

# Upward Planar Morphs



G. Da Lozzo G. Di Battista F. Frati M. Patrignani V. Roselli



GRAPH DRAWING 2018

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# What are we talking about?

Morphs

Preliminaries

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Tools

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Plane *st*-Graphs

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Upward Plane Graphs

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Upward Planar Morphs

# What are we talking about?

## Morphs

Transform one drawing into another by moving vertices by preserving at *any time* the properties of the drawings.

Preliminaries

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Upward Plane Graphs

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# What are we talking about?

## Morphs

If vertices move at *uniform speed* along straight-line trajectories,  
the morph is *linear*

# What are we talking about?

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Sometimes linear morphs do not preserve the properties of the input drawings: some intermediate *steps* are necessary

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## Linear Morphs

A *linear morph* is a morph such that vertices move at uniform speed along straight-line trajectories.

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## Linear Morphs

A *linear morph* is a morph such that vertices move at uniform speed along straight-line trajectories.

## Morphs

A *morph* is a finite sequence of linear morphs, called steps.

The *complexity* of a morphing algorithm is given by the number of intermediate morphing steps.

# Morphs

## Some literature

1914-17: Tietze, Smith, Veblen, [...]

- existential proofs for polygons

1944-83: Cairns, Thomassen, [...]

- existential proofs for triangulations and planar graphs

2006: Lubiw et al., SODA

- morphs of orthogonal drawings in a polynomial number of steps

2013: Alamdari et al., SODA

- first algorithm for morphing general planar graph drawings in polynomially many steps

2014: Angelini et al., ICALP

- optimal (linear) algorithm for planar graphs

2014: Barrera-Cruz et al., GD

- polynomial algorithm for Schnyder drawings of triangulations

2015: Angelini et al., SoCG

- optimal (linear) algorithm for convex drawings

2018: van Goethem & Verbeek, SoCG

- polynomial algorithm for orthogonal drawings

Preliminaries

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Plane *st*-Graphs

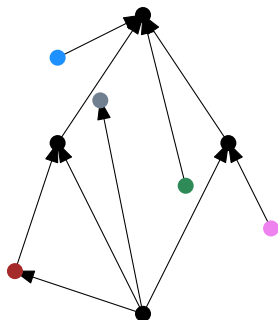
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Upward Plane Graphs

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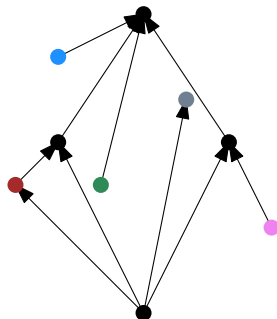
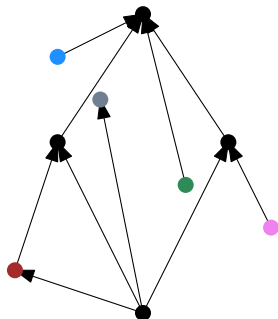
## Upward Plane Graphs



- natural extension of planarity to directed graphs
- edges drawn *upward*
- vertices with no incoming edges are *sources*
- vertices with no outgoing edges are *sinks*

# What are we talking about?

## Upward Plane Graphs



Upward Planar Morphs are defined for *upward-equivalent* drawings.  
Upward equivalence is a *necessary condition*.

# Upward Drawings

## Some literature

1988: Di Battista & Tamassia

- drawing algorithms

1992: Di Battista et al.

- area requirements

1994-98: Bertolazzi et al.

- drawing algorithms for triconnected, upward planarity test for single-source

2002: Garg & Tamassia

- upward planarity test

...

# Upward Planar Morphs

## Our results

Given two upward-equivalent planar straight-line drawings, there *always exists* a morph between them such that *all the intermediate drawings* of the morph are upward planar and straight-line.

Upward Planar Graphs	$O(n^2) - \Omega(n)$ steps
Reduced Upward Planar Graphs	$\Theta(n)$ steps
Planar st-Graphs	$O(n)$ steps
Reduced Planar st-Graphs	$\Theta(1)$ steps

# Definitions & Tools

Preliminaries

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Plane *st*-Graphs

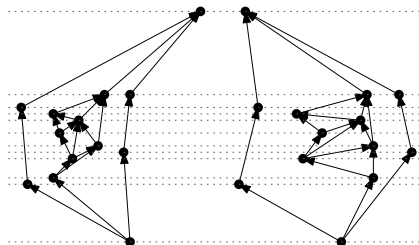
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Upward Plane Graphs

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# Left-to-Right equivalence

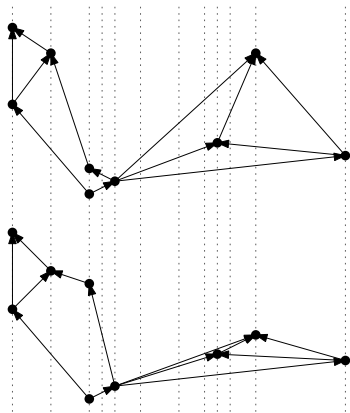
Two upward-equivalent drawings  $\Gamma_0$  and  $\Gamma_1$  of a graph are LR-equivalent if any horizontal line intersects the same sequence of vertices and/or edges in both drawings.



For any two LR-equivalent drawings of  $G$ , there exists a *1-step* upward planar morph between them.

# Bottom-to-Top equivalence

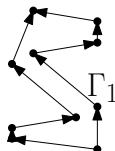
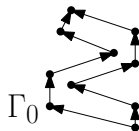
Two upward-equivalent drawings  $\Gamma_0$  and  $\Gamma_1$  of a graph are BT-equivalent if any vertical line intersects the same sequence of vertices and/or edges in both drawings.



For any two LR-equivalent drawings of  $G$ , there exists a *1-step* upward planar morph between them.

# HVH pairs

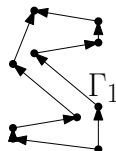
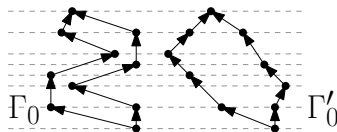
Two upward-equivalent planar drawings  $\Gamma_0$  and  $\Gamma_1$  of  $G$  are an HVH-pair if there exist  $\Gamma'_0$  and  $\Gamma'_1$  such that:



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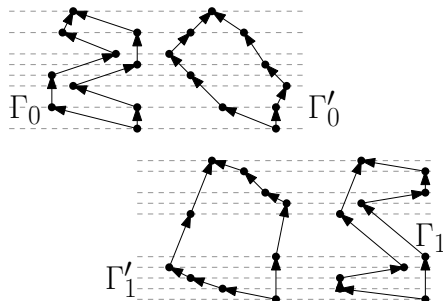
- $\Gamma_0$  and  $\Gamma'_0$  are LR-equivalent



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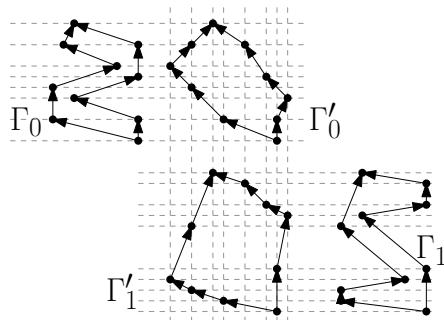
- $\Gamma_0$  and  $\Gamma'_0$  are LR-equivalent
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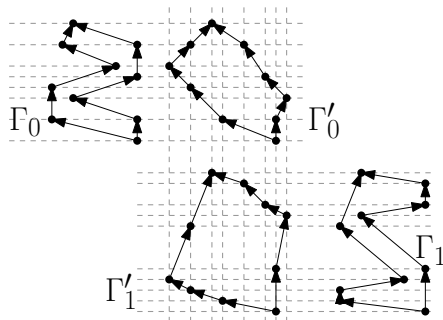
- $\Gamma_0$  and  $\Gamma'_0$  are LR-equivalent
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- $\Gamma_0$  and  $\Gamma'_0$  are LR-equivalent
- $\Gamma_1$  and  $\Gamma'_1$  are LR-equivalent
- $\Gamma'_0$  and  $\Gamma'_1$  are BT-equivalent



## Lemma

For any two upward planar drawings that form an HVH pair, there exists a **3-step** upward planar morph between them.

# Plane *st*-Graphs

Preliminaries

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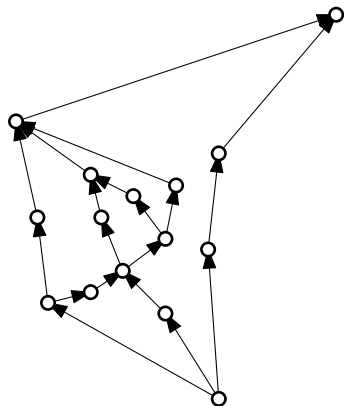
Plane *st*-Graphs

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Upward Plane Graphs

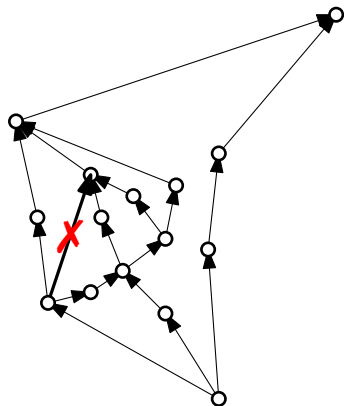
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# Reduced Plane $st$ -Graphs



- $G$  is a DAG with exactly one source and one sink, and they lie on the outer face

# Reduced Plane $st$ -Graphs

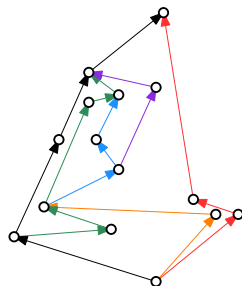
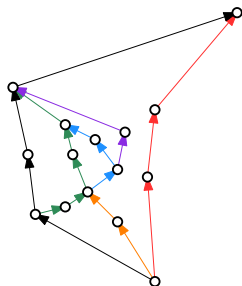


- $G$  is a DAG with exactly one source and one sink, and they lie on the outer face
- reduced = no transitive edges

# Reduced Plane $st$ -Graphs

## Theorem

For any two upward-equivalent planar drawings of a reduced  $st$ -graph  $G$ , there exists a **3-step** upward planar morph between them.



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Upward Plane Graphs  
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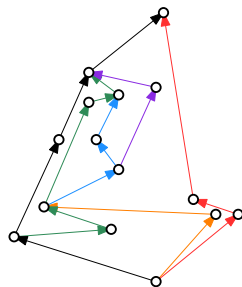
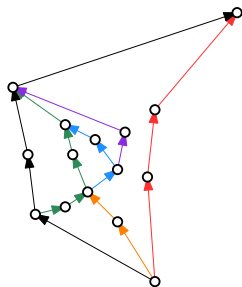
Upward Planar Morphs

# Reduced Plane $st$ -Graphs

## Theorem

For any two upward-equivalent planar drawings of a reduced  $st$ -graph  $G$ , there exists a **3-step** upward planar morph between them.

Proof: we show that they are an  $HVH$ -pair



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Plane  $st$ -Graphs  
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Upward Plane Graphs  
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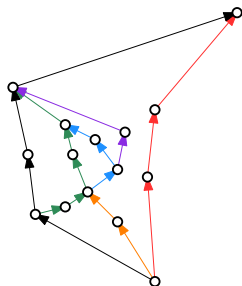
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# Reduced Plane *st*-Graphs

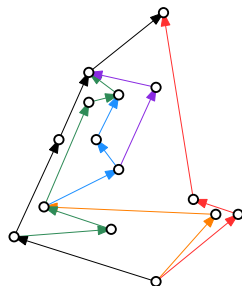
## Theorem

For any two upward-equivalent planar drawings of a reduced *st*-graph  $G$ , there exists a *3-step* upward planar morph between them.

Proof: we show that they are an *HVH*-pair



- assume  $G$  biconnected (or augment it)
- exploit an ear-decomposition for constructing  $\Gamma'_0$  and  $\Gamma'_1$



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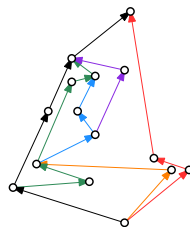
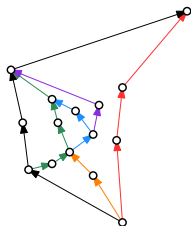
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Upward Planar Morphs

# Reduced Plane $st$ -Graphs

Construction of  $\Gamma'_0$  and  $\Gamma'_1$



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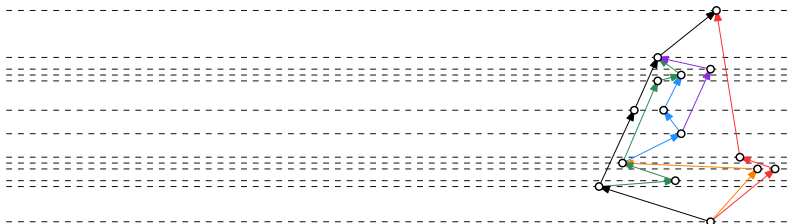
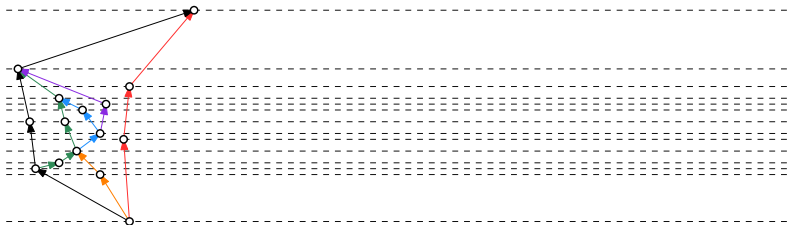
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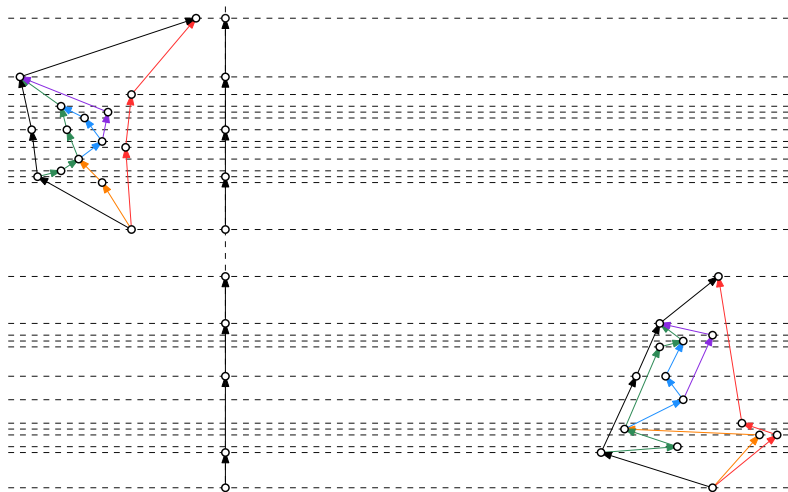
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Upward Plane Graphs

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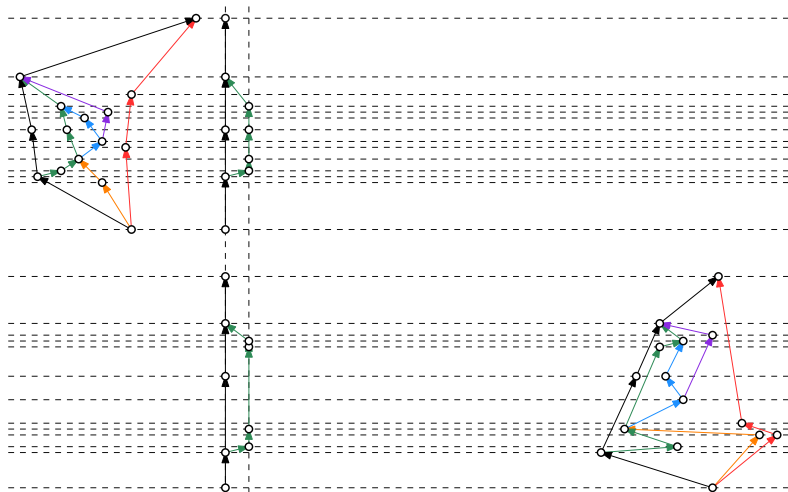
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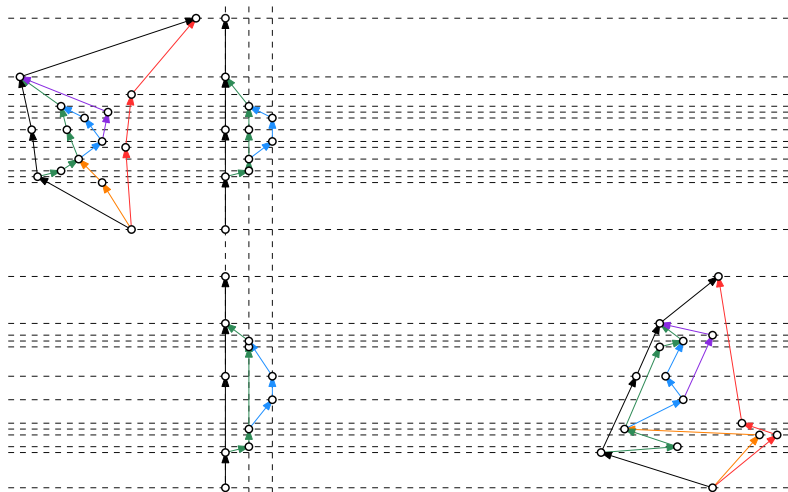
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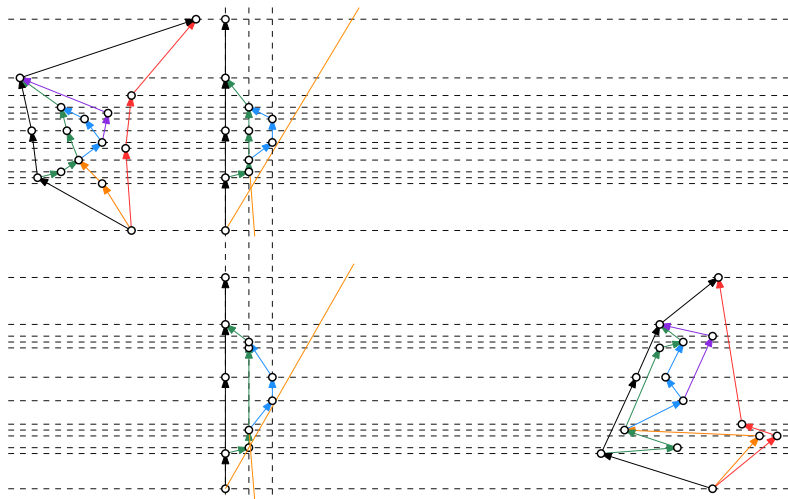
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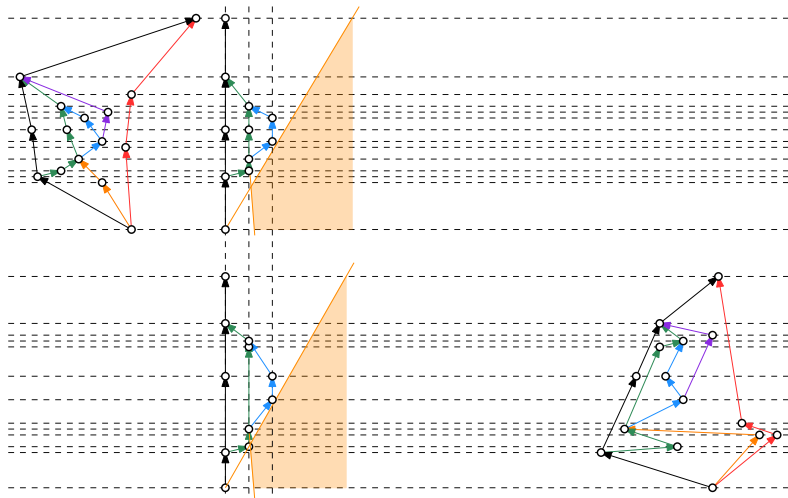
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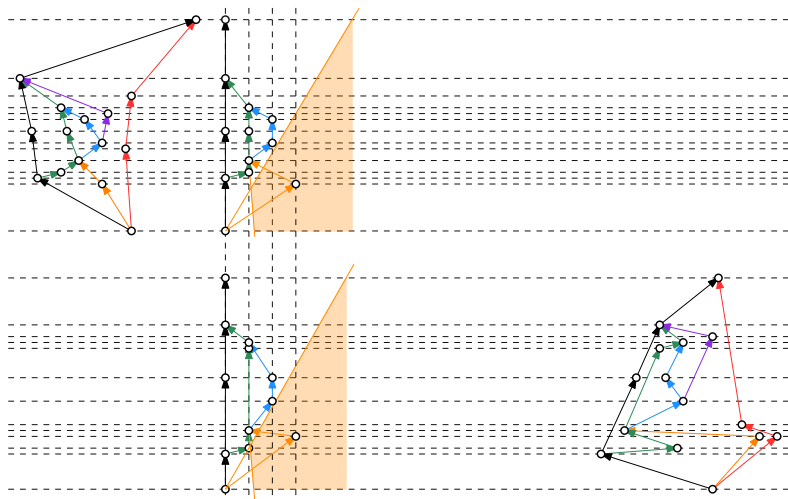
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Plane  $st$ -Graphs

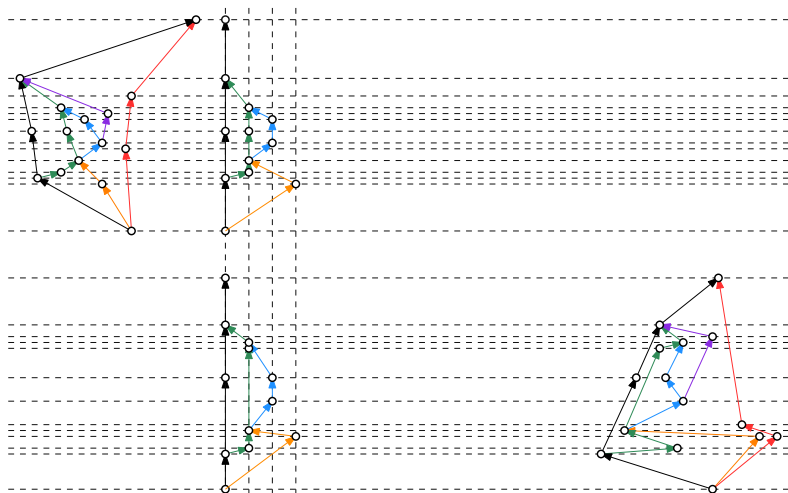
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Plane  $st$ -Graphs

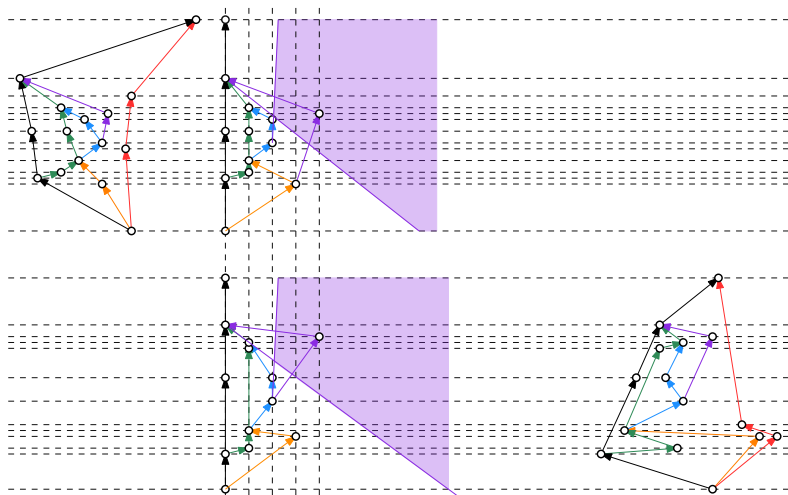
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Plane  $st$ -Graphs

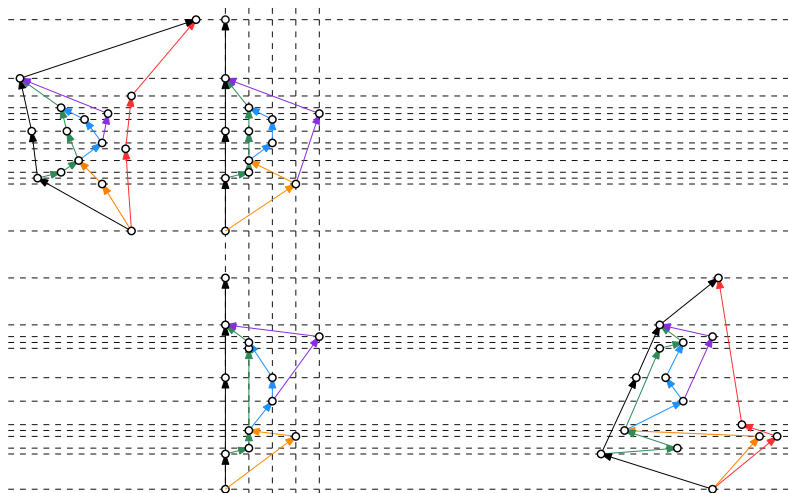
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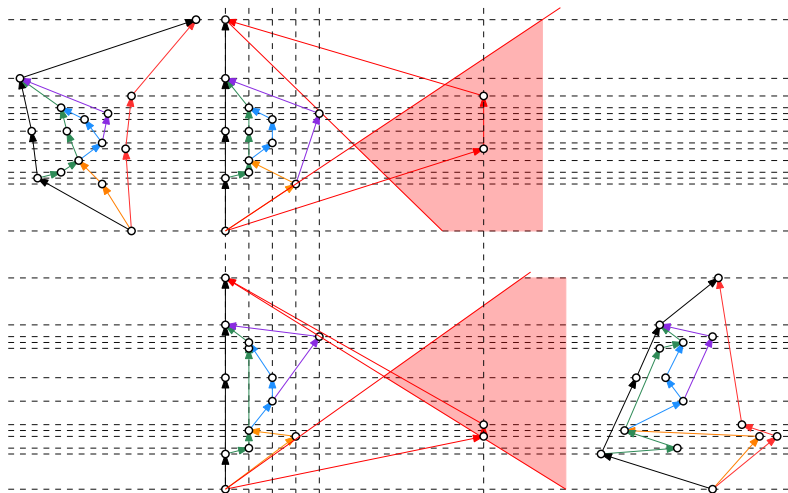
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Plane  $st$ -Graphs

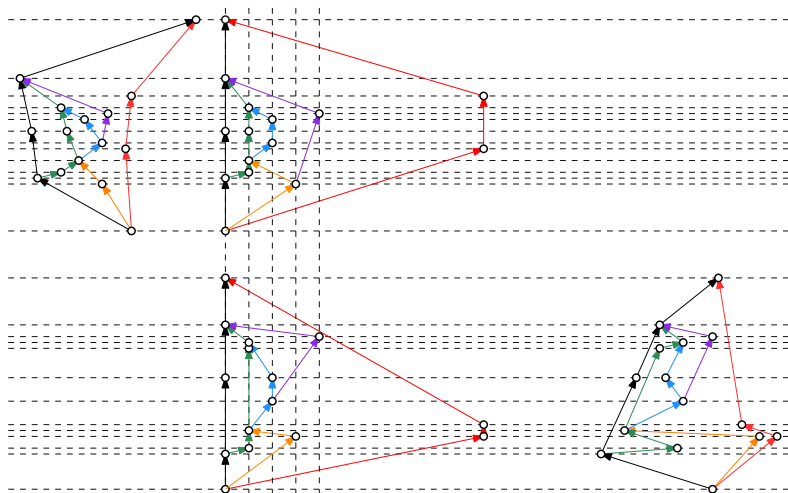
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# Reduced Plane $st$ -Graphs

Construction of  $\Gamma'_0$  and  $\Gamma'_1$



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Upward Plane Graphs

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# Reduced Plane *st*-Graphs

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Upward Plane Graphs

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# Reduced Plane $st$ -Graphs

## Theorem

For any two upward-equivalent planar drawings of an  $n$ -vertex plane  $st$ -graph  $G$ , there exists a  $O(n)$ -step upward planar morph between them.

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- augment  $G$  to be a maximal  $st$ -graph, call it  $G^+$

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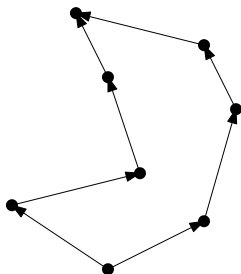
## Proof

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- compute upward planar drawings  $\Gamma_0^+$  and  $\Gamma_1^+$  of  $G^+$  LR-equivalent to  $\Gamma_0$  and  $\Gamma_1$ , respectively
- contract an internal low-degree vertex  $v$  to one of its neighbors in  $\Gamma_0^+$  and  $\Gamma_1^+$ , recursively compute a morph of the obtained drawings, add  $v$  back to the obtained morph

# Reduced Plane $st$ -Graphs

Augmentation to  $G^+$  and computation of  $\Gamma_0^+$  and  $\Gamma_1^+$

For each internal face  $f$  of  $G$ :



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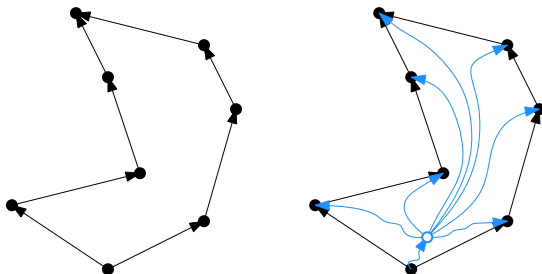
Upward Plane Graphs

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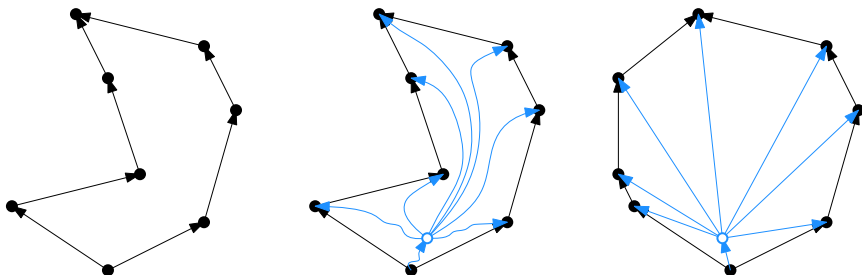
Upward Plane Graphs

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# Reduced Plane $st$ -Graphs

Augmentation to  $G^+$  and computation of  $\Gamma_0^+$  and  $\Gamma_1^+$

For each internal face  $f$  of  $G$ :



Use the algorithm for drawing hierarchical plane graphs with assigned  $y$ -coordinates by Hong & Nagamochi

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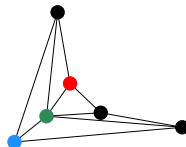
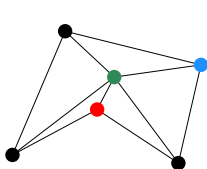
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# Reduced *st*-Graphs

How to morph maximal plane graphs

Edge contractions are widely used in morphing algorithms



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Plane *st*-Graphs

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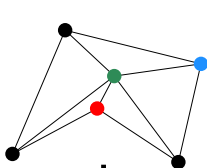
Upward Plane Graphs

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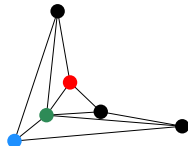
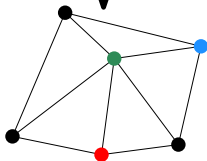
# Reduced *st*-Graphs

How to morph maximal plane graphs

Edge contractions are widely used in morphing algorithms



lin. morph



Preliminaries  
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Tools  
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Plane *st*-Graphs  
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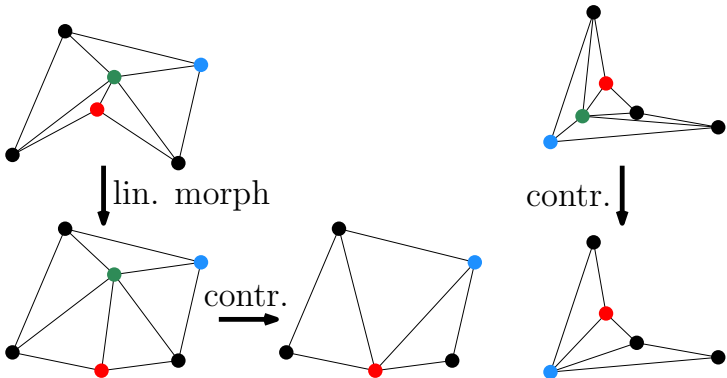
Upward Plane Graphs  
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Upward Planar Morphs

# Reduced *st*-Graphs

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Preliminaries  
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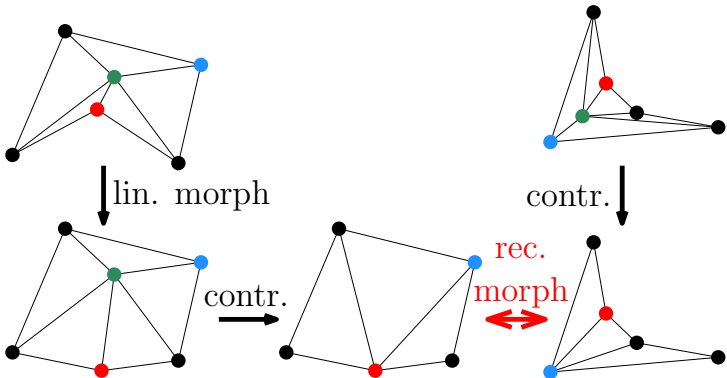
Plane *st*-Graphs  
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Upward Plane Graphs  
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# Reduced *st*-Graphs

How to morph maximal plane graphs

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Preliminaries  
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Plane *st*-Graphs  
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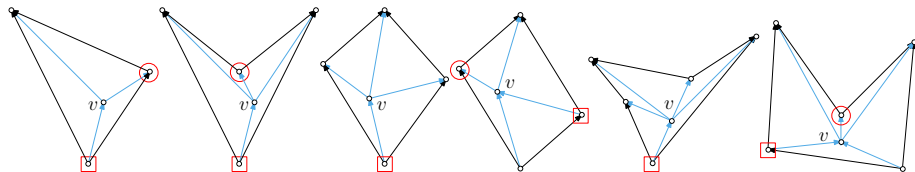
Upward Plane Graphs  
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# Reduced Plane $st$ -Graphs

## Distinguished Neighbors & Contractions

A neighbor  $u$  of  $v$  is a *distinguished* neighbor of  $v$  if it is:

- either a predecessor of  $v$  s.t.  $G$  has a path from any other predecessor  $w$  to  $v$  through  $u$ , or
- a successor of  $v$  s.t.  $G$  has a path from  $v$  through  $u$  to any other successor  $w$



Preliminaries  
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Plane  $st$ -Graphs  
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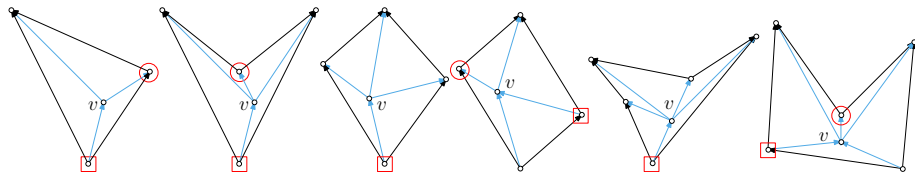
Upward Plane Graphs  
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# Reduced Plane $st$ -Graphs

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With a slight variation of the standard technique, we contract  $v$  on a distinguished neighbor

# Upward Plane Graphs

Preliminaries

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Plane *st*-Graphs

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Upward Plane Graphs

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# (Reduced) Upward Plane Graphs

## Theorem

For any two upward equivalent upward planar drawings of an  $n$ -vertex upward plane graph  $G$ , there exists a  $O(n^2)$ -step upward planar morph between them.

# (Reduced) Upward Plane Graphs

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## High-level Idea

- augment  $G$ ,  $\Gamma_0$ , and  $\Gamma_1$  to *obtain a (reduced)  $st$ -graph*  $G^+$
- compute an upward planar morph  $\mathcal{M}$  between  $\Gamma_0^+$  and  $\Gamma_1^+$

# (Reduced) Upward Plane Graphs

## Theorem

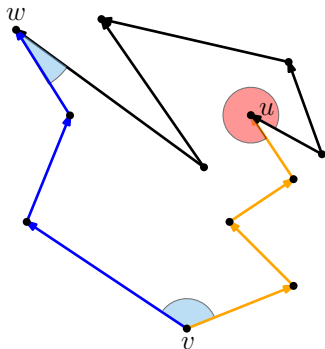
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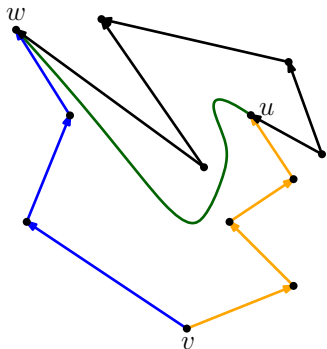
- augment  $G$ ,  $\Gamma_0$ , and  $\Gamma_1$  to *obtain a (reduced)  $st$ -graph*  $G^+$
- compute an upward planar morph  $\mathcal{M}$  between  $\Gamma_0^+$  and  $\Gamma_1^+$
- restrict  $\mathcal{M}$  to the vertices and edges of  $G$

# (Reduced) Upward Plane Graphs

- $G$  is not an  $st$ -graph  $\implies$  some faces have *more than one* source and one sink
- *Large* angles must occur at some of such vertices

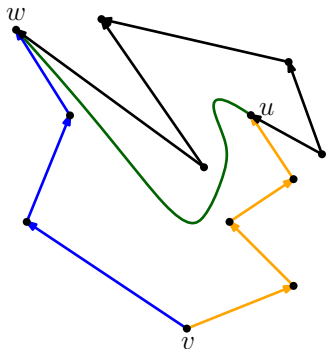


# (Reduced) Upward Plane Graphs



- $G$  is not an  $st$ -graph  $\implies$  some faces have *more than one* source and one sink
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- *splitting* the face at a Large vertex solves the problem locally
- so, add edge  $uw$

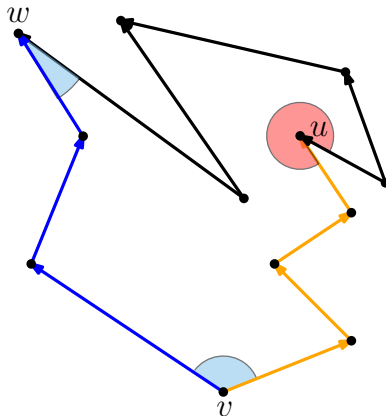
# (Reduced) Upward Plane Graphs



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- *Large* angles must occur at some of such vertices
- *splitting* the face at a Large vertex solves the problem locally
- so, add edge  $uw$
- assume that modifying the drawing for inserting  $uw$  costs  $T_{uw}(n)$  morphing steps
- $G$  contains  $O(n)$  Large vertices  
 $\implies$  turning  $G$  into a (reduced)  $st$ -graph takes  $O(n \cdot T_{uw}(n))$  steps

# Upward Plane Graphs

Removing Large angles



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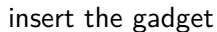
Plane  $st$ -Graphs

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Upward Plane Graphs

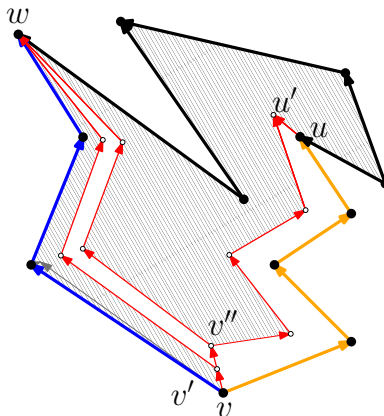
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## Removing Large angles



# Upward Plane Graphs

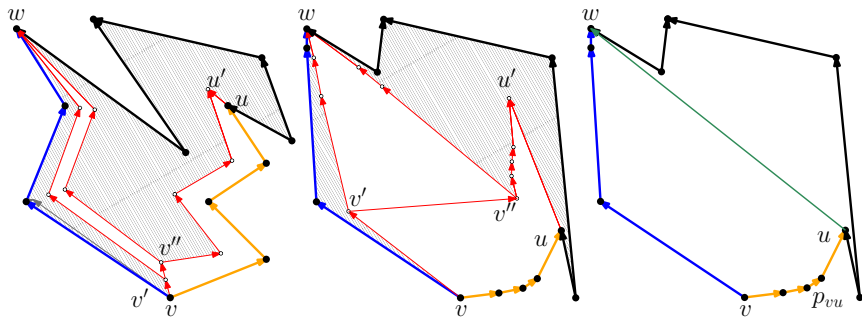
Removing Large angles



triangulate all the faces except those induced by the gadget

# Upward Plane Graphs

Removing Large angles



Preliminaries  
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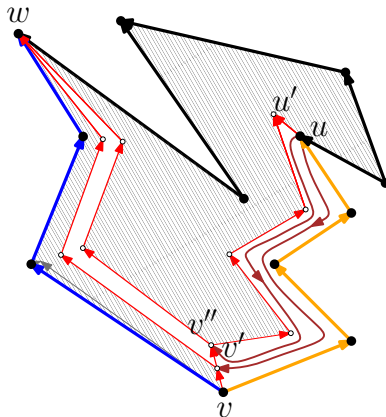
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Plane  $st$ -Graphs  
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Upward Plane Graphs  
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# Upward Plane Graphs

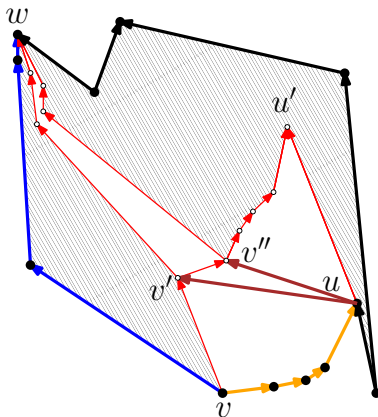
## Removing Large angles



insert edges  $uv'$  and  $uv''$  to obtain an  $st$ -graph  $G^*$   
where  $u$  is a predecessor of both  $v'$  and  $v''$

# Upward Plane Graphs

Removing Large angles



compute an upward drawing of  $G^*$

Preliminaries  
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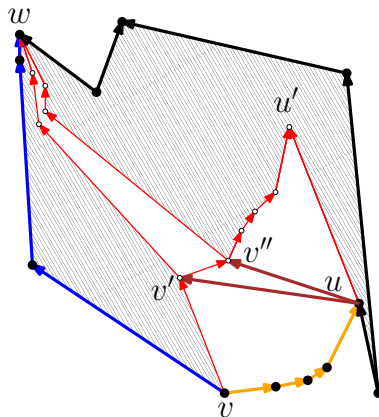
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Plane  $st$ -Graphs  
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Upward Plane Graphs  
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# Upward Plane Graphs

Removing Large angles

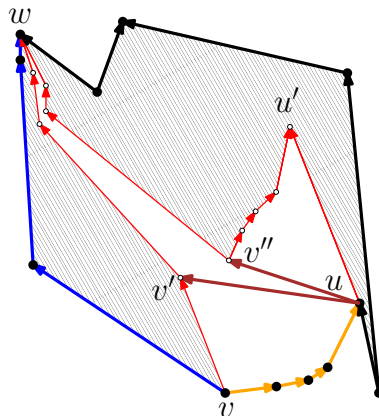


compute an upward drawing of  $G^*$

...but even by removing  $v'v''$ ,  $u$  might not have visibility on  $w$

# Upward Plane Graphs

Removing Large angles



remove edge  $v'v'' \dots$

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Plane  $st$ -Graphs

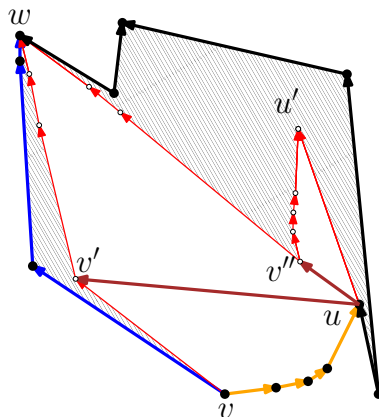
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Upward Plane Graphs

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# Upward Plane Graphs

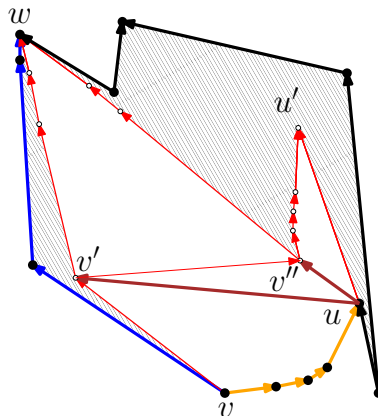
Removing Large angles



compute a convex upward planar drawing of the obtained  $st$ -graph by preserving the  $y$ -coordinates (apply Hong & Nagamochi)

# Upward Plane Graphs

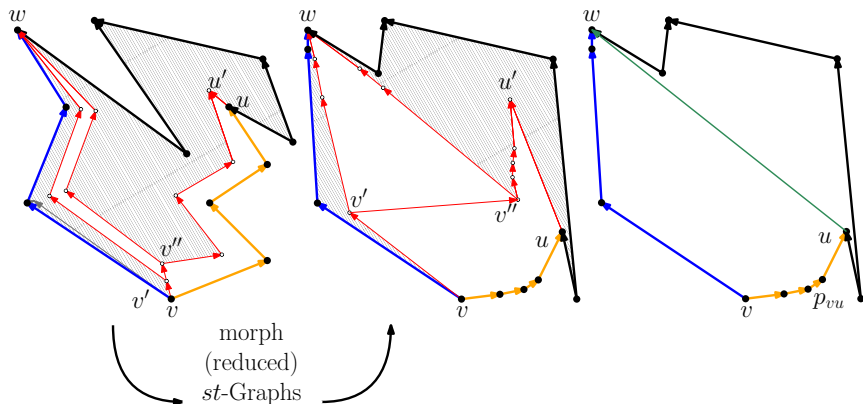
Removing Large angles



add back  $v'v''$  and obtain an upward planar drawing of  $G^*$

# Upward Plane Graphs

Removing Large angles



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Plane  $st$ -Graphs  
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Upward Plane Graphs  
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# Upward Planar Morphs

## Conclusions and Open Problems

Upward Planar Graphs  $O(n^2) - \Omega(n)$  steps  $O(n)?$

Reduced Upward Planar Graphs  $\Theta(n)$  steps

Planar st-Graphs  $O(n)$  steps  $O(1)?$

Reduced Planar st-Graphs  $\Theta(1)$  steps

??-Graphs  $o(n)$  steps

Preliminaries  
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Plane st-Graphs  
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Upward Plane Graphs  
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# Upward Planar Morphs

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??-Graphs  $o(n)$  steps

# Thank you!