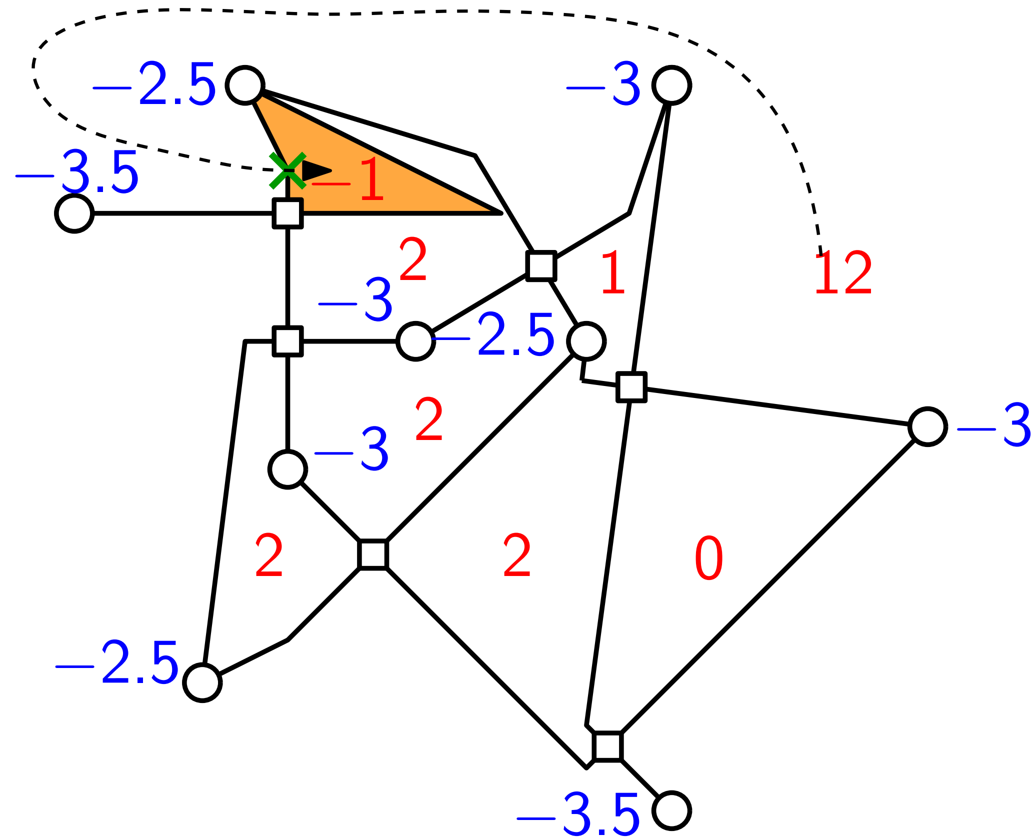
  - Transfer charges from these large faces to lenses



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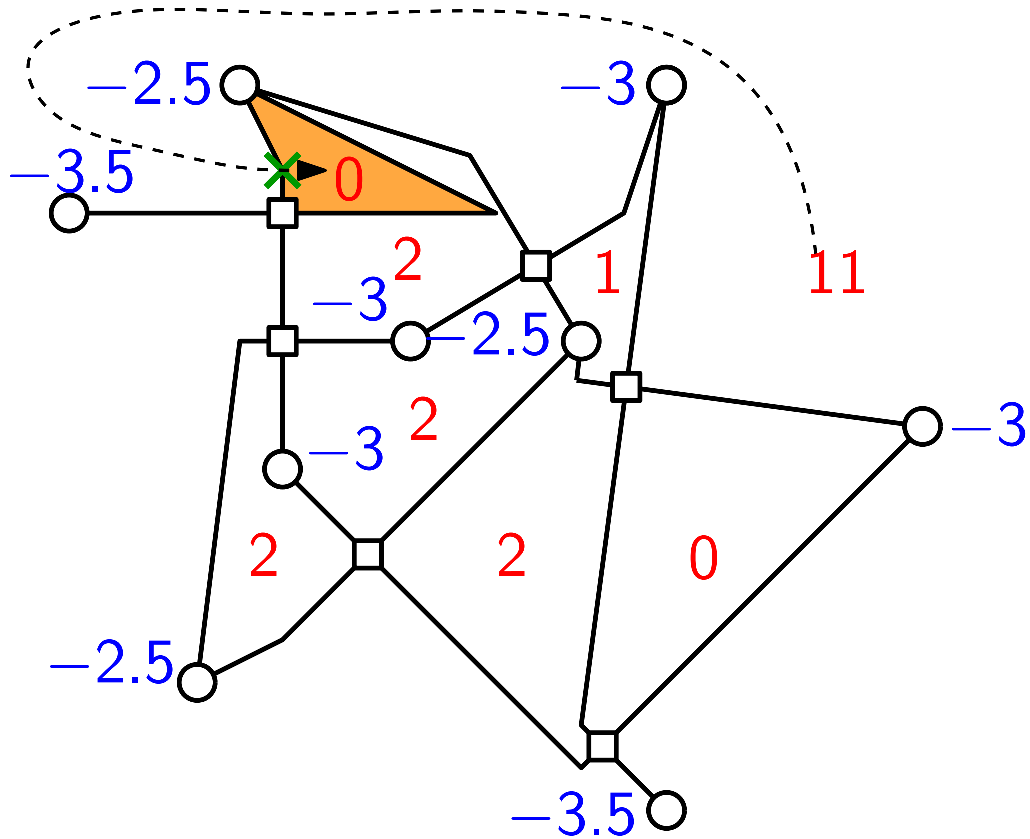

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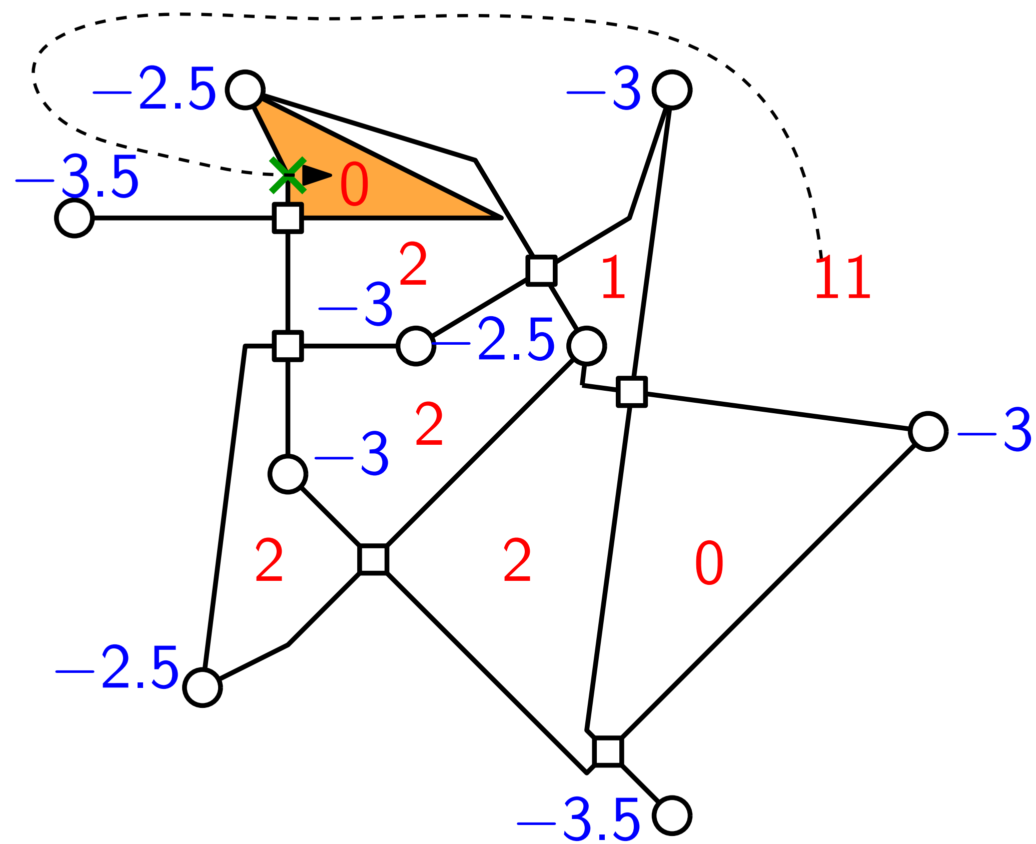

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Arikushi et al. 2012

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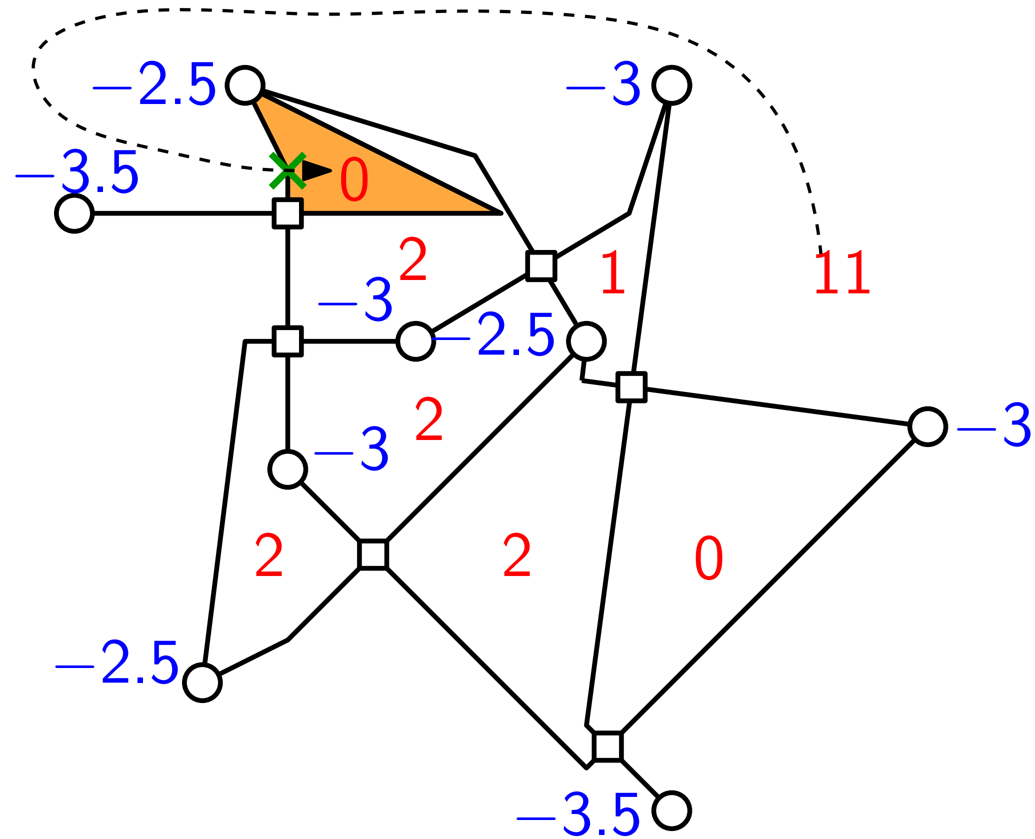

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Arikushi et al. 2012

- Some maths magic happens

$$|E_1| - 4n = \sum \tfrac{1}{2}\deg(v) - 4$$

Arikushi et al. 2012

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$$|E_1| - 4n \leq \sum ch''(v) + \sum \underbrace{ch''(f)}_{\geq 0}$$

Arikushi et al. 2012

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Arikushi et al. 2012

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- Improvement: Consider planar subgraph G_0

Arikushi et al. 2012

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  - Induction: If G_0 can be triangulated with k edges then $|E_1| \leq 8k$

Arikushi et al. 2012

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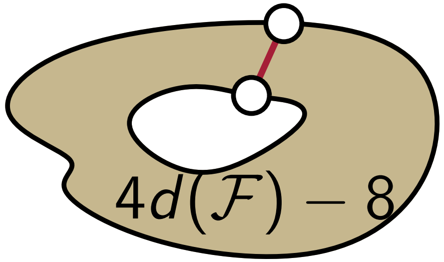

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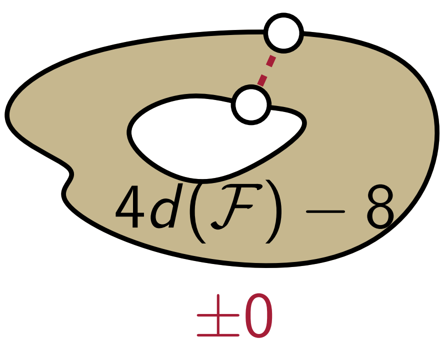

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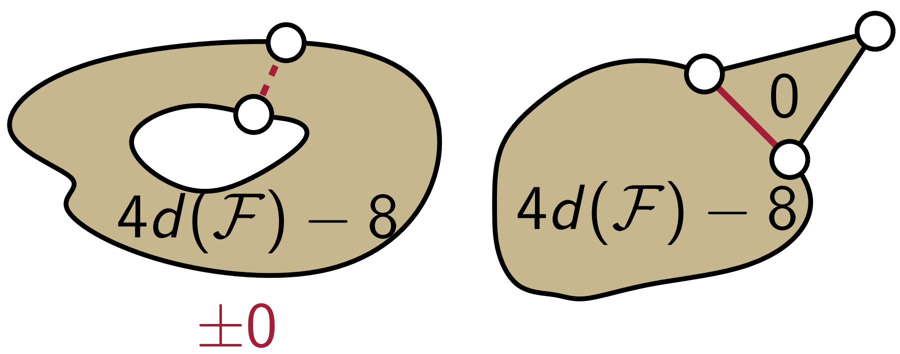

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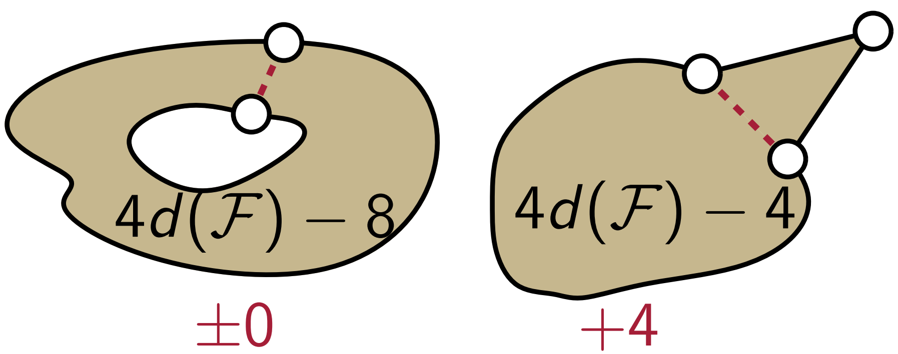

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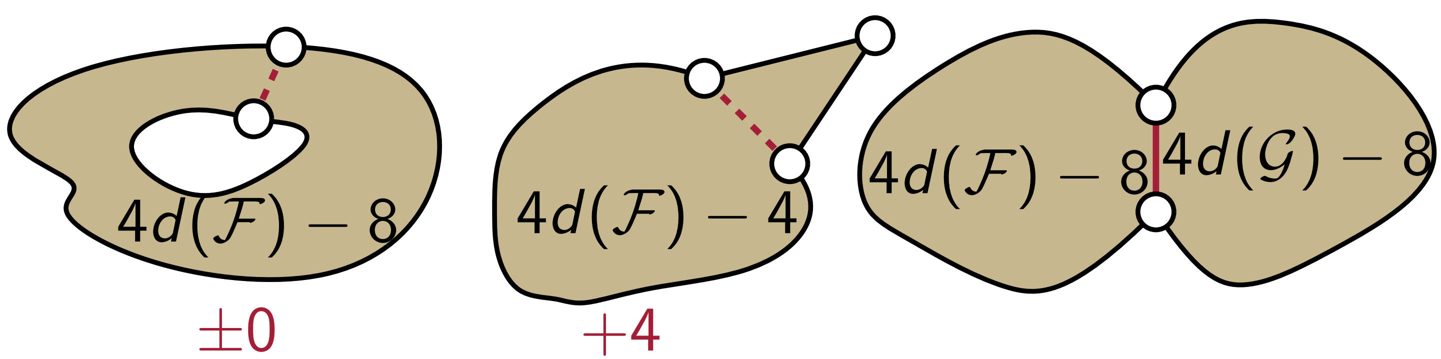

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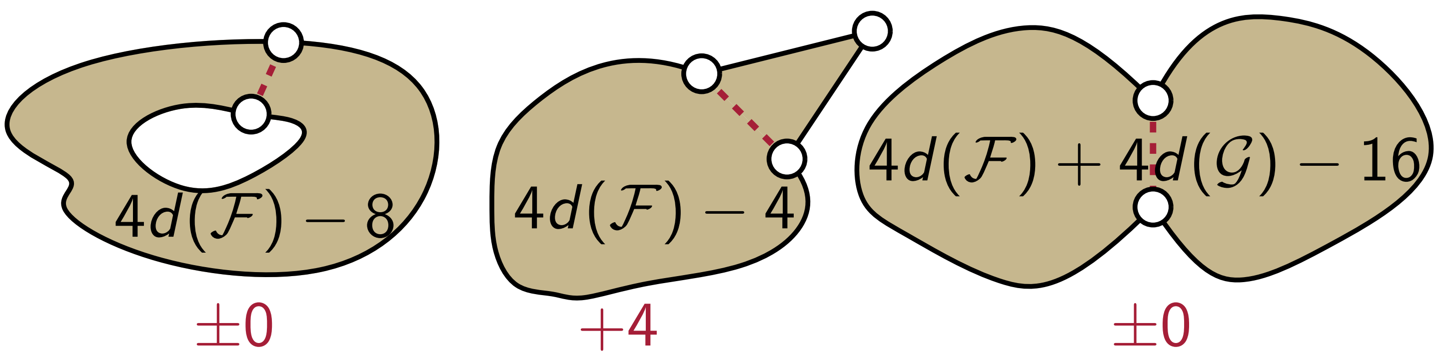

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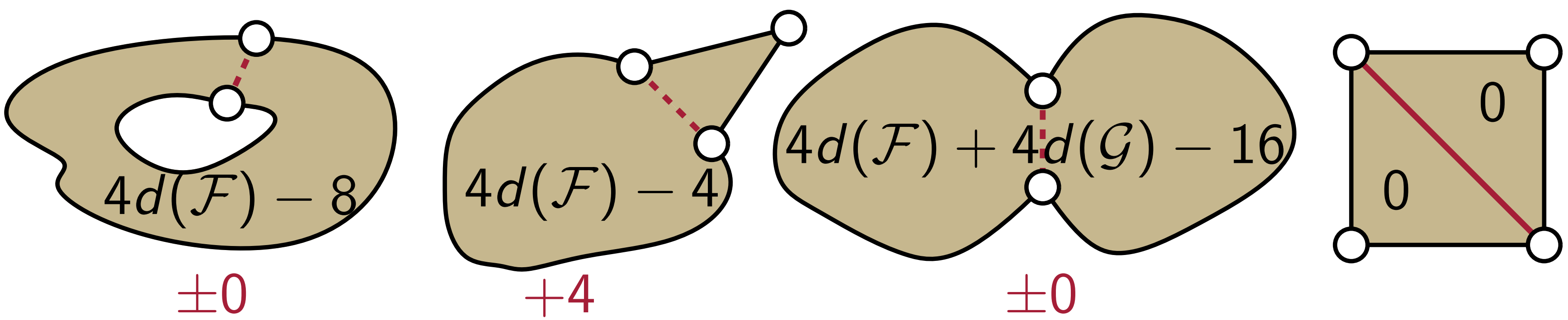

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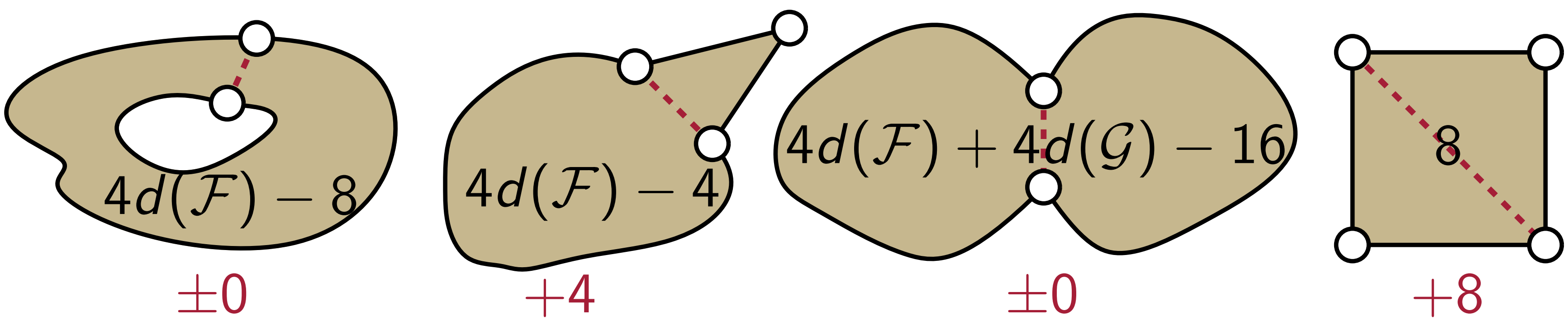

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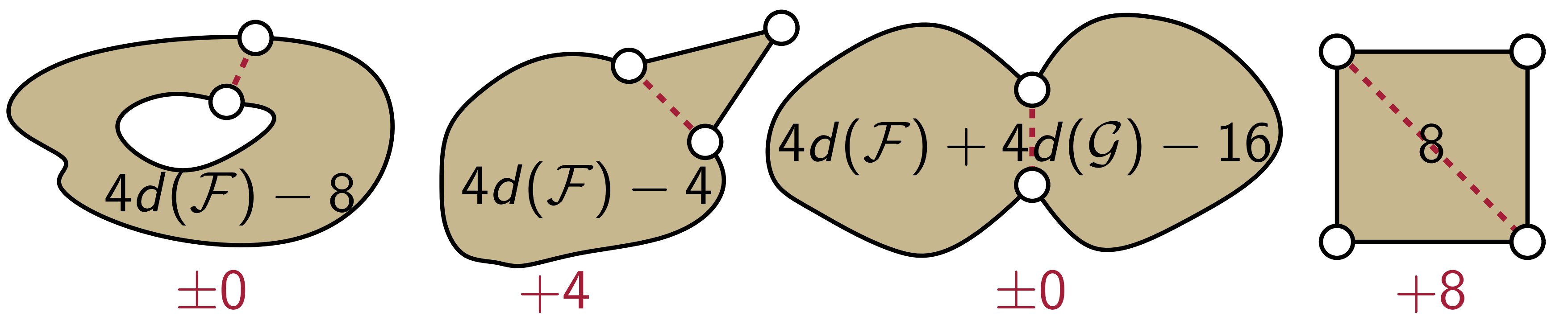

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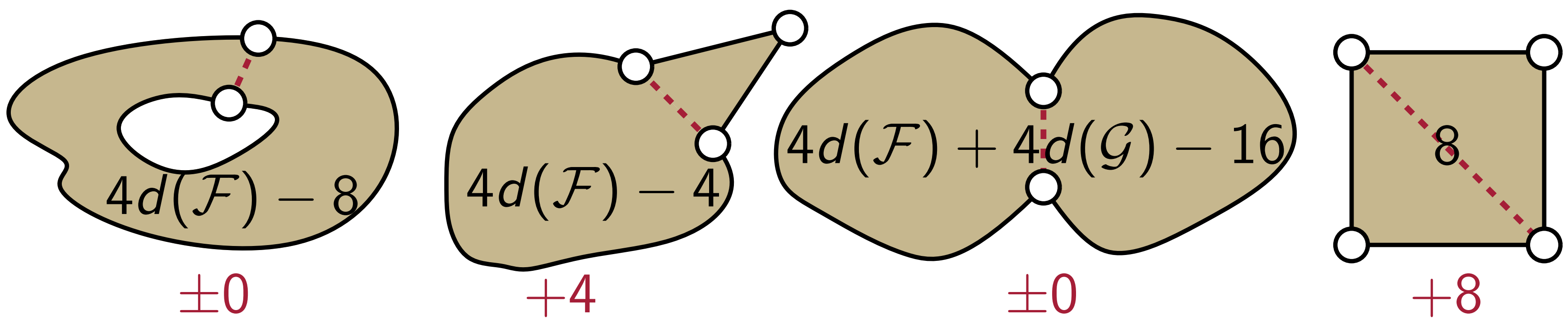

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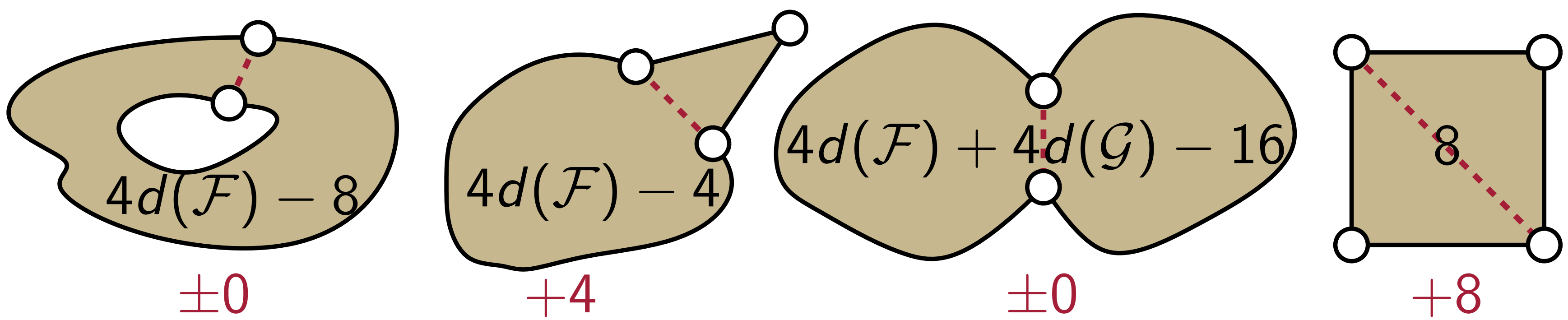

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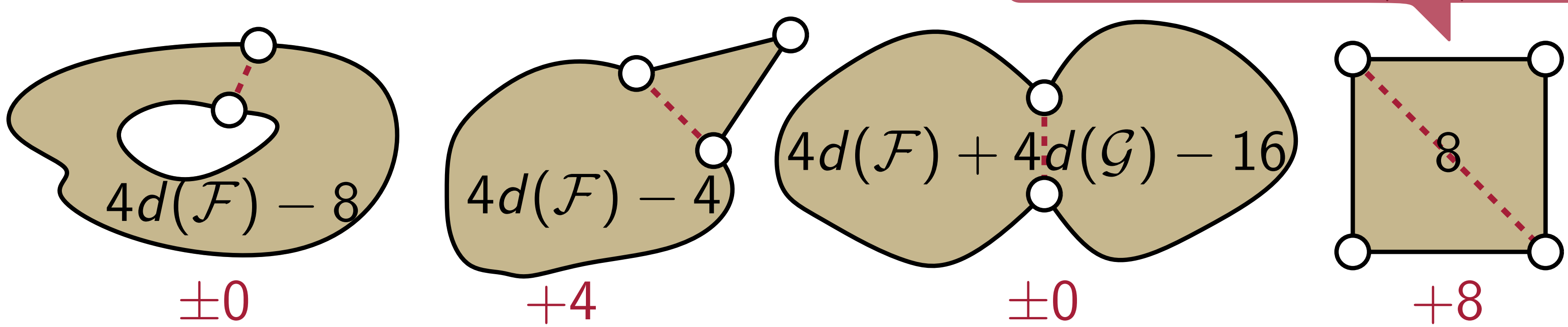
Step 2:
better analysis of small faces



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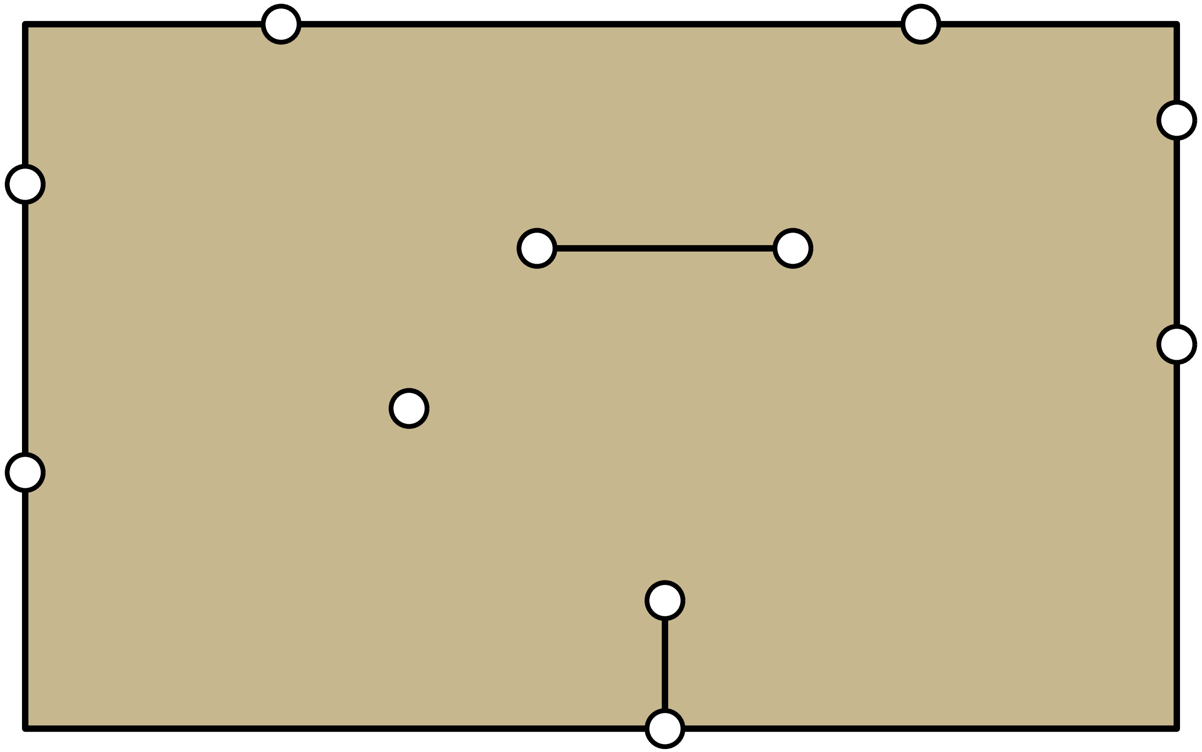

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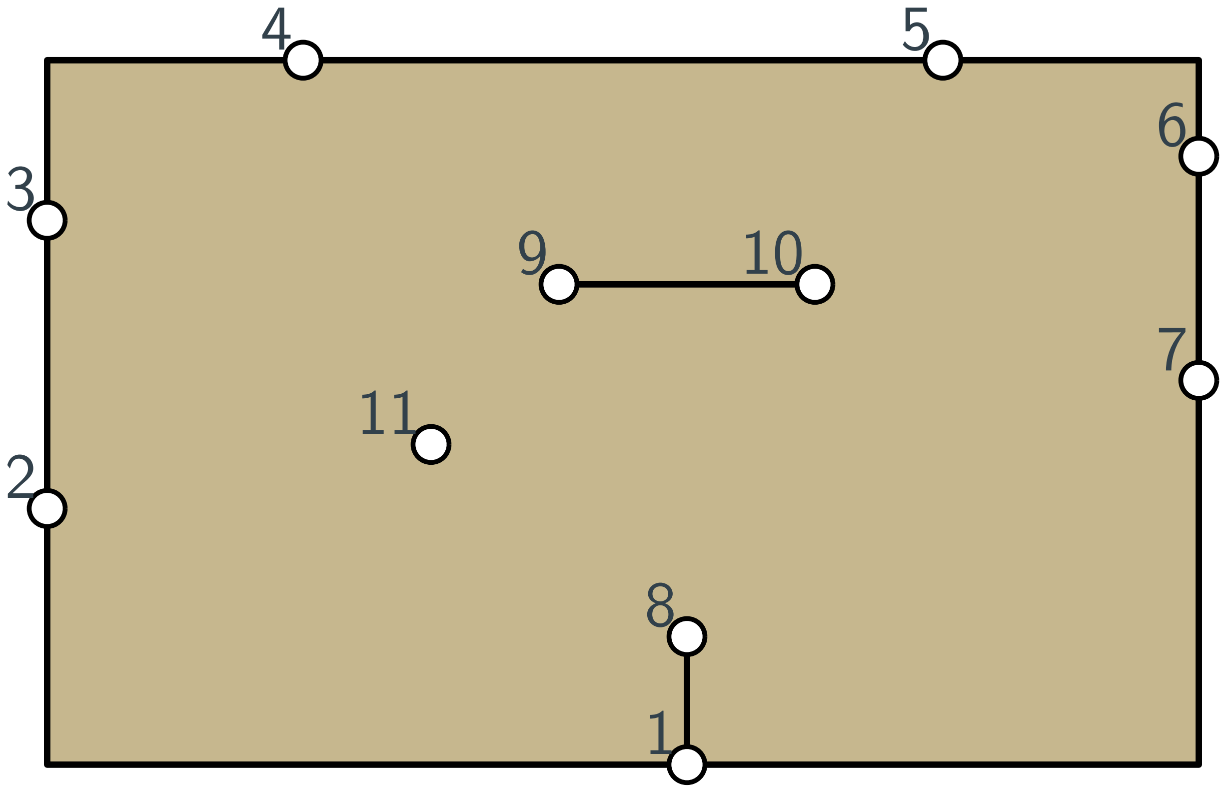

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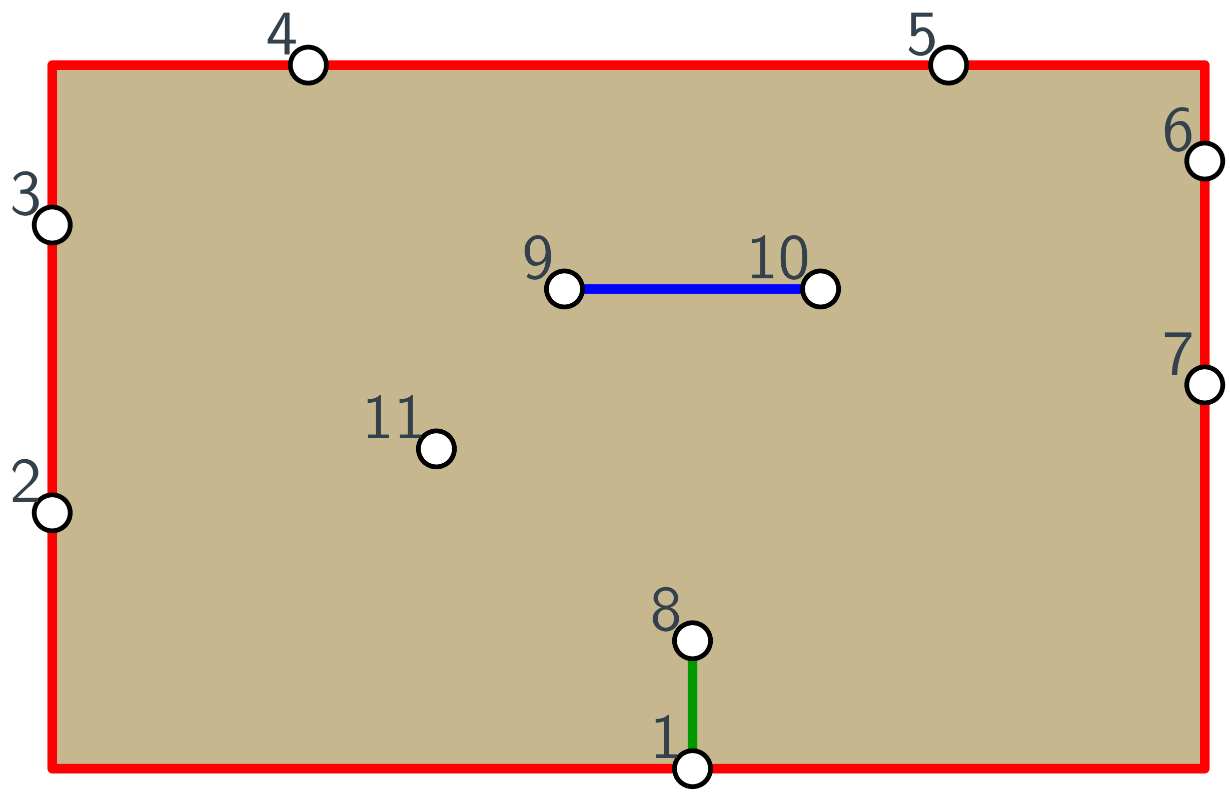

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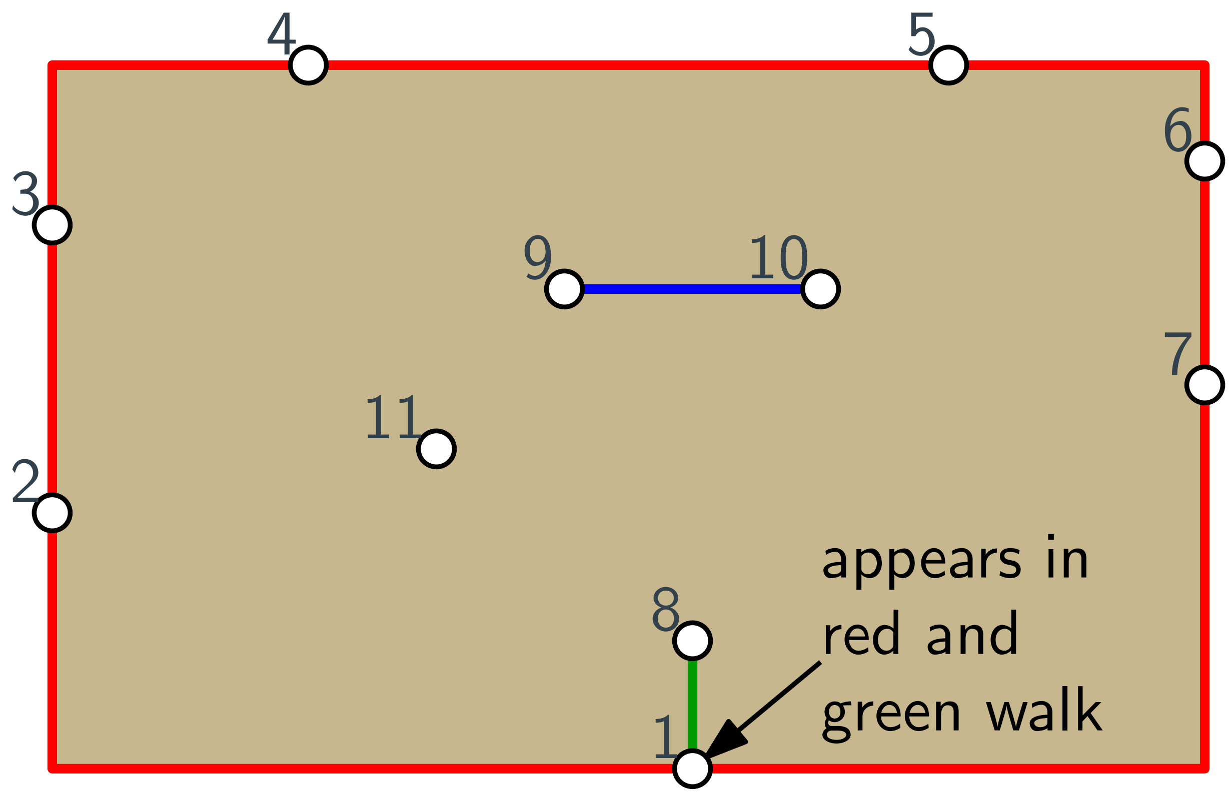

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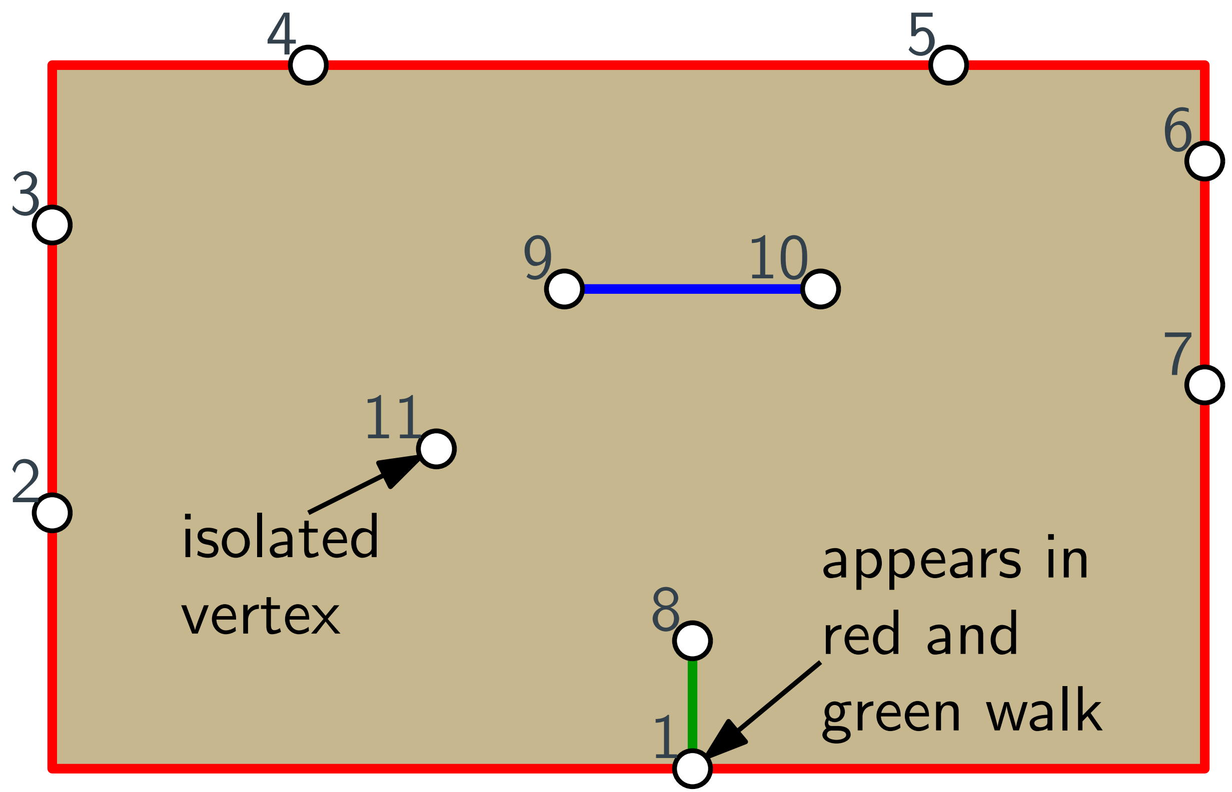

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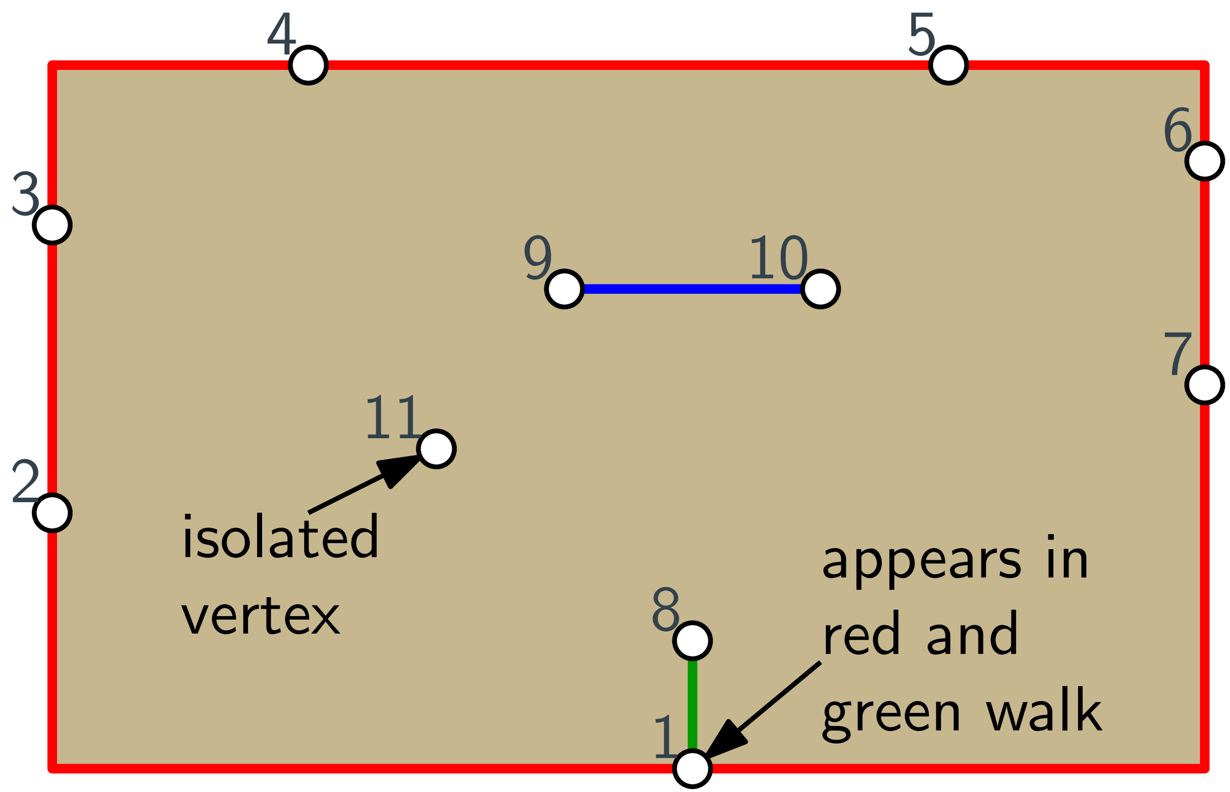

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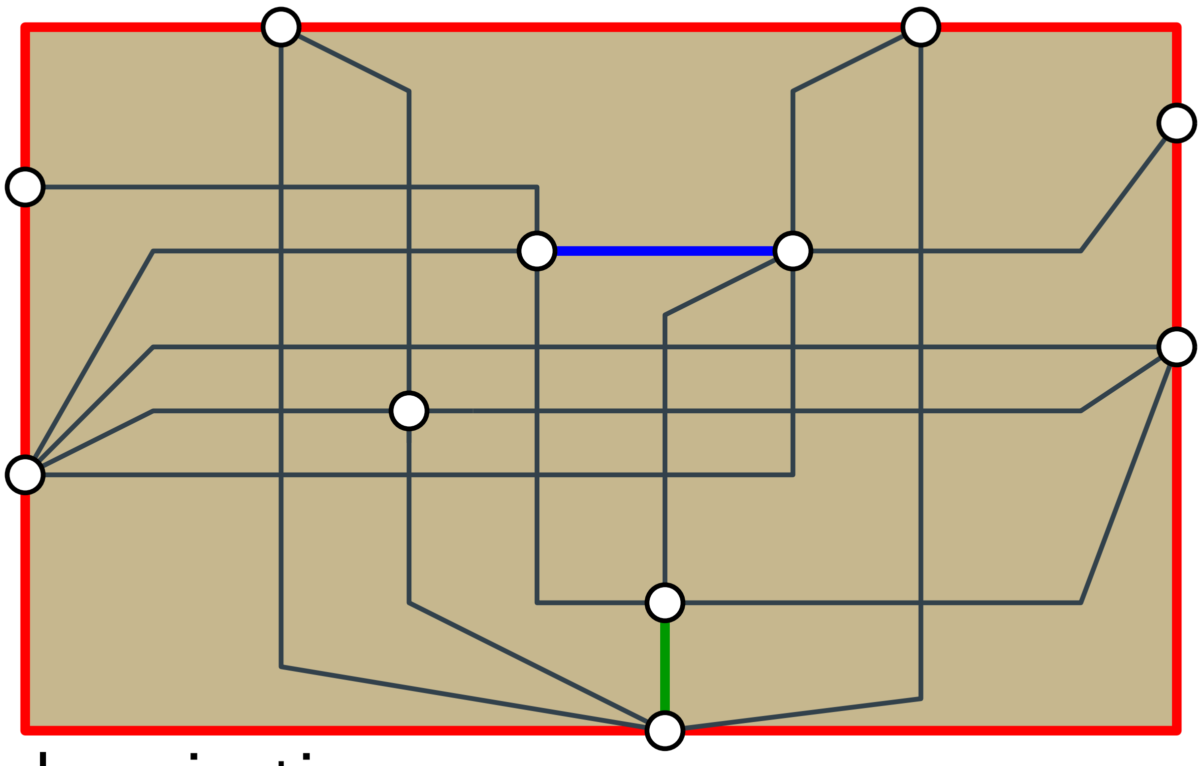
Good Faces



- Consider planarization

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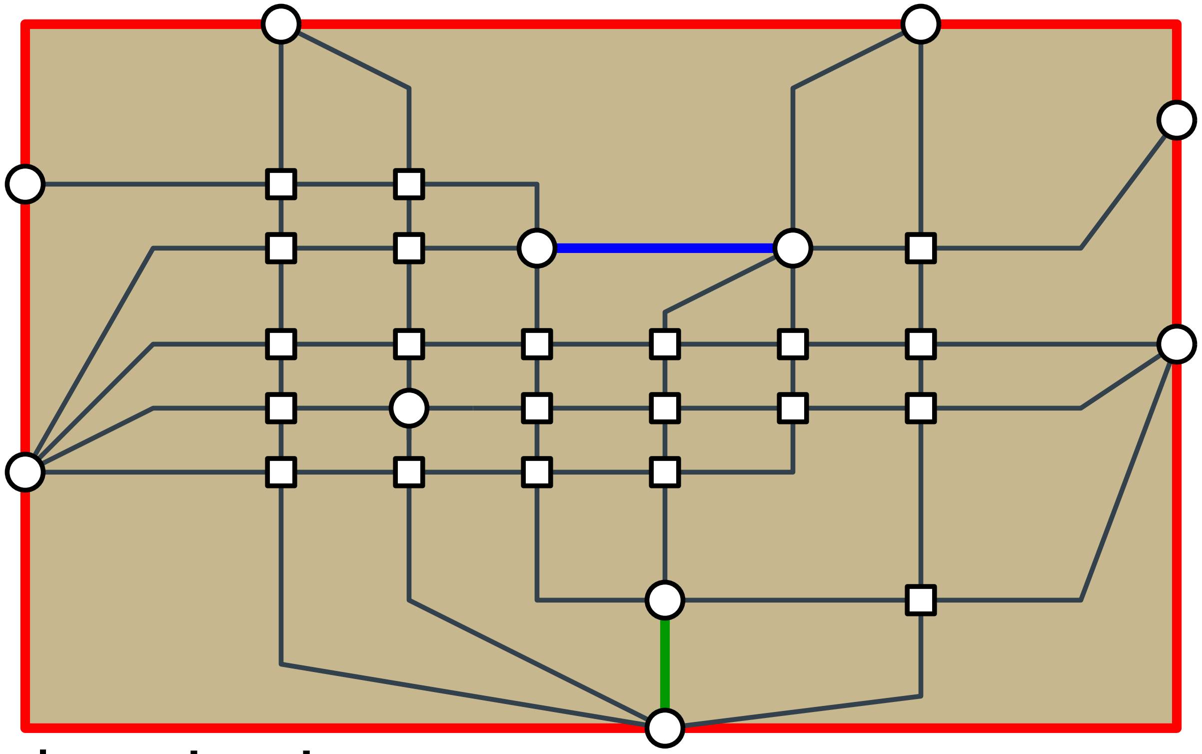

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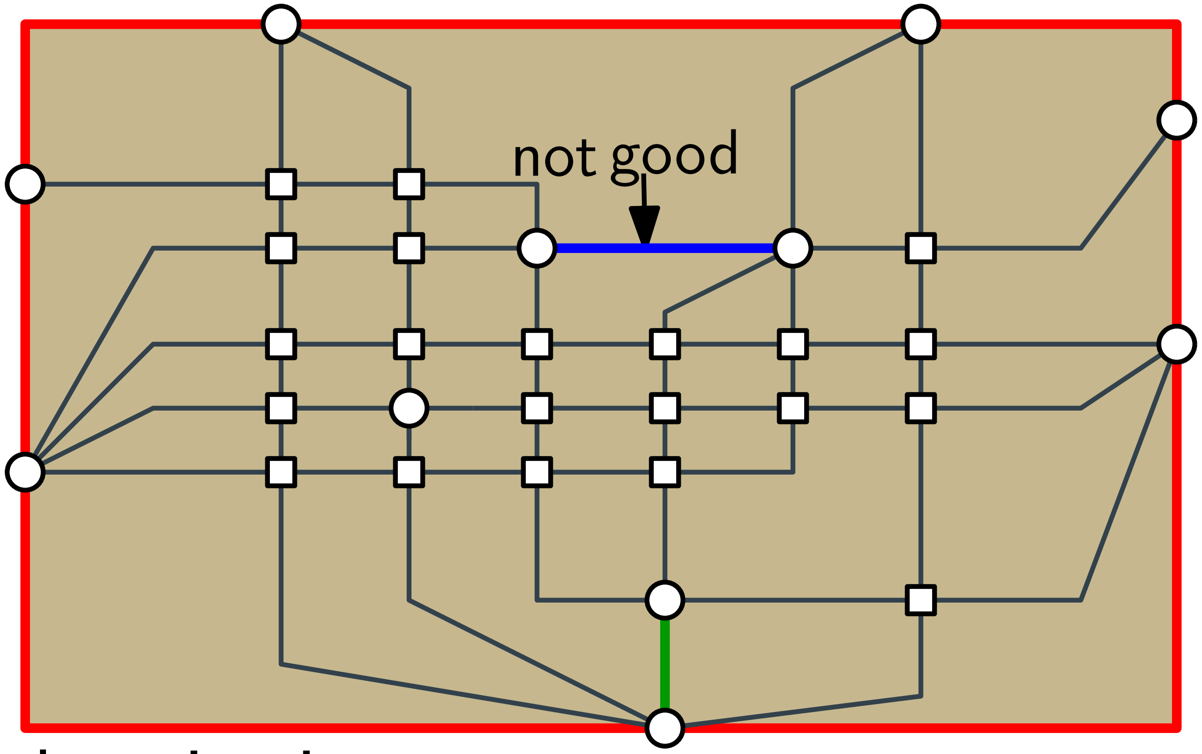

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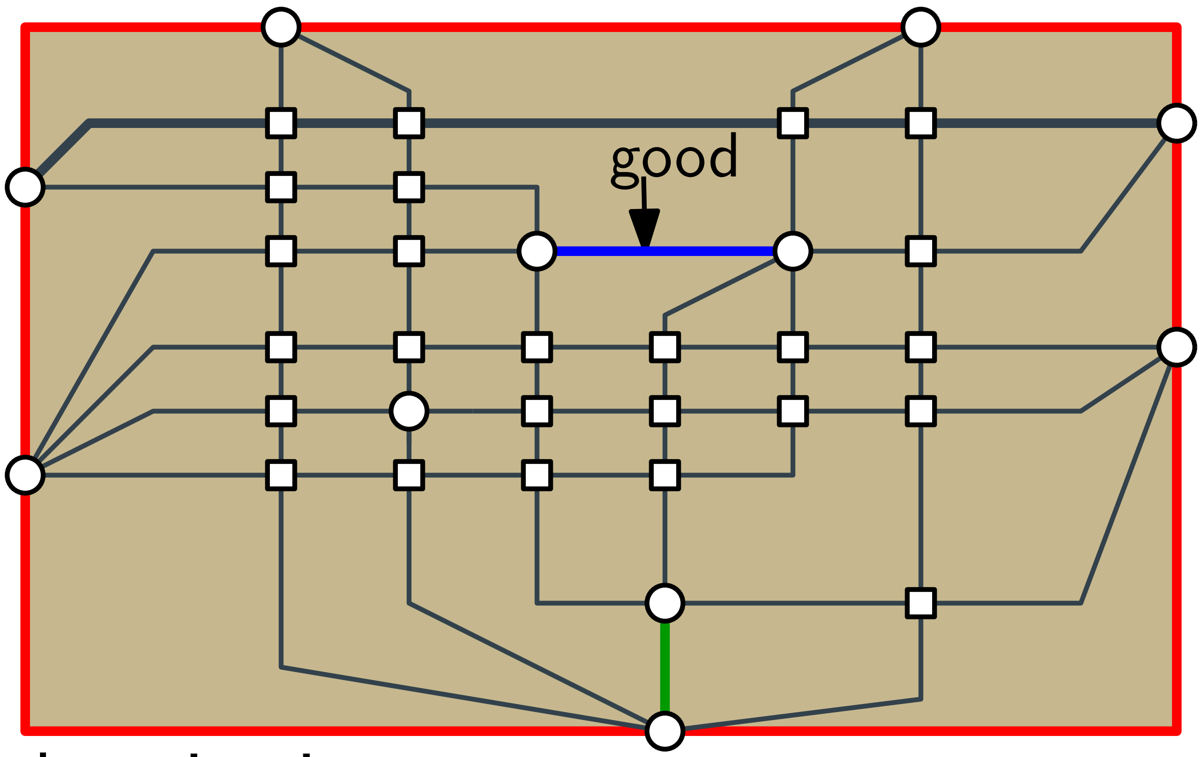

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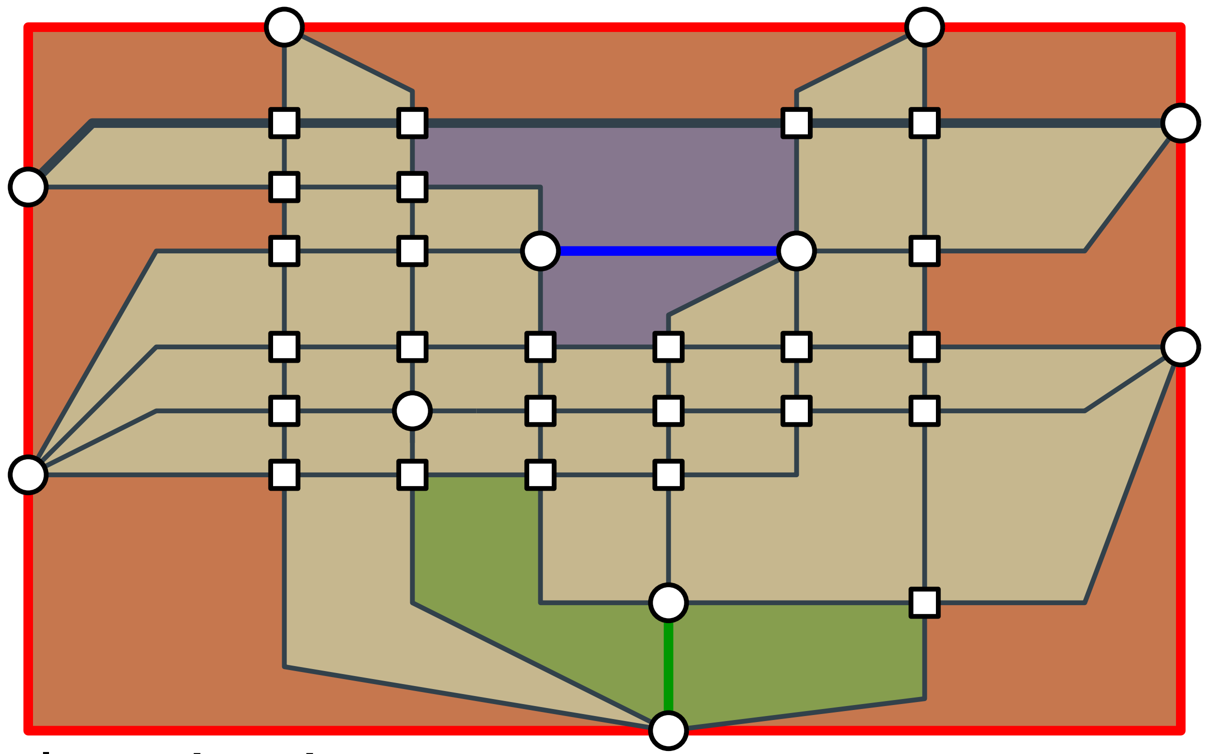

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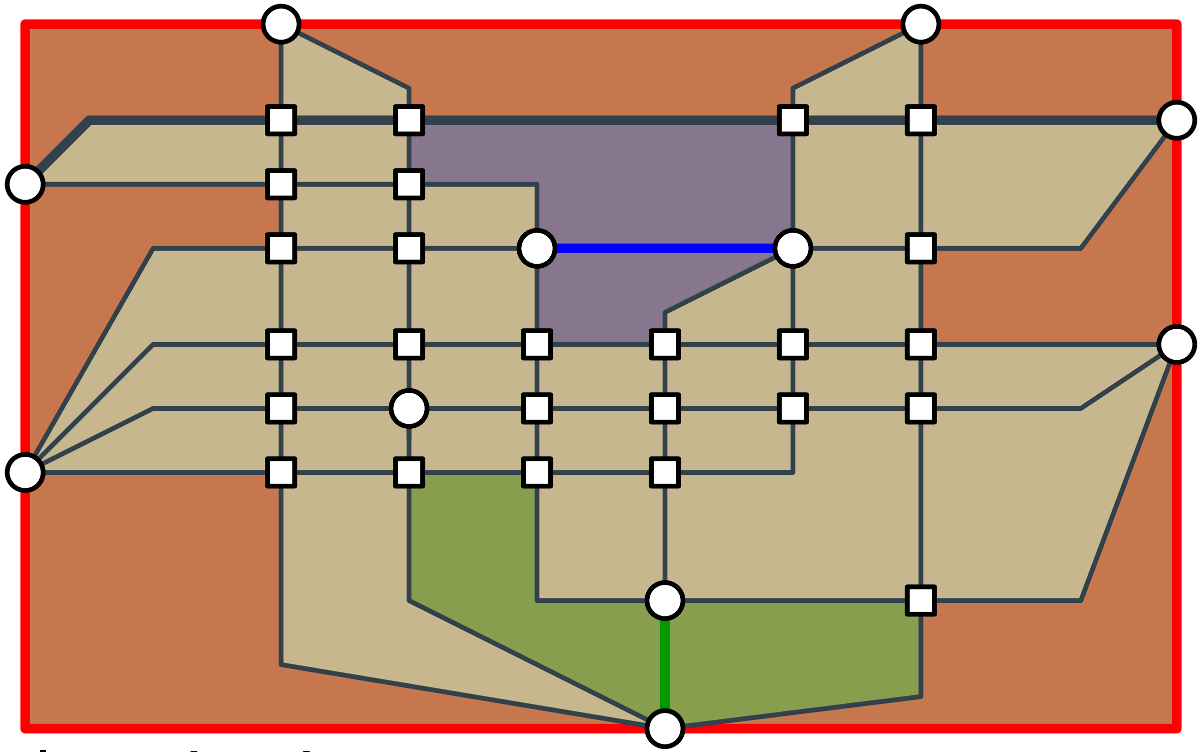

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  - These faces have at least $2\ell(\mathcal{F}) - 4b(\mathcal{F})$ initial charge (Recall: $ch(f) = \ell(f) - 4$)

Number of Intersected Edges in a Good Face

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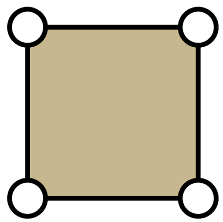

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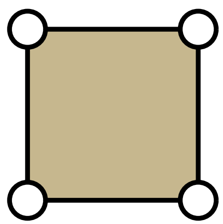

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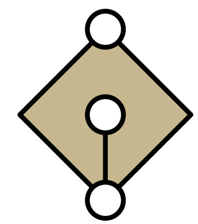

1 edges missing to K_3

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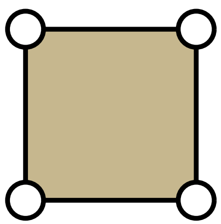
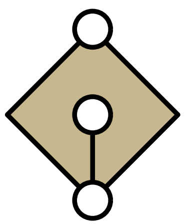


2 edges missing to K_4

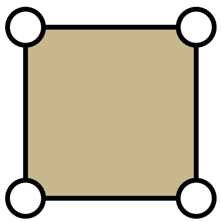


1 edges missing to K_3

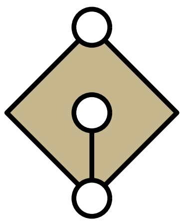
  - two edges:

Small Faces

- K_n contains $\binom{n}{2}$ edges
- All bounded planar faces $\mathcal{F}$ which can be triangulated with
  - one edge:

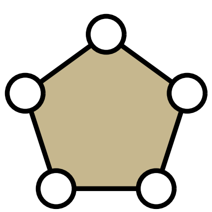


2 edges missing to K_4



1 edges missing to K_3

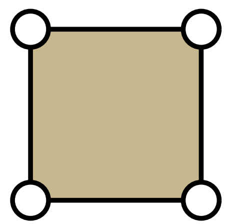
  - two edges:



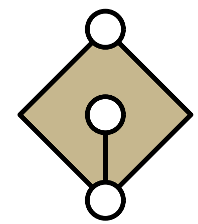
5 edges missing to K_5

Small Faces

- K_n contains $\binom{n}{2}$ edges
- All bounded planar faces $\mathcal{F}$ which can be triangulated with
  - one edge:

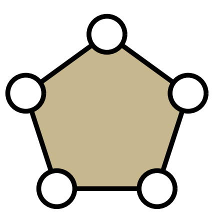


2 edges missing to K_4

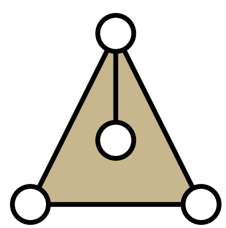


1 edges missing to K_3

  - two edges:



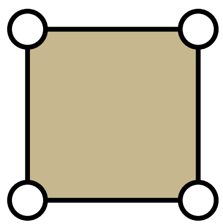
5 edges missing to K_5



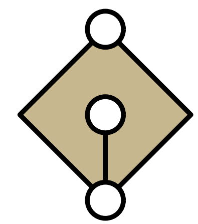
2 edges missing to K_4

Small Faces

- K_n contains $\binom{n}{2}$ edges
- All bounded planar faces $\mathcal{F}$ which can be triangulated with
  - one edge:

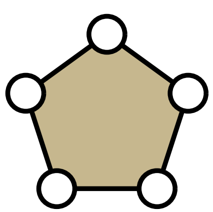


2 edges missing to K_4

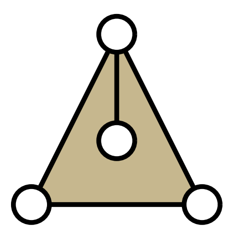


1 edges missing to K_3

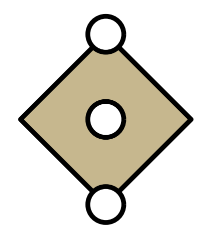
  - two edges:



5 edges missing to K_5



2 edges missing to K_4



2 edges missing to K_3

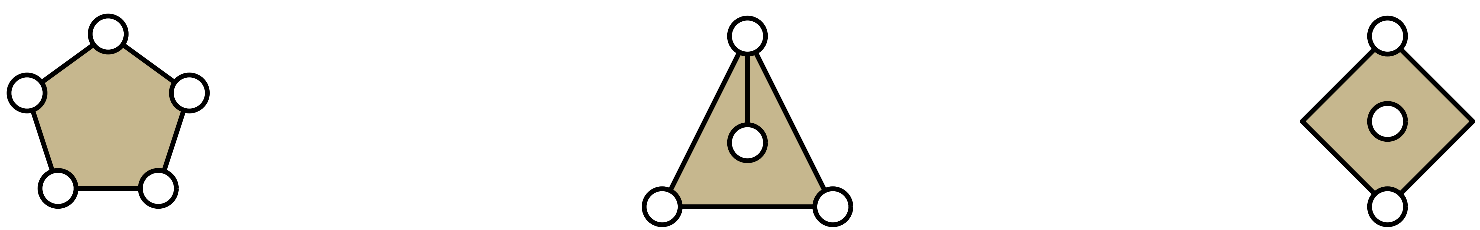
Small Faces

- K_n contains $\binom{n}{2}$ edges
- All bounded planar faces $\mathcal{F}$ which can be triangulated with
  - one edge:

2 edges missing to K_4 1 edges missing to K_3

We assume 8/3 intersected edges here.

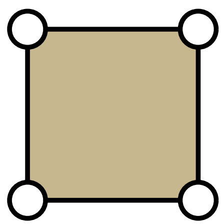
  - two edges:

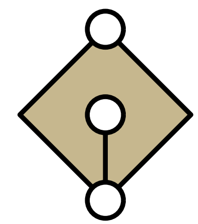


5 edges missing to K_5 2 edges missing to K_4 2 edges missing to K_3

Small Faces

- K_n contains $\binom{n}{2}$ edges
- All bounded planar faces $\mathcal{F}$ which can be triangulated with
  - one edge:

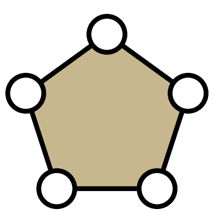


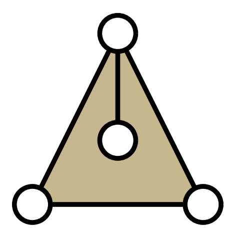


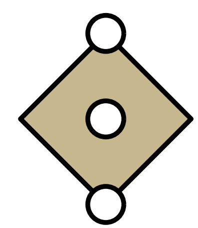
2 edges missing to K_4 1 edges missing to K_3

We assume $8/3$ intersected edges here.

  - two edges:







5 edges missing to K_5 2 edges missing to K_4 2 edges missing to K_3

We assume $16/3$ intersected edges here.

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

- Now: Assume that G_0 is obtained from triangulation T

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

- Now: Assume that G_0 is obtained from triangulation T
  - Induction: removal of k edges from $T \Rightarrow$ at most $8/3k$ intersected edges

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

- Now: Assume that G_0 is obtained from triangulation T
  - Induction: removal of k edges from $T \Rightarrow$ at most $8/3k$ intersected edges
  - Actually, we show that $|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}')$ if $\mathcal{F}'$ can be triangulated with $t(\mathcal{F}')$ edges

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

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  - Actually, we show that $|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}')$ if $\mathcal{F}'$ can be triangulated with $t(\mathcal{F}')$ edges
  - Remove edges based on a BFS traversal of the dual

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

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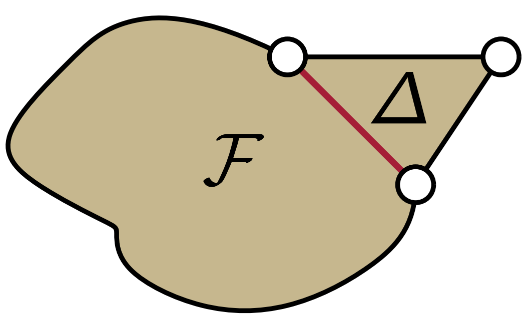

Improving the Upper Bound

- So far we know:

$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

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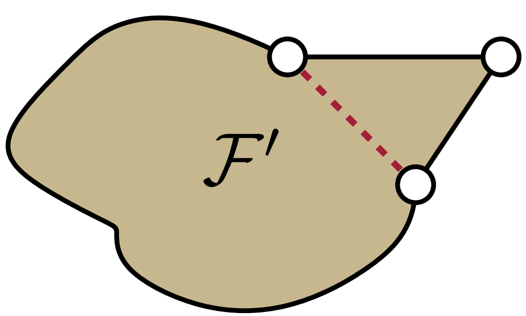

Improving the Upper Bound

- So far we know:

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- Now: Assume that G_0 is obtained from triangulation T
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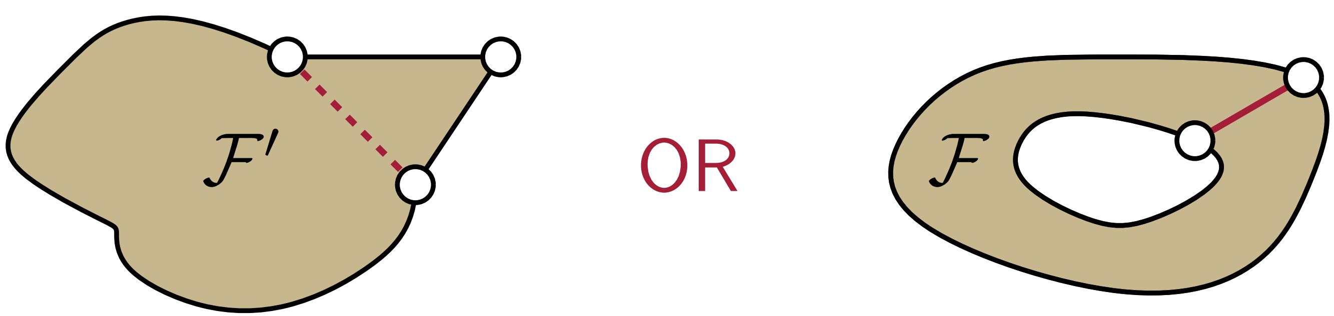
  - Remove edges based on a BFS traversal of the dual



Improving the Upper Bound

- So far we know:

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- Now: Assume that G_0 is obtained from triangulation T
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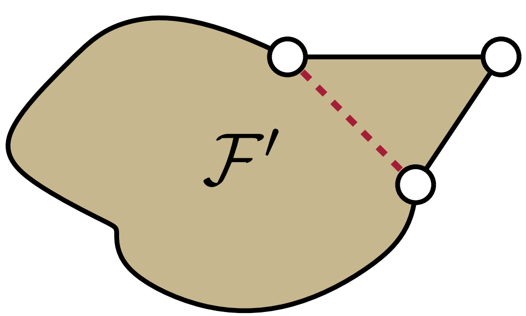
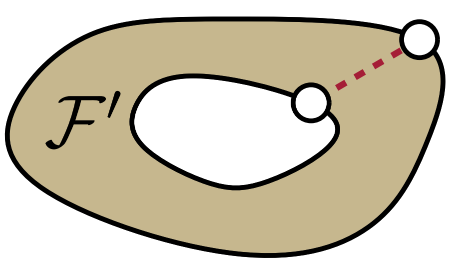
  - Remove edges based on a BFS traversal of the dual



OR



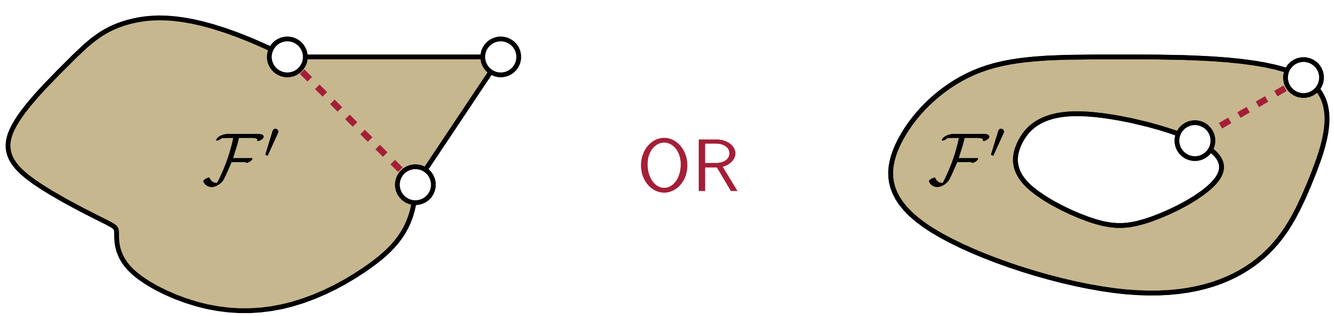
Improving the Upper Bound

Done for $t(\mathcal{F}') \in \{0, 1, 2\}$

- So far we know:

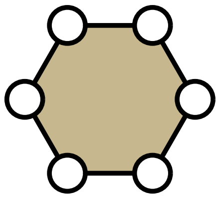
$$|E_1| \leq 8/3|\mathcal{F}_1| + 16/3|\mathcal{F}_2| + \sum_{\mathcal{F} \in \mathcal{F}_{3+}} 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8$$

- Now: Assume that G_0 is obtained from triangulation T
  - Induction: removal of k edges from $T \Rightarrow$ at most $8/3k$ intersected edges
  - Actually, we show that $|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}')$ if $\mathcal{F}'$ can be triangulated with $t(\mathcal{F}')$ edges
  - Remove edges based on a BFS traversal of the dual



$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

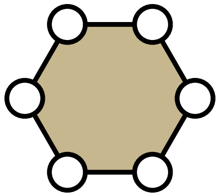
$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



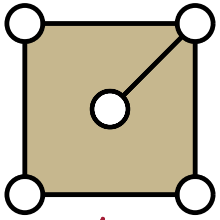
$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$

$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



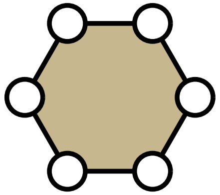
$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$



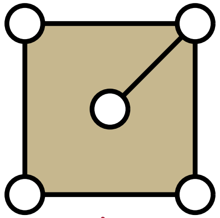
$d(\mathcal{F}') = 5$
$m(\mathcal{F}') = 1$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 2$
$|E_1(\mathcal{F}')| \leq 8$

$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

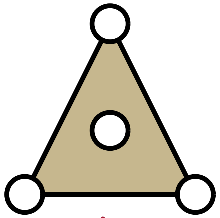
$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$



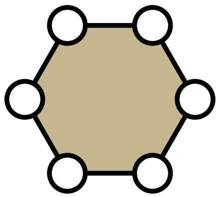
$d(\mathcal{F}') = 5$
$m(\mathcal{F}') = 1$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 2$
$|E_1(\mathcal{F}')| \leq 8$



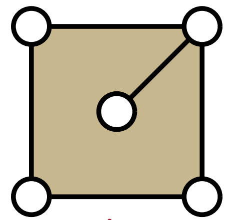
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 1$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 6$

$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

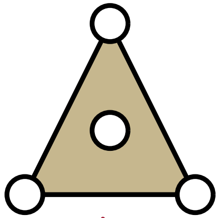
$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



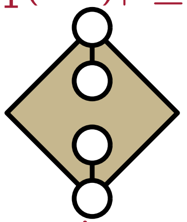
$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$



$d(\mathcal{F}') = 5$
$m(\mathcal{F}') = 1$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 2$
$|E_1(\mathcal{F}')| \leq 8$



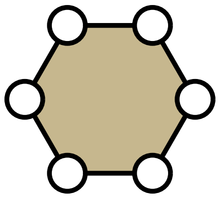
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 1$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 6$



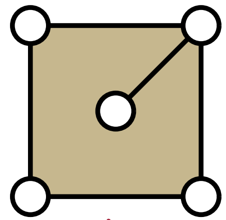
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$

$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

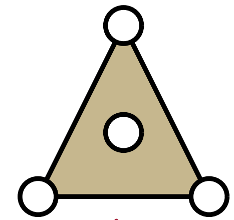
$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



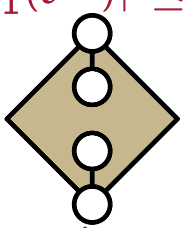
$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$



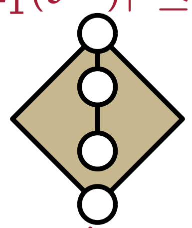
$d(\mathcal{F}') = 5$
$m(\mathcal{F}') = 1$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 2$
$|E_1(\mathcal{F}')| \leq 8$



$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 1$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 6$



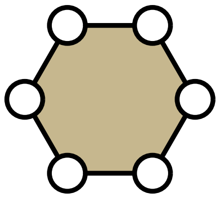
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$



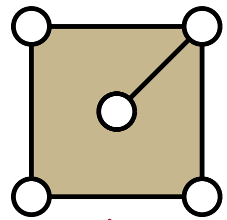
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$

$t(\mathcal{F}') = 3 \Rightarrow |E_1(\mathcal{F}')| \leq 8$

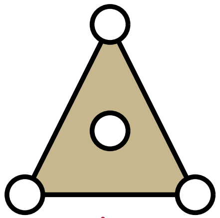
$$|E_1(\mathcal{F}')| \leq 2d(\mathcal{F}') - 2m(\mathcal{F}') + 2i(\mathcal{F}') + 4b(\mathcal{F}') - 8$$



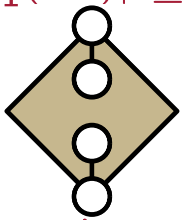
$d(\mathcal{F}') = 6$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 8$



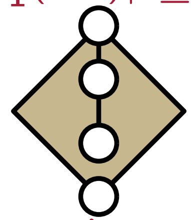
$d(\mathcal{F}') = 5$
$m(\mathcal{F}') = 1$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 2$
$|E_1(\mathcal{F}')| \leq 8$



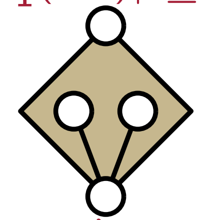
$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 0$
$i(\mathcal{F}') = 1$
$b(\mathcal{F}') = 1$
$|E_1(\mathcal{F}')| \leq 6$



$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$



$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$



$d(\mathcal{F}') = 4$
$m(\mathcal{F}') = 2$
$i(\mathcal{F}') = 0$
$b(\mathcal{F}') = 3$
$|E_1(\mathcal{F}')| \leq 8$

$t(\mathcal{F}') > 3$

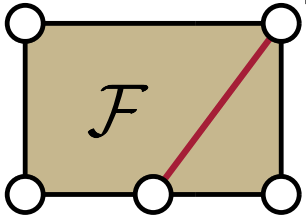
- Induction hypothesis:
  $$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:

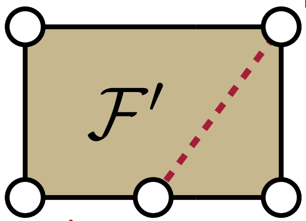


$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



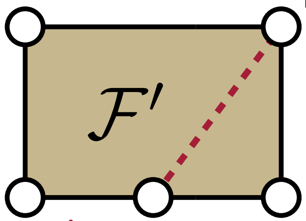
$$d(\mathcal{F}') = d(\mathcal{F}) + 1$$
$$m(\mathcal{F}') = m(\mathcal{F})$$
$$i(\mathcal{F}') = i(\mathcal{F})$$
$$b(\mathcal{F}') = b(\mathcal{F})$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$$

$t(\mathcal{F}') > 3$

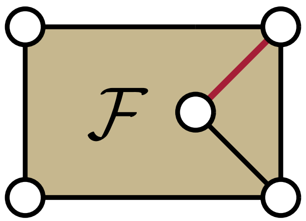
- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$d(\mathcal{F}') = d(\mathcal{F}) + 1$
$m(\mathcal{F}') = m(\mathcal{F})$
$i(\mathcal{F}') = i(\mathcal{F})$
$b(\mathcal{F}') = b(\mathcal{F})$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$

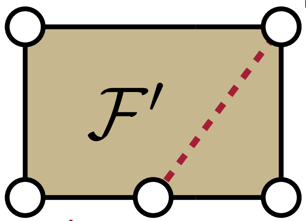


$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$$d(\mathcal{F}') = d(\mathcal{F}) + 1$$
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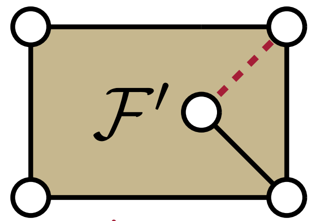
$$i(\mathcal{F}') = i(\mathcal{F})$$
$$b(\mathcal{F}') = b(\mathcal{F})$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$$



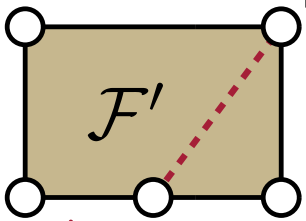
$$d(\mathcal{F}') = d(\mathcal{F})$$
$$m(\mathcal{F}') = m(\mathcal{F}) + 1$$
$$i(\mathcal{F}') = i(\mathcal{F})$$
$$b(\mathcal{F}') = b(\mathcal{F}) + 1$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$$

$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



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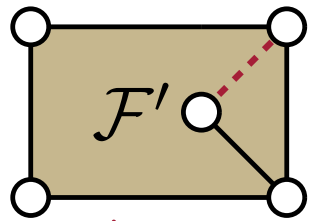

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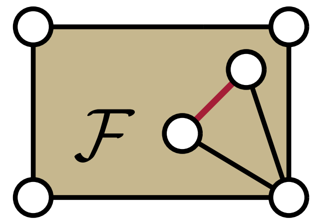
$$i(\mathcal{F}') = i(\mathcal{F})$$
$$b(\mathcal{F}') = b(\mathcal{F}) + 1$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$$

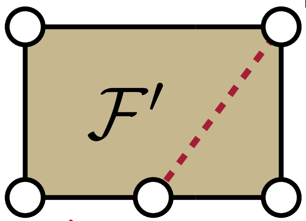


$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$d(\mathcal{F}') = d(\mathcal{F}) + 1$
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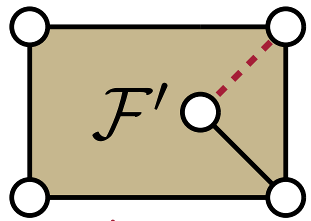
$b(\mathcal{F}') = b(\mathcal{F})$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$



$d(\mathcal{F}') = d(\mathcal{F})$
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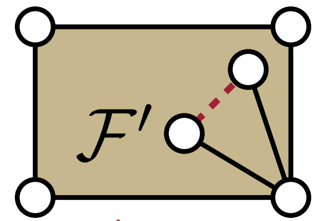

$d(\mathcal{F}') = d(\mathcal{F})$
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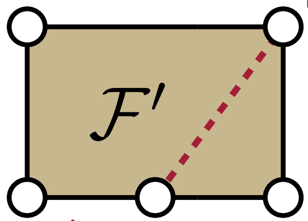
$b(\mathcal{F}') = b(\mathcal{F}) + 1$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$

$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$d(\mathcal{F}') = d(\mathcal{F}) + 1$
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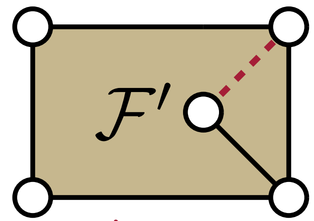

$d(\mathcal{F}') = d(\mathcal{F})$
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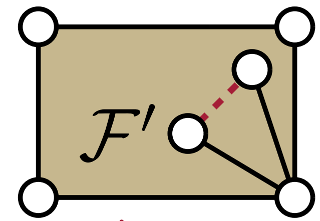

$d(\mathcal{F}') = d(\mathcal{F})$
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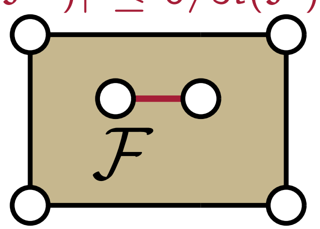
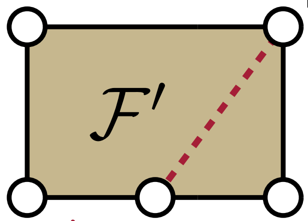


$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$$d(\mathcal{F}') = d(\mathcal{F}) + 1$$
$$m(\mathcal{F}') = m(\mathcal{F})$$
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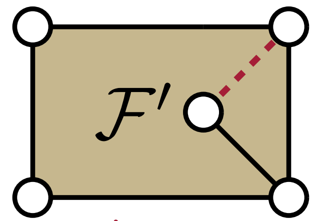

$$d(\mathcal{F}') = d(\mathcal{F})$$
$$m(\mathcal{F}') = m(\mathcal{F}) + 1$$
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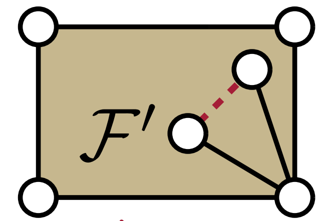
$$b(\mathcal{F}') = b(\mathcal{F}) + 1$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$$



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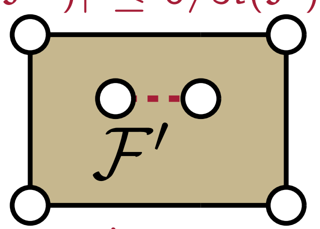


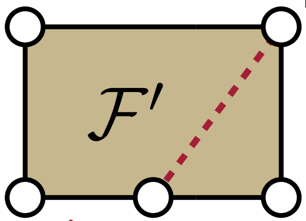
$$d(\mathcal{F}') = d(\mathcal{F})$$
$$m(\mathcal{F}') = m(\mathcal{F})$$
$$i(\mathcal{F}') = i(\mathcal{F}) + 2$$
$$b(\mathcal{F}') = b(\mathcal{F}) - 1$$
$$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F})$$

$t(\mathcal{F}') > 3$

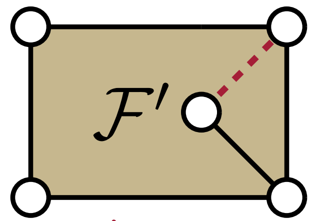
- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



$d(\mathcal{F}') = d(\mathcal{F}) + 1$
$m(\mathcal{F}') = m(\mathcal{F})$
$i(\mathcal{F}') = i(\mathcal{F})$
$b(\mathcal{F}') = b(\mathcal{F})$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$



$d(\mathcal{F}') = d(\mathcal{F})$
$m(\mathcal{F}') = m(\mathcal{F}) + 1$
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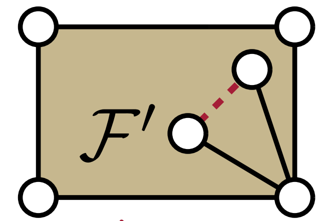

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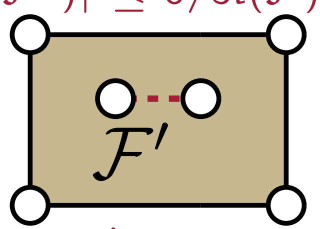

$d(\mathcal{F}') = d(\mathcal{F})$
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$i(\mathcal{F}') = i(\mathcal{F}) + 2$
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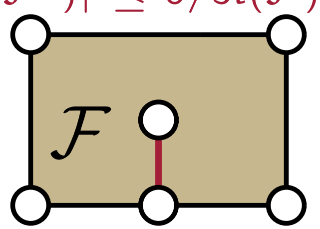
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F})$

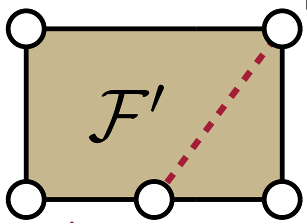


$t(\mathcal{F}') > 3$

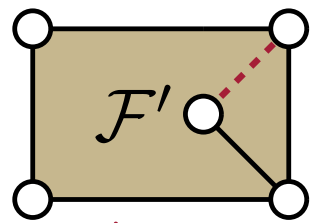
- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

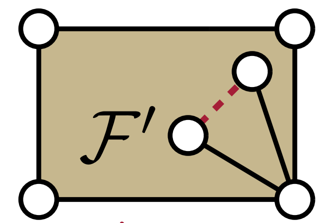
- Induction step:



$d(\mathcal{F}') = d(\mathcal{F}) + 1$
$m(\mathcal{F}') = m(\mathcal{F})$
$i(\mathcal{F}') = i(\mathcal{F})$
$b(\mathcal{F}') = b(\mathcal{F})$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$



$d(\mathcal{F}') = d(\mathcal{F})$
$m(\mathcal{F}') = m(\mathcal{F}) + 1$
$i(\mathcal{F}') = i(\mathcal{F})$
$b(\mathcal{F}') = b(\mathcal{F}) + 1$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F}) + 2$



$d(\mathcal{F}') = d(\mathcal{F})$
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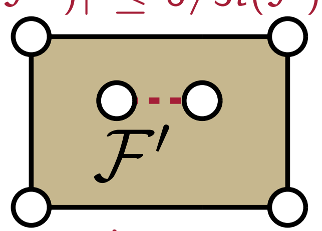
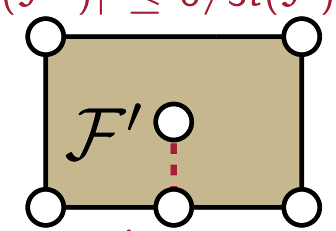


$d(\mathcal{F}') = d(\mathcal{F})$
$m(\mathcal{F}') = m(\mathcal{F})$
$i(\mathcal{F}') = i(\mathcal{F}) + 2$
$b(\mathcal{F}') = b(\mathcal{F}) - 1$
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F})$



$d(\mathcal{F}') = d(\mathcal{F})$
$m(\mathcal{F}') = m(\mathcal{F}) - 1$
$i(\mathcal{F}') = i(\mathcal{F}) + 1$
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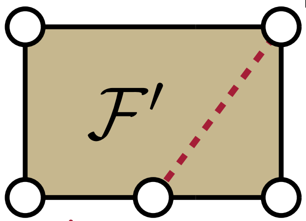
$|E_1(\mathcal{F}')| \leq 8/3t(\mathcal{F})$

$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



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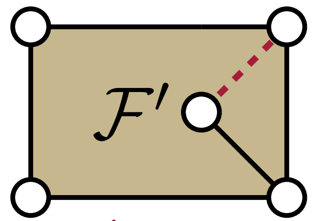

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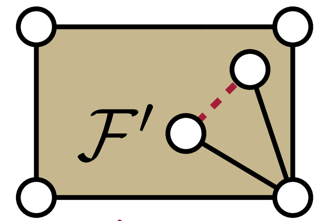

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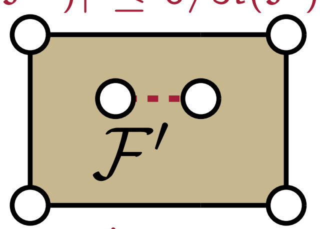

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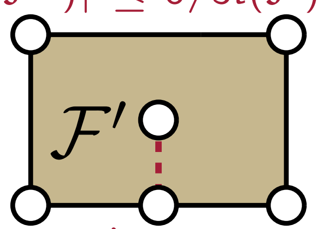

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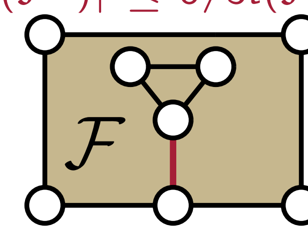
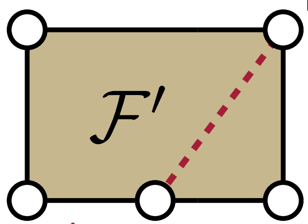


$t(\mathcal{F}') > 3$

- Induction hypothesis:

$$|E_1(\mathcal{F})| \leq 2d(\mathcal{F}) - 2m(\mathcal{F}) + 2i(\mathcal{F}) + 4b(\mathcal{F}) - 8 \leq 8/3t(\mathcal{F})$$

- Induction step:



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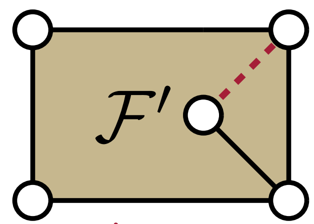

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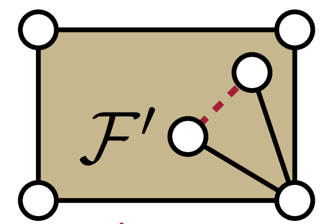

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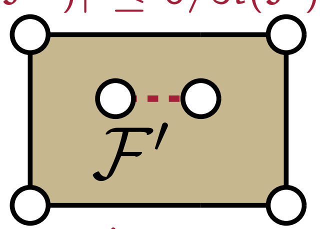

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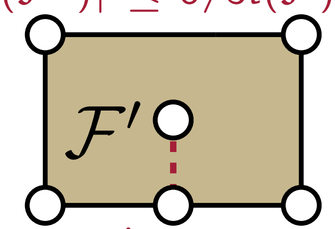

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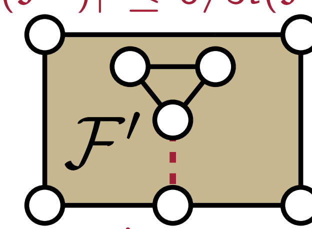

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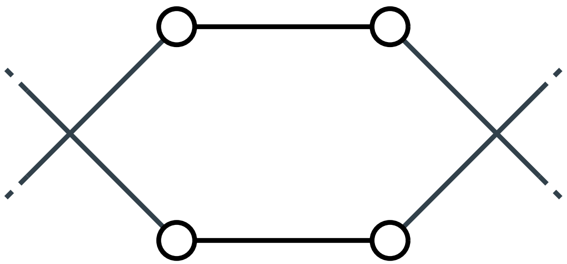

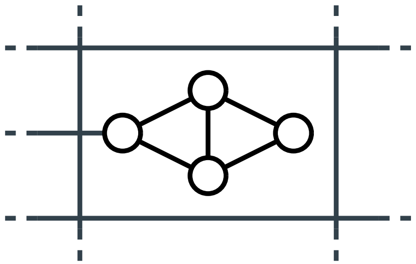

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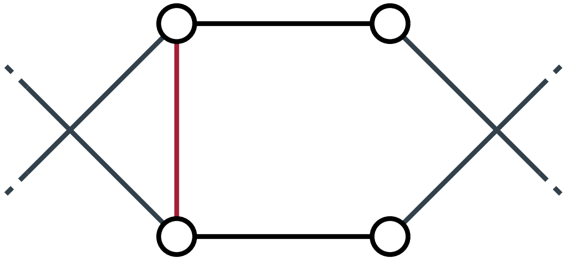

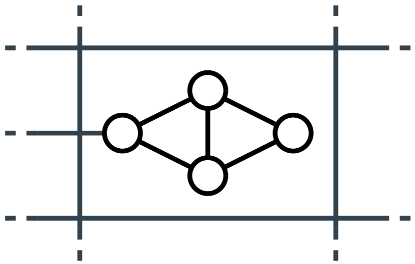

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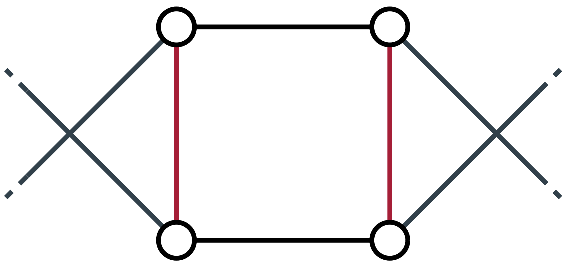

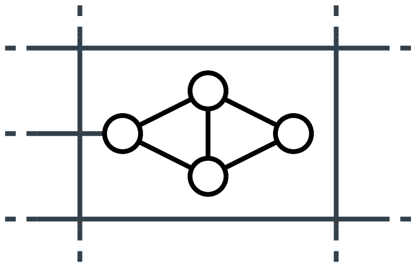

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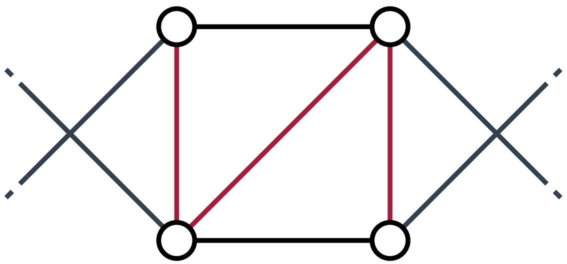

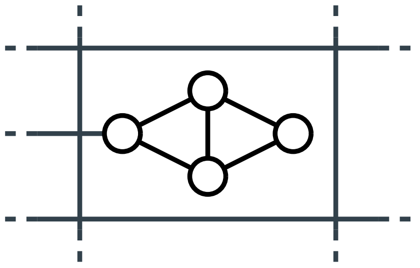

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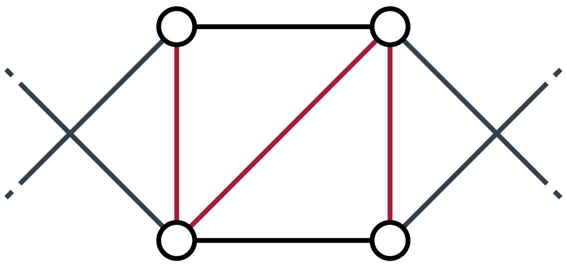

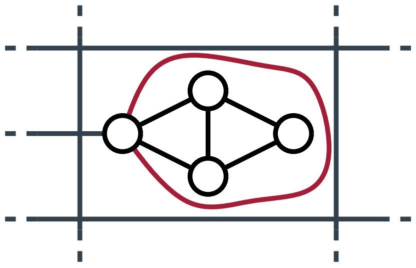

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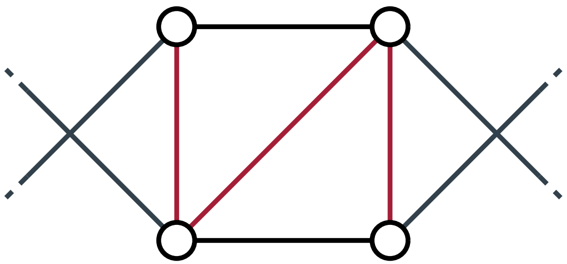

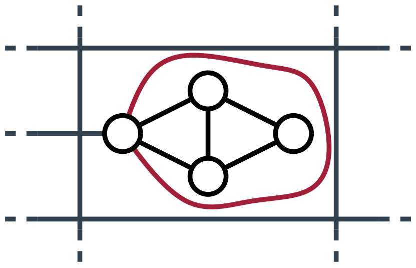

- Layout subgraphs separated by selfloops individually

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Thank you for your attention!