Universal Slope Sets for Upward Planar Drawings

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k-bend planar slope number

The *k-bend planar slope number* $\mathsf{psn}_k(G)$ of a planar graph G is the minimum number of slopes needed to construct a drawing of G that:

- is planar
- has at most k bends per edge

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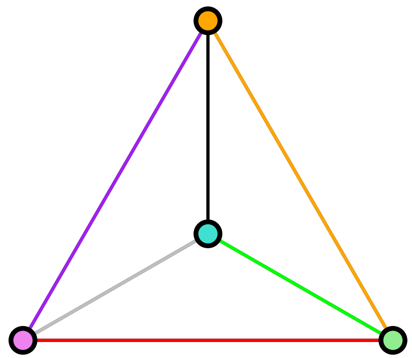

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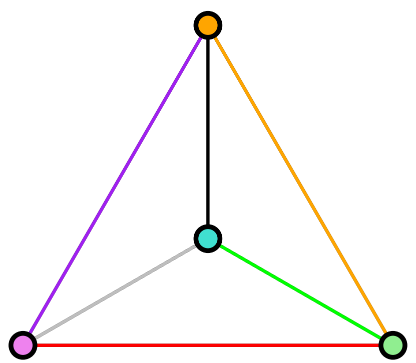

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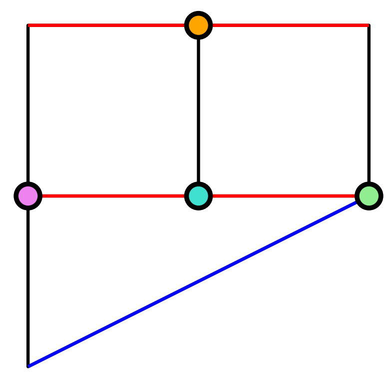

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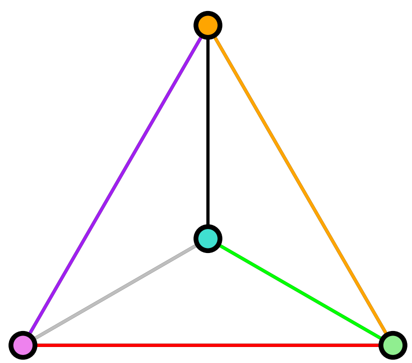

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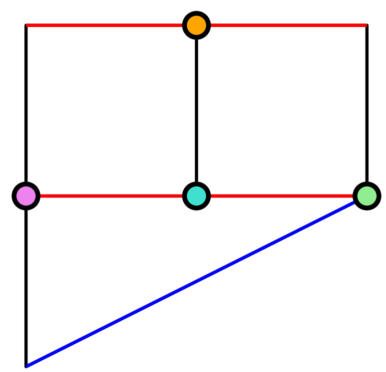

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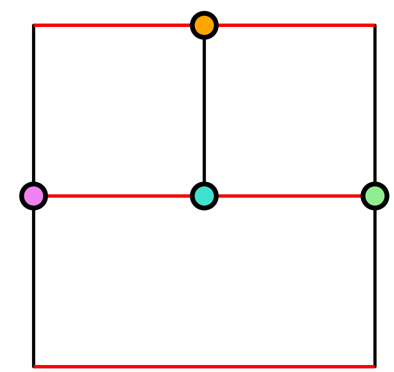


$\mathsf{psn}_2(K_4) = 2$

k-bend planar slope number: known results

For every planar graph G

- $\mathsf{psn}_0(G) = O(K^{\Delta})$ (for a constant K)
- $\mathsf{psn}_0(G) = \Omega(\Delta)$
- $\mathsf{psn}_1(G) \leq 2\Delta$
- in the worst case $\mathsf{psn}_1(G) \geq \frac{3(\Delta-1)}{4}$
- $\mathsf{psn}_2(G) = \lceil \frac{\Delta}{2} \rceil$

Keszegh, Pach, Pálvölgyi, GD 2010, SIDMA 2013

For every planar graph G, $\mathsf{psn}_1(G) \leq \frac{3(\Delta-1)}{2}$

Knauer and Walczak, LATIN 2016

For every planar graph G, $\mathsf{psn}_1(G) \leq \Delta - 1$

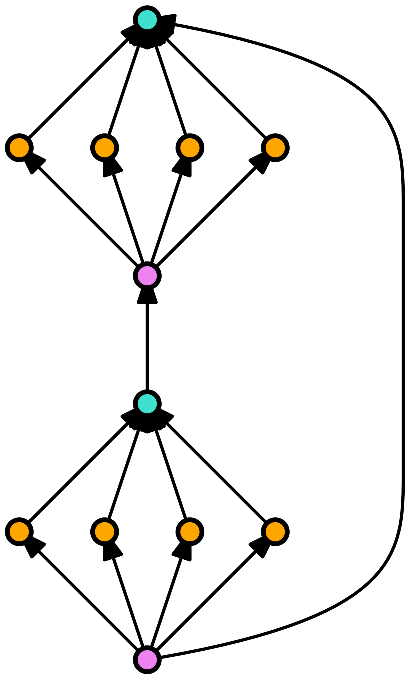
Angelini et al., SoCG 2017

k-bend *upward* planar slope number

The *k-bend upward planar slope number* $\mathsf{upsn}_k(G)$ of an upward planar graph G is the minimum number of slopes needed to construct a drawing that:

- is planar
- has at most k bends per edge
- is upward

Non-upward vs. upward slope number



G

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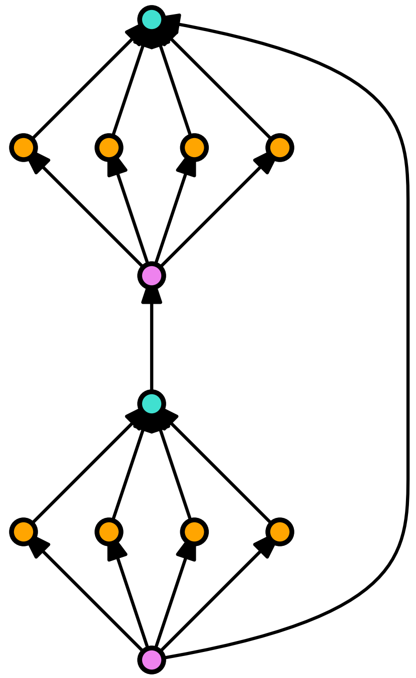

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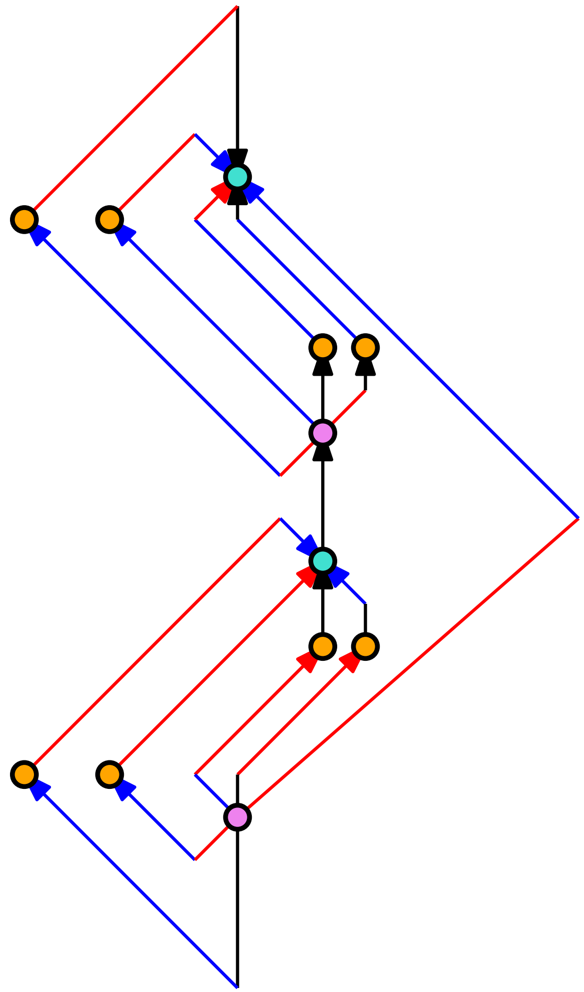

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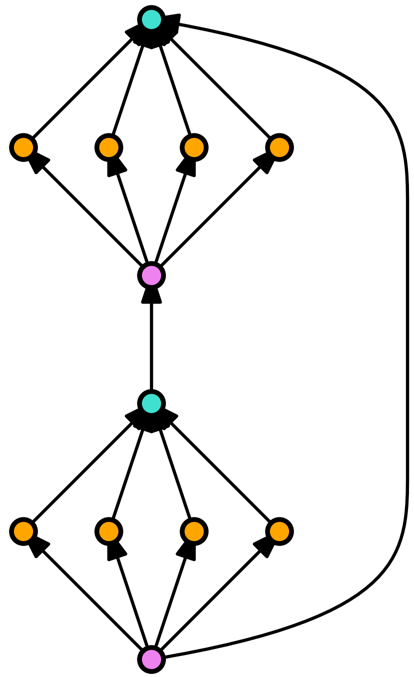

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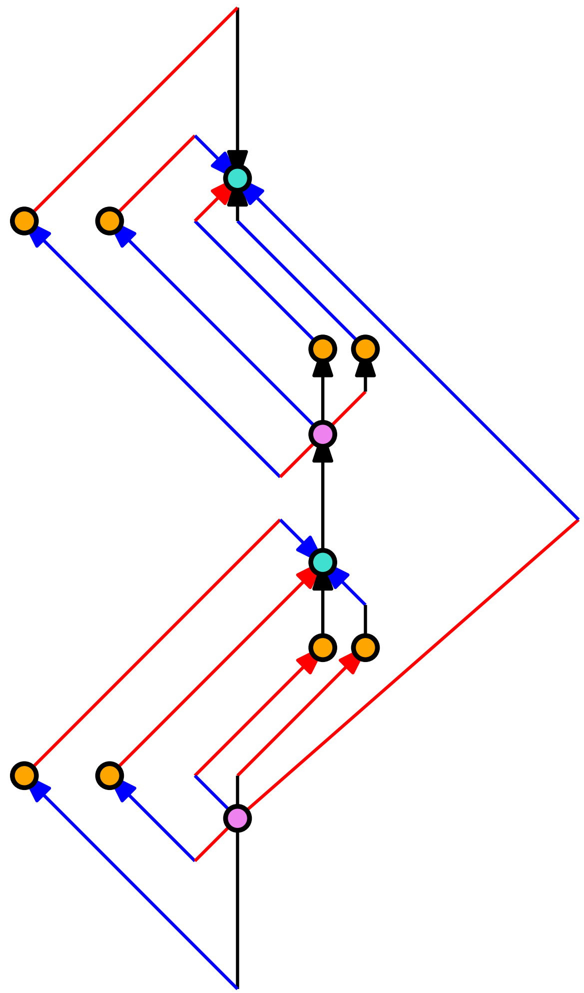

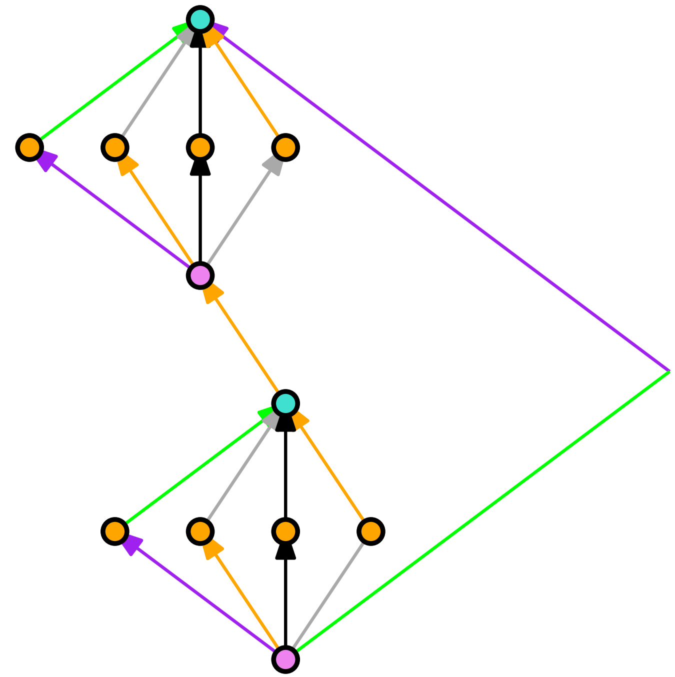

$\mathsf{upsn}_1(G) = 3$ $\mathsf{upsn}_1(G) = 5$

k-bend upward planar slope number: known results

For every planar poset P, $\mathsf{upsn}_1(P) \leq \Delta$, which is worst-case optimal

Czyzowicz, Pelc, Rival, Urrutia, Order 1990

For every series-parallel digraph G, $\mathsf{upsn}_1(G) \leq \Delta$, which is worst-case optimal

Di Giacomo, Liotta, Montecchiani, GD 2016

What does it mean upward?

Every edge is drawn as a curve monotonically increasing in the y-direction

Every edge is drawn as a curve monotonically non-decreasing in the y-direction

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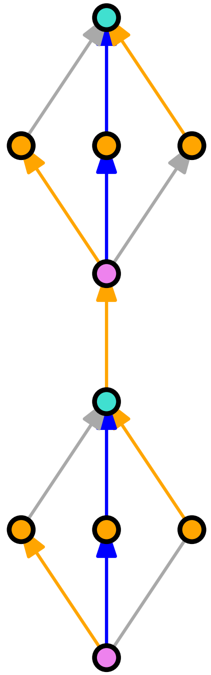

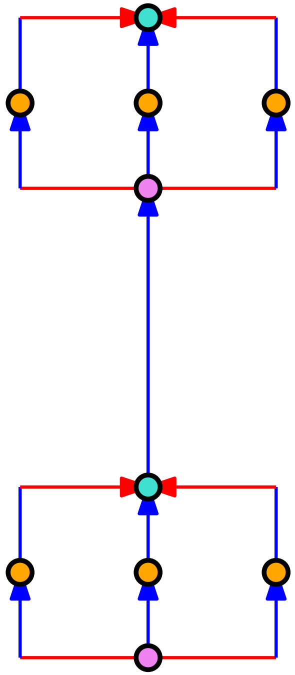

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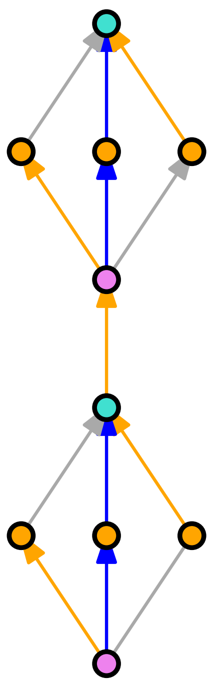

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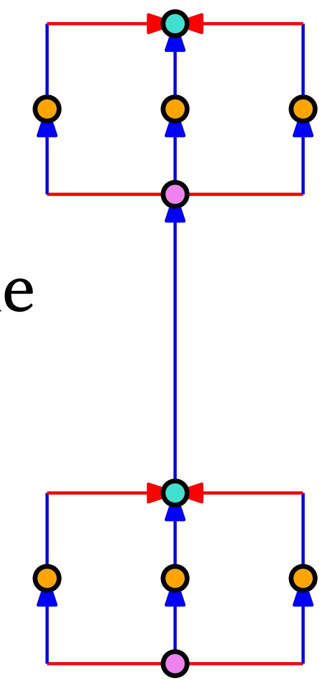

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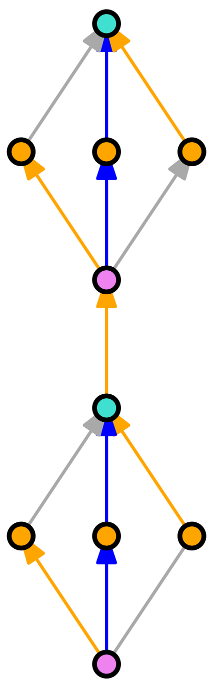

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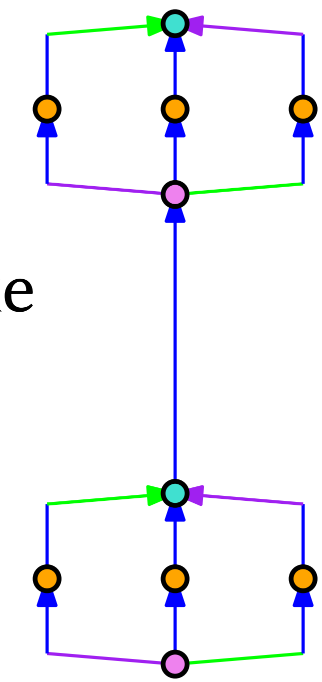

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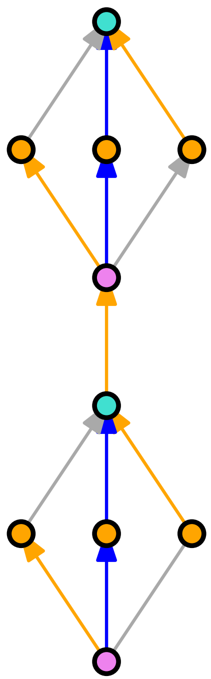
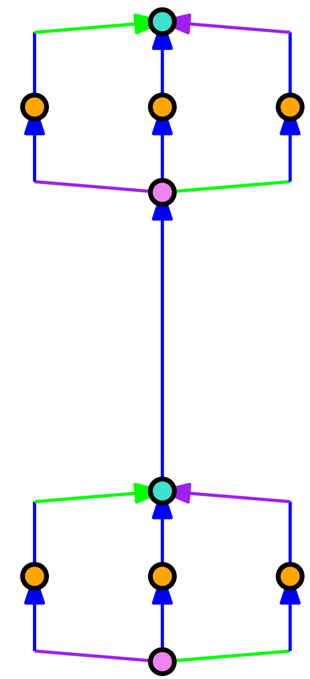


In this case, however, the number of slopes increases by one



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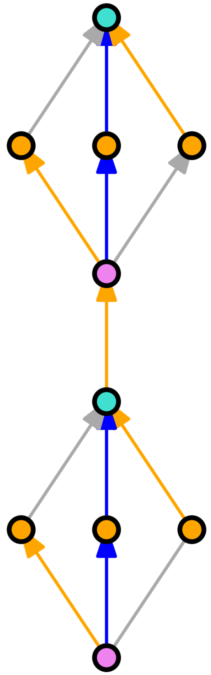
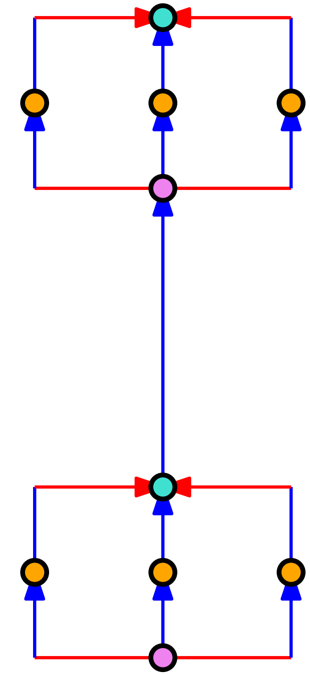


In the rest of the talk I will use the non-decreasing model



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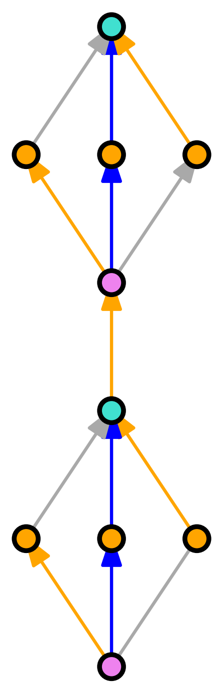

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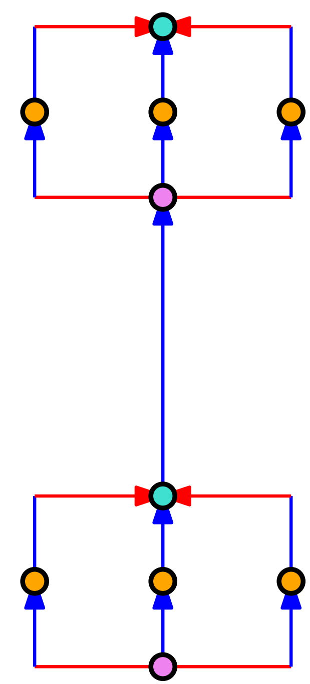
All the constructions can also be used in the increasing model with one slope more



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The results above are based on linear time algorithms

1-bend upward planar drawings of bitonic st-graphs

Bitonic planar st-graph [1]

- A *bitonic planar st-graph* is an embedded planar DAG that admits a *bitonic st-ordering*, i.e., numbering σ of its vertices s.t.

[1] Gronemann, GD 2016

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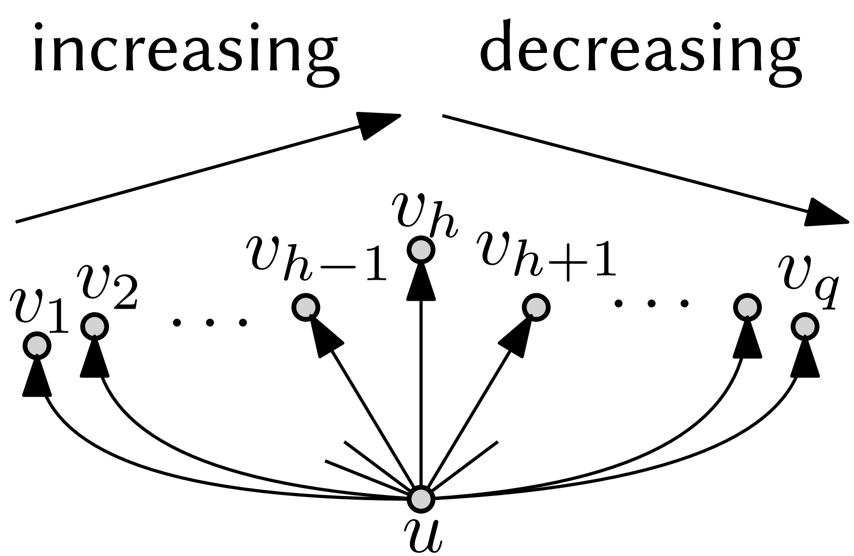

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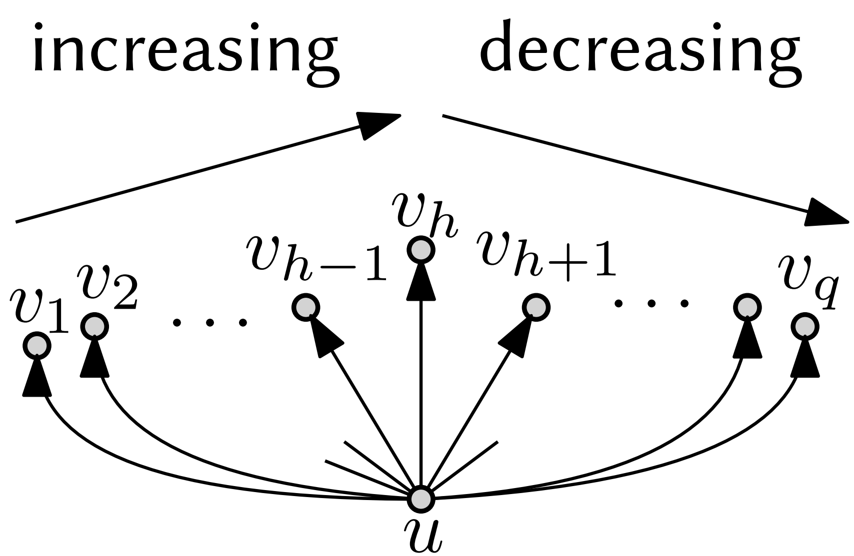

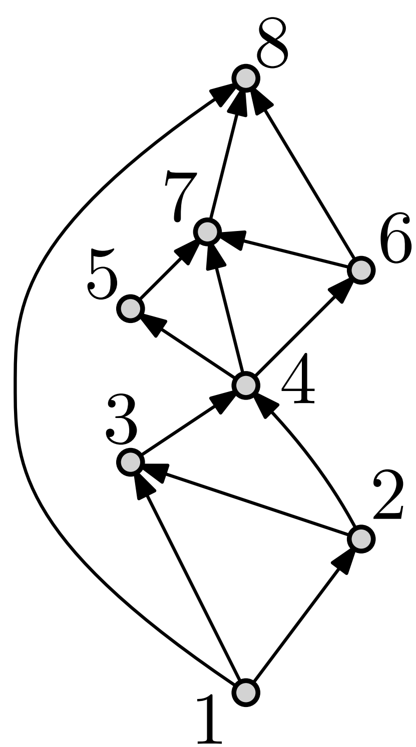

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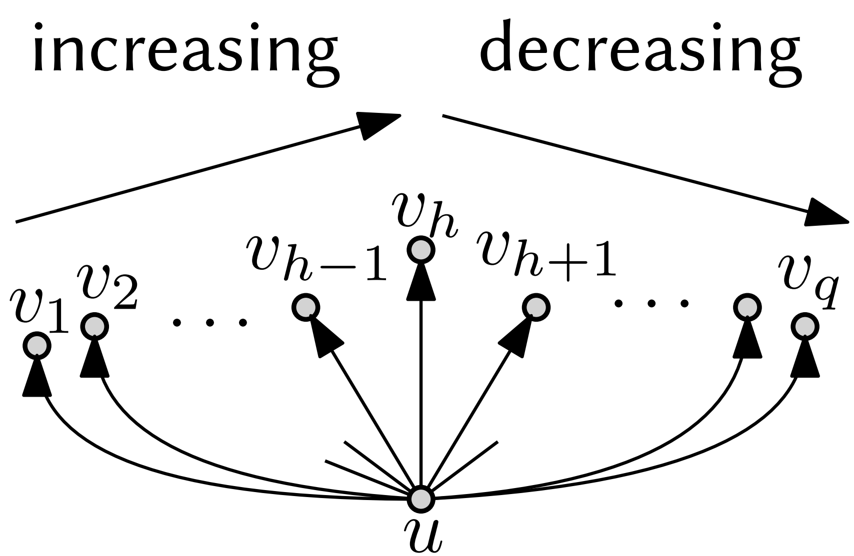

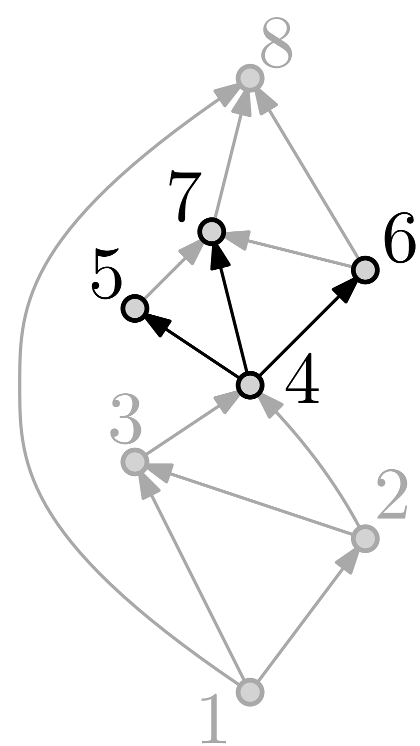

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The drawing algorithm: overview

INPUT: a bitonic planar st-graph G, a set of Δ slopes $\mathcal{S}$ including the horizontal
OUTPUT: a 1-bend upward planar drawing Γ that uses only the slopes in $\mathcal{S}$

1 - Compute a bitonic st-ordering σ of G

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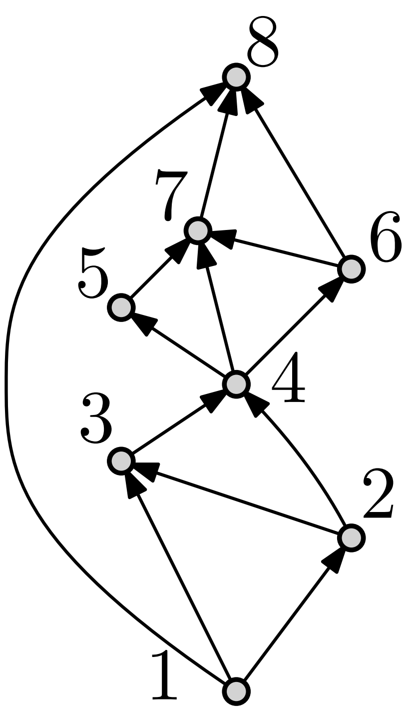

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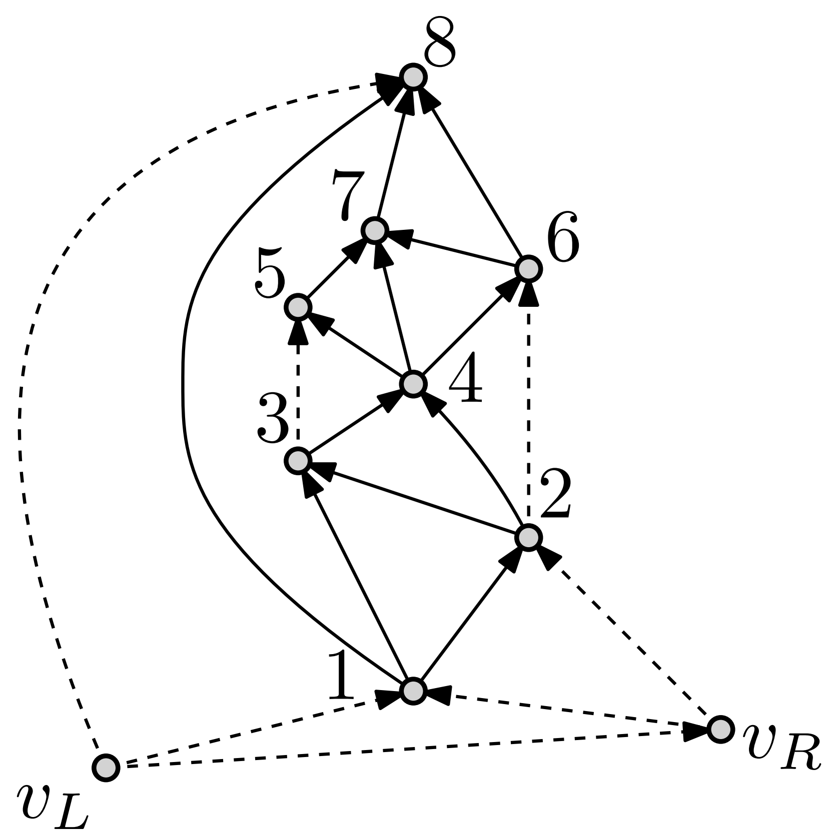
Augment G so that each vertex has at least two predecessors (see [1])



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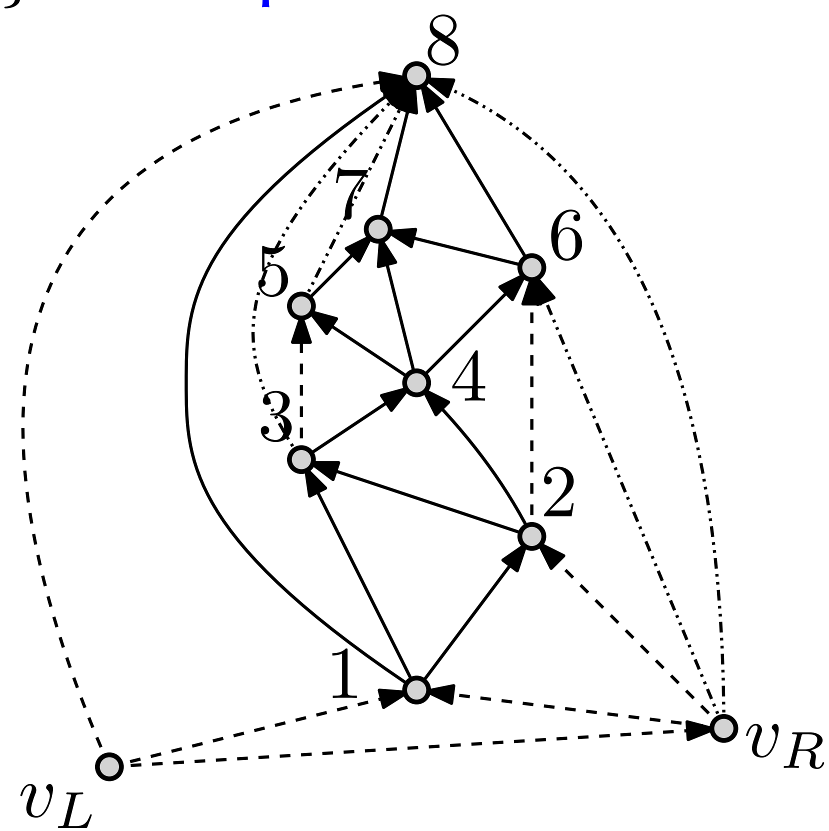
Triangulate G;

$\chi = \{v_L, v_R, v_1, v_2, \ldots, v_n\}$ is an upward canonical ordering



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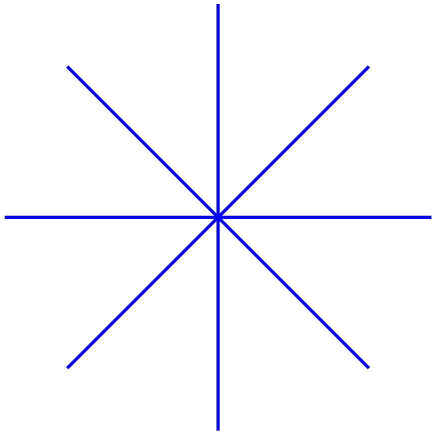
The set of slopes

Let $\mathcal{S} = \{\rho_1, \rho_2, \ldots, \rho_\Delta\}$ be the given set of slopes

- These slopes will be called *real slopes*



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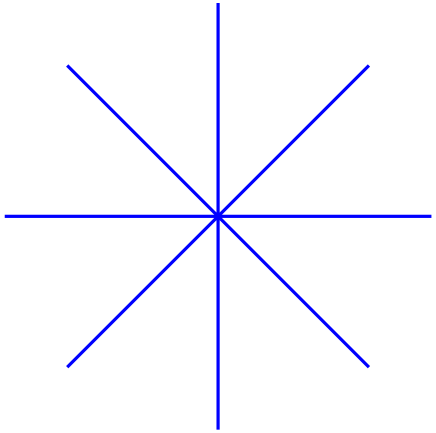

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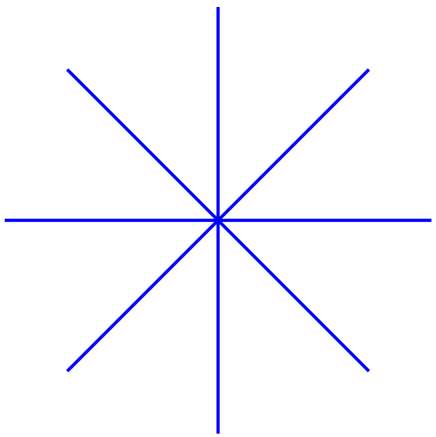
For each pair of consecutive slopes in $\mathcal{S}$ we add Δ^* *dummy slopes* between them



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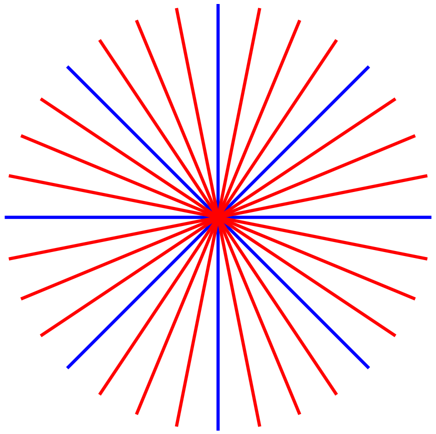

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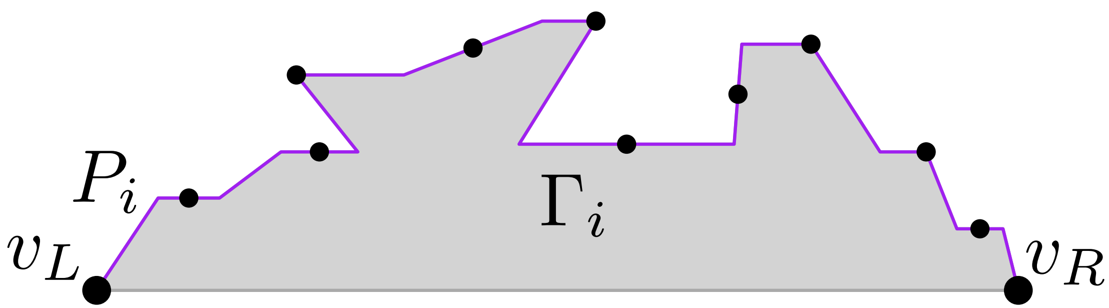
I1 – Γ_i is a 1-bend planar upward drawing whose real edges uses only the slopes in $\mathcal{S}$

I2 – Every edge in the upper boundary P_i of Γ_i contains a horizontal segment



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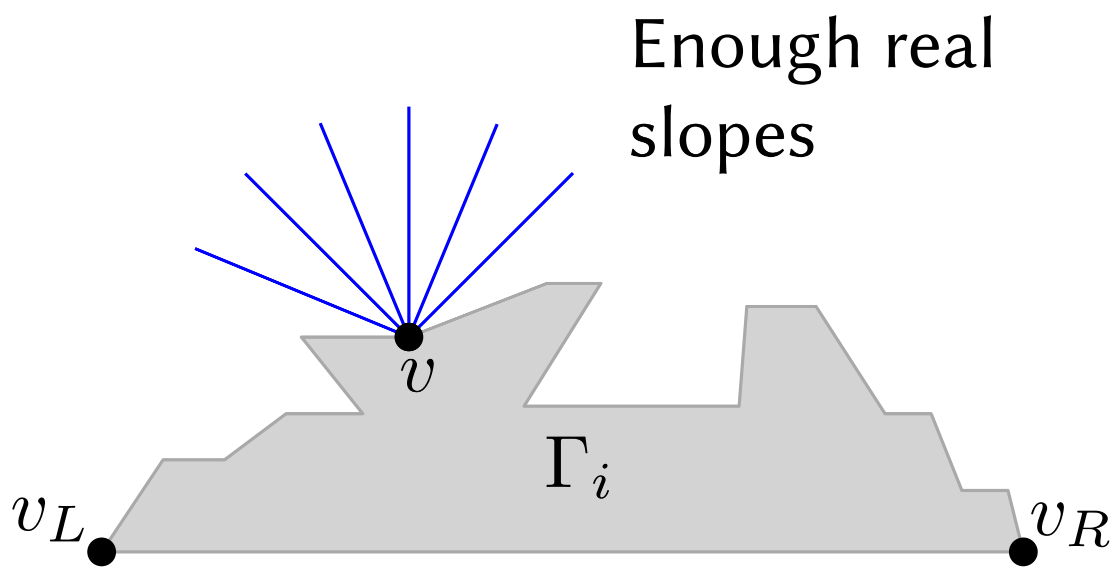

Construction of Γ

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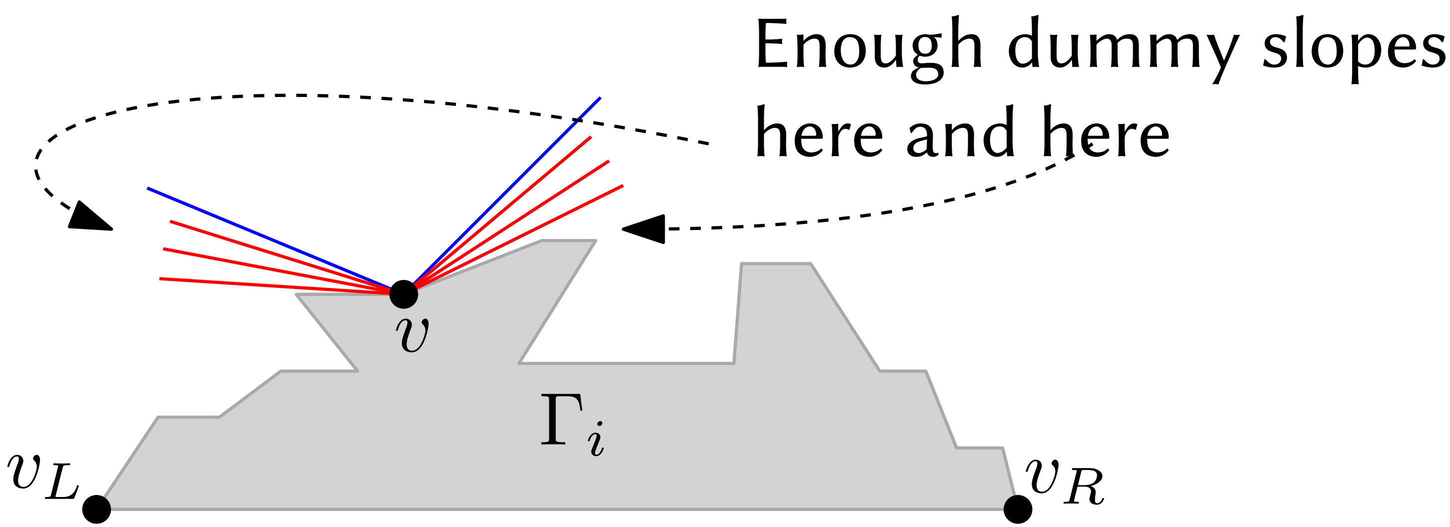
I4 – For each vertex v the number of dummy slopes above v that are before the first real slope and are free are at least the number of dummy edges incident on v that have still to be drawn



A crucial lemma

Let Γ_i be a drawing that satisfies **I1-I4**;
let (u, v) be an edge of P_i such that u is before v;
let λ be a positive number.

There exists Γ_i' such that:

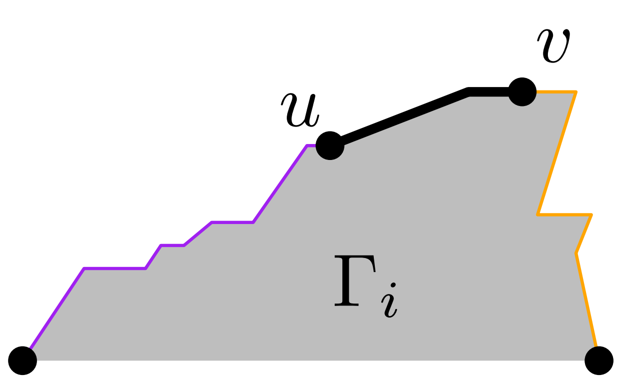
- satisfies **I1-I4**
- the horizontal distance between u and v is increased by λ
- the horizontal distance between any two other consecutive vertices along P_i is not changed

A crucial lemma

Let Γ_i be a drawing that satisfies **I1-I4**;
let (u, v) be an edge of P_i such that u is before v;
let λ be a positive number.

There exists Γ'_i such that:

- satisfies **I1-I4**
- the horizontal distance between u and v is increased by λ
- the horizontal distance between any two other consecutive vertices along P_i is not changed



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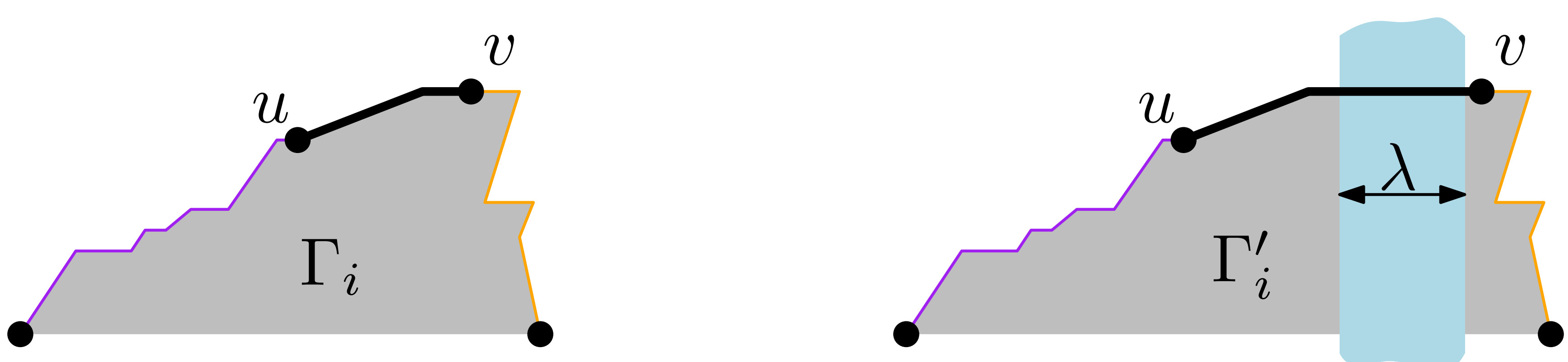
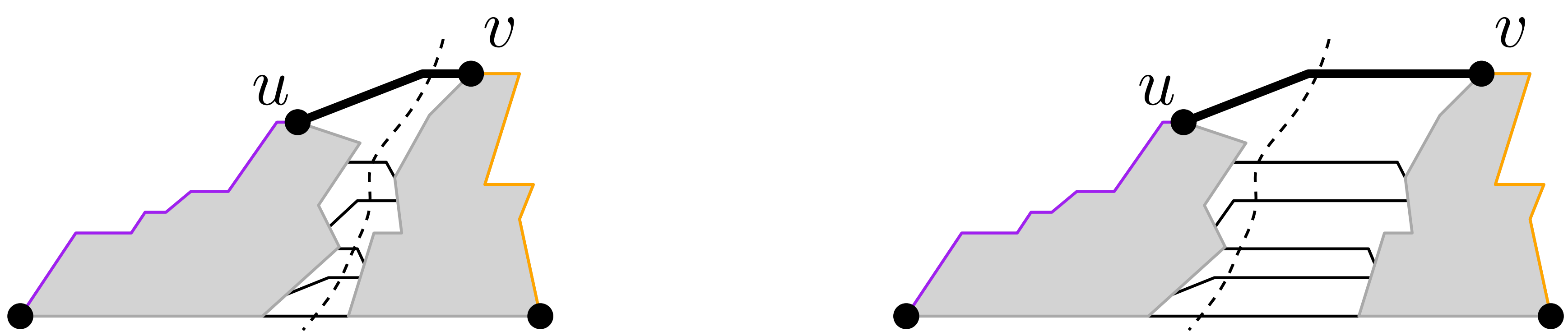


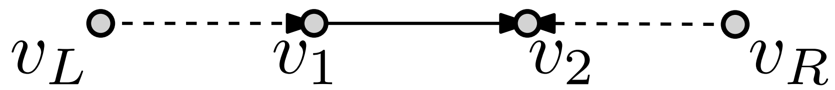
Sketch of proof

By using **I2** and induction we can prove that there is a cut of horizontal edges



Construction of Γ

We draw G_2 as a horizontal path (ignoring some dummy edges)



Construction of Γ

Addition of v_i $(2 < i < n)$

Construction of Γ

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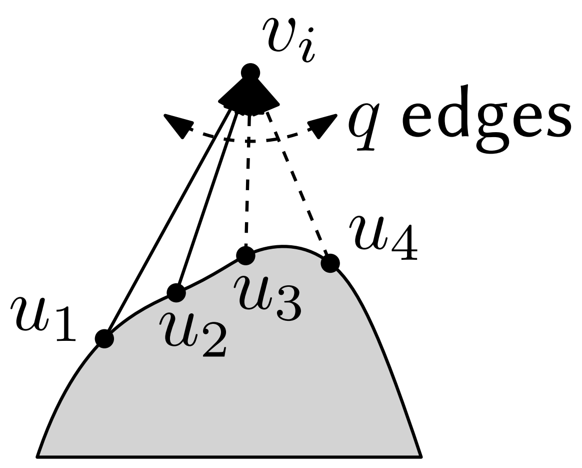

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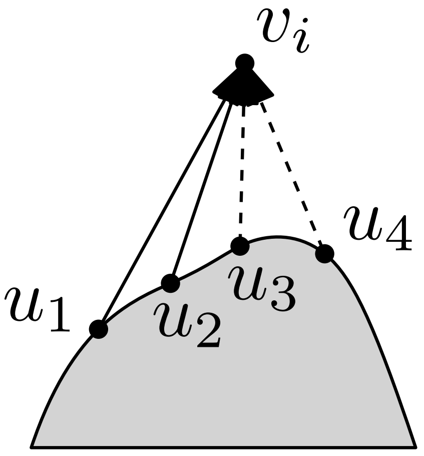

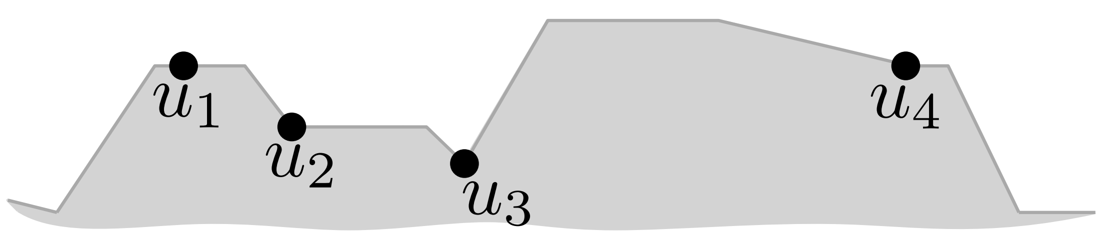

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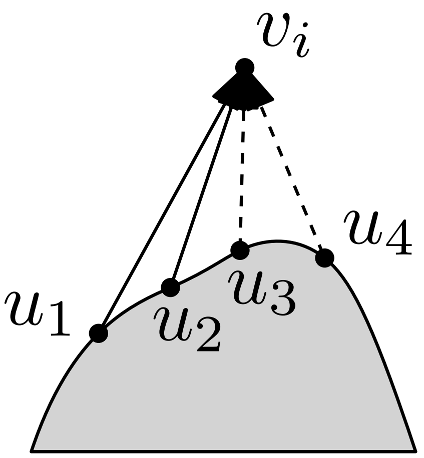
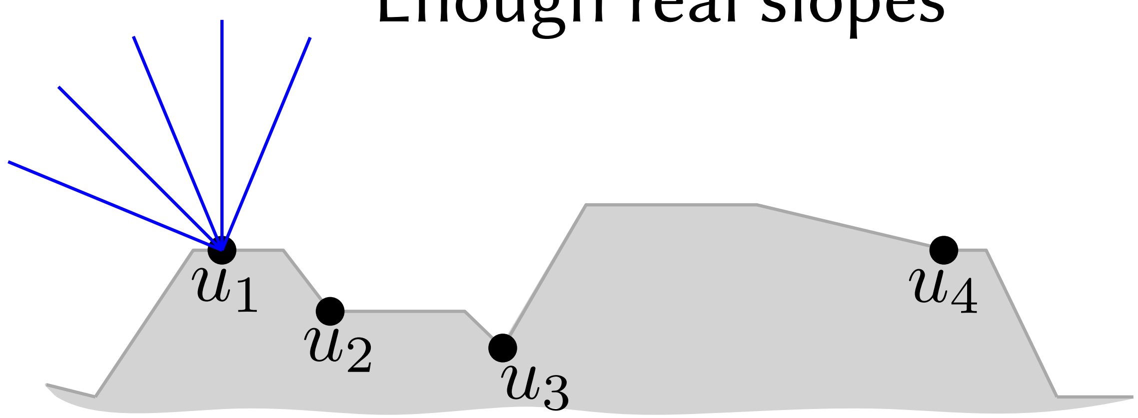
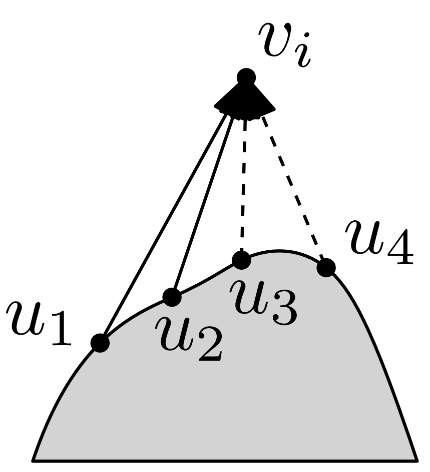


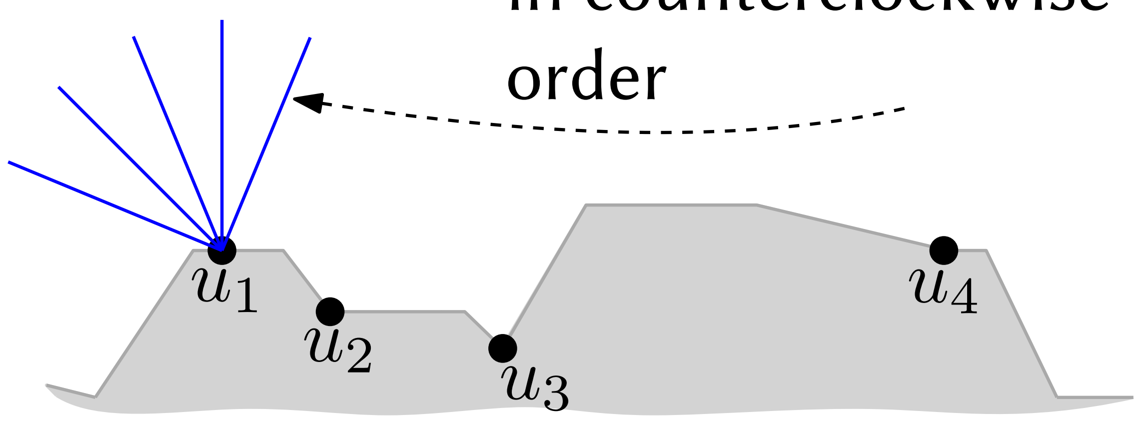


Construction of Γ

Addition of v_i $(2 < i < n)$



We choose the first one in counterclockwise order



Construction of Γ

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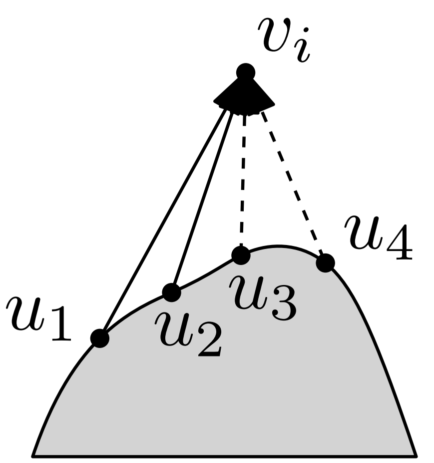
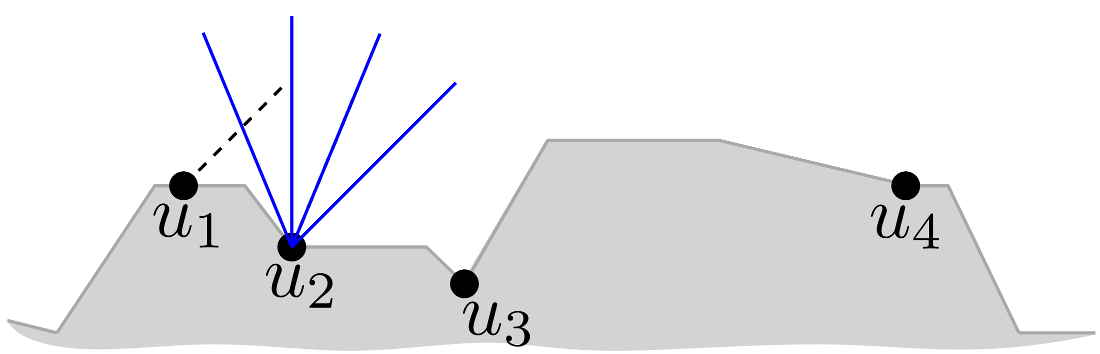


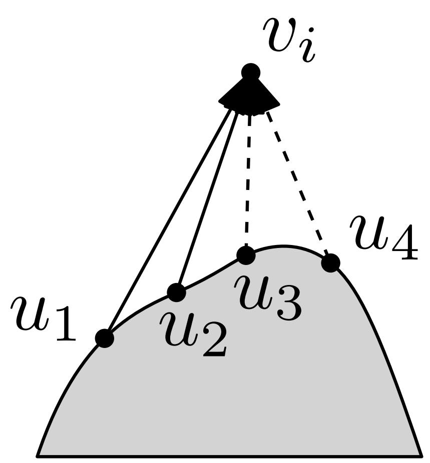
Enough real slopes

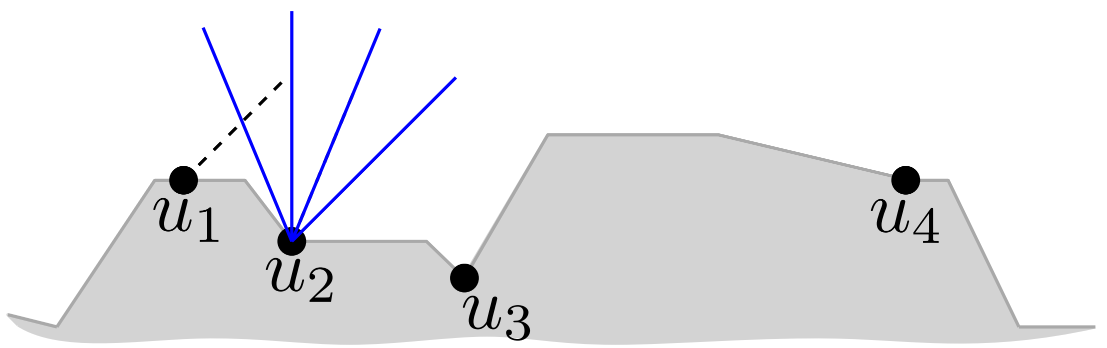


Construction of Γ

Addition of v_i $(2 < i < n)$

We arbitrarily choose one of them





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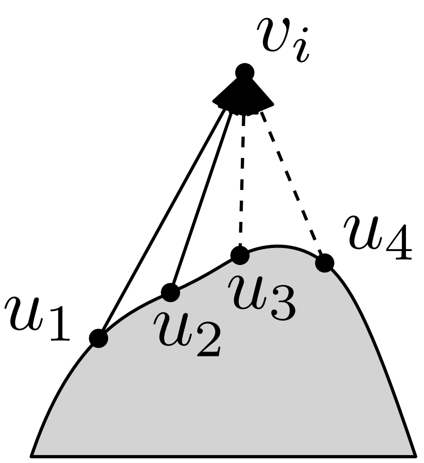

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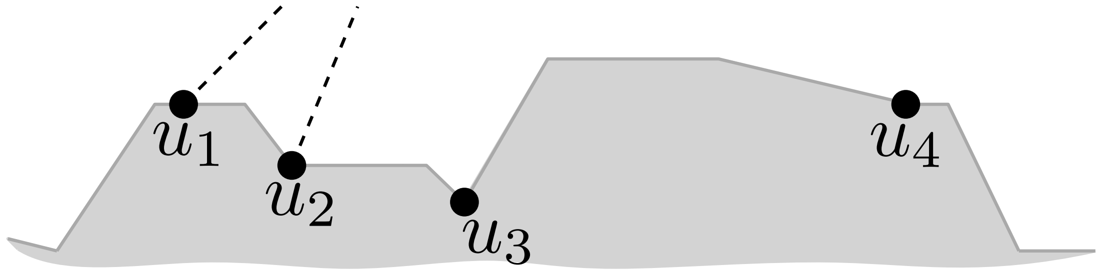

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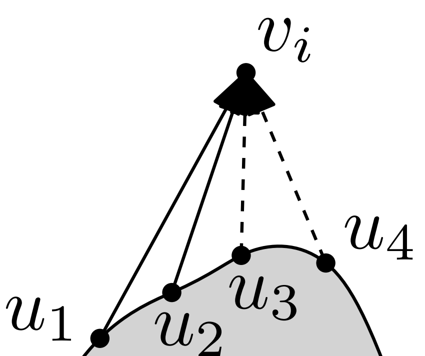

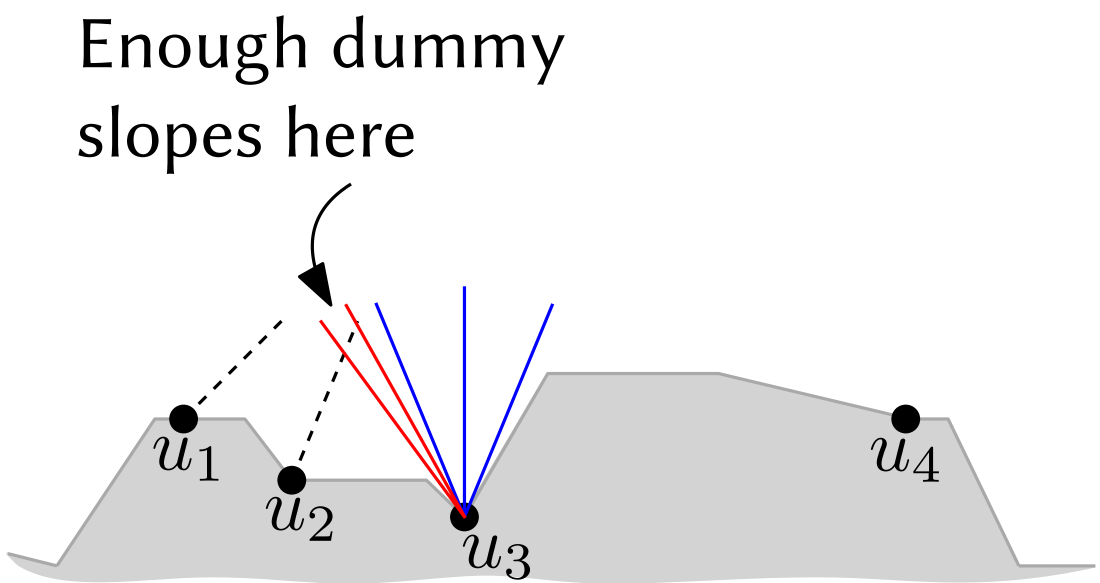

Construction of Γ

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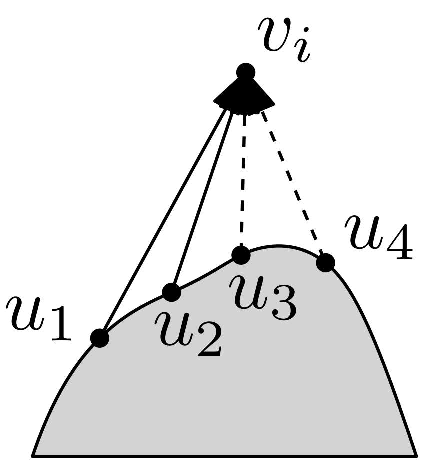
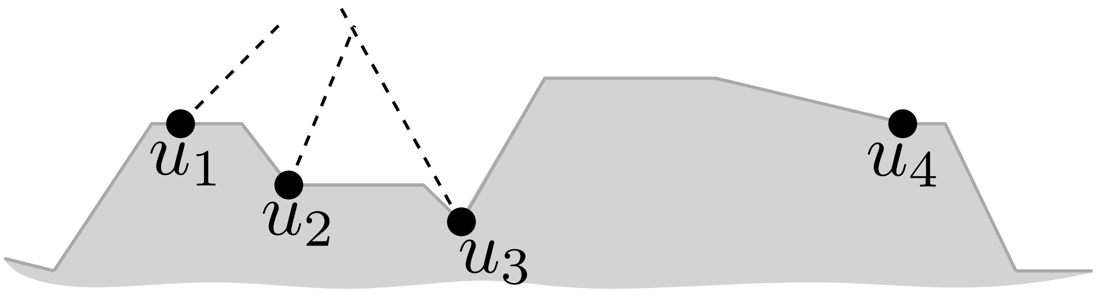


We arbitrarily choose one of them



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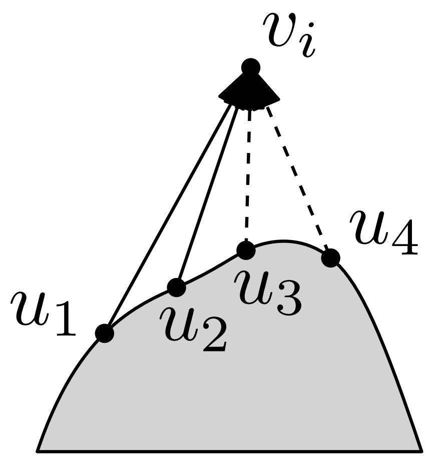

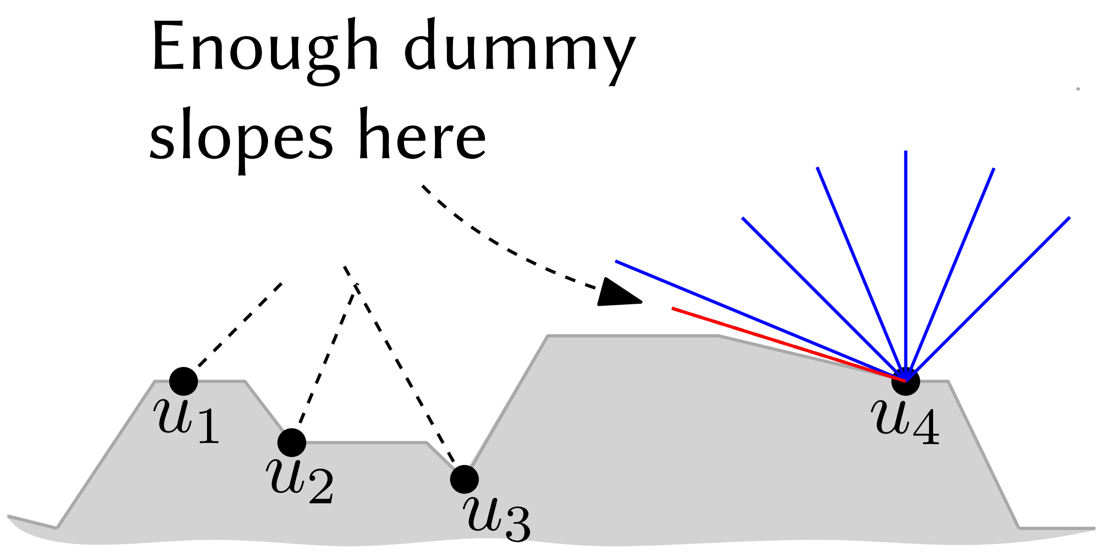

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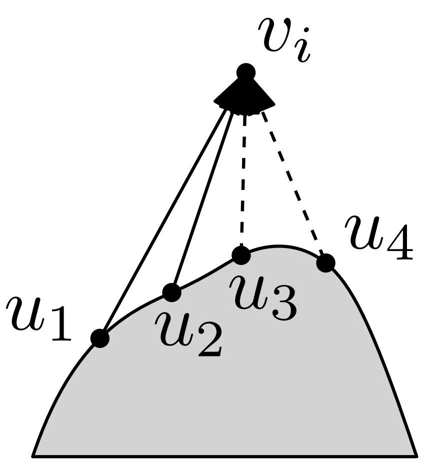

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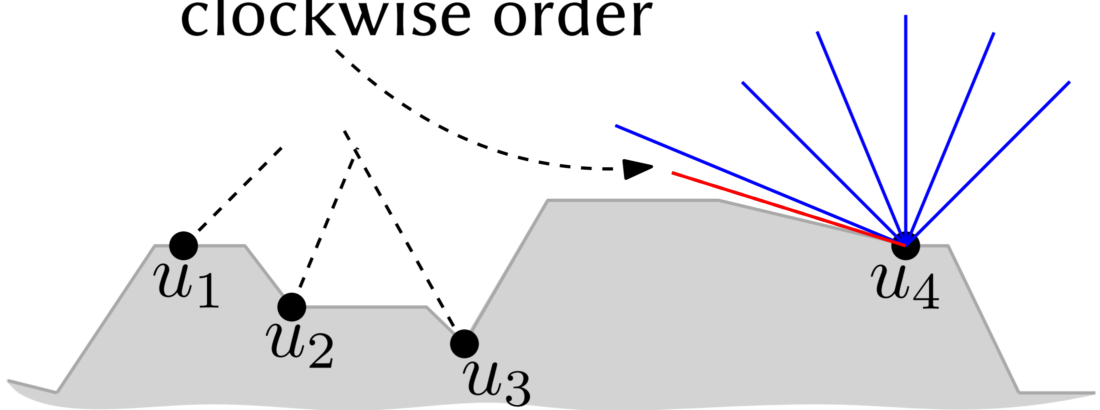

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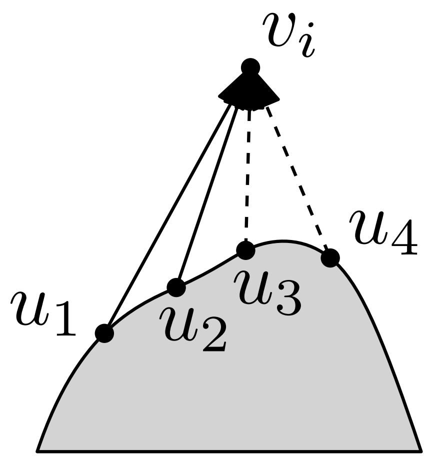

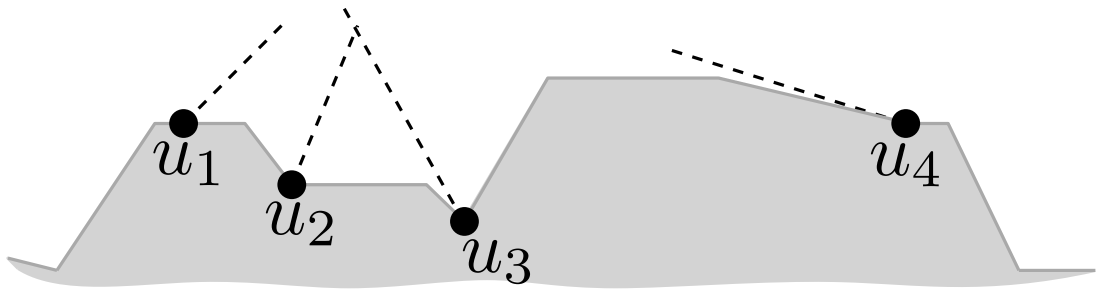

Construction of Γ

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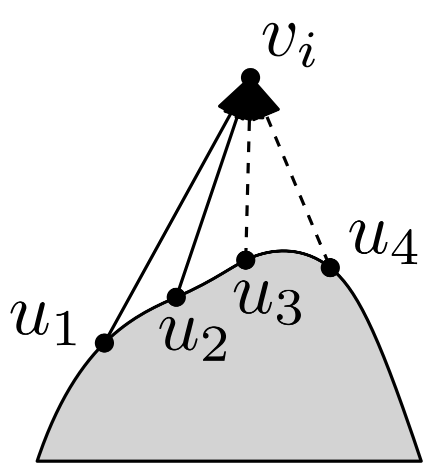

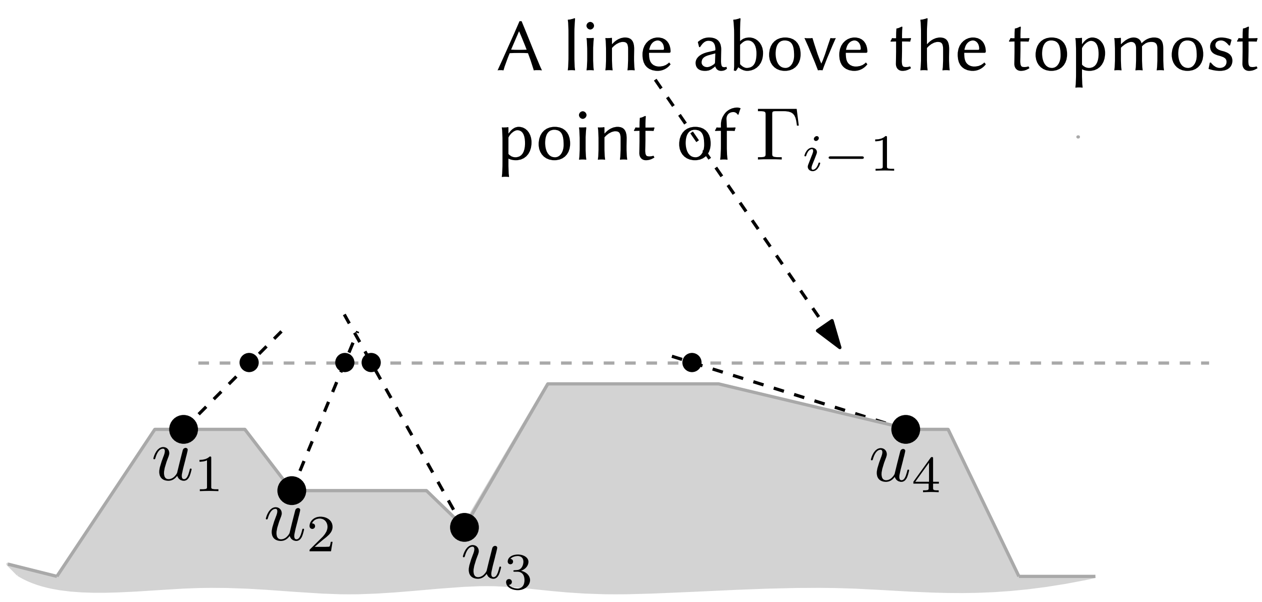

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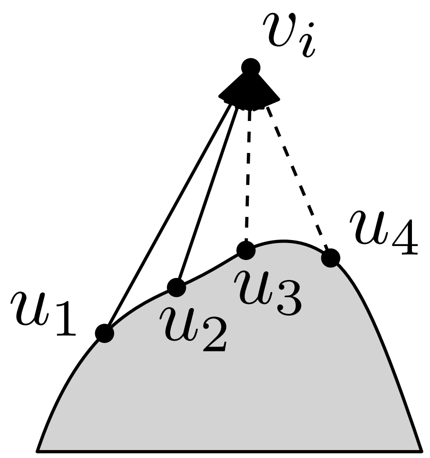

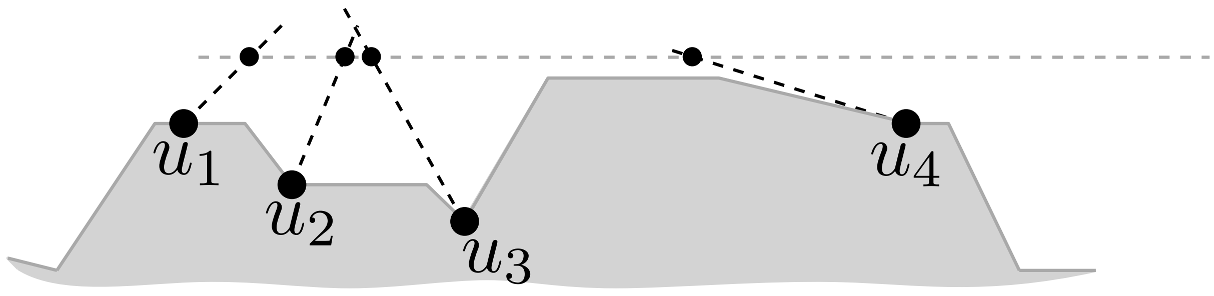

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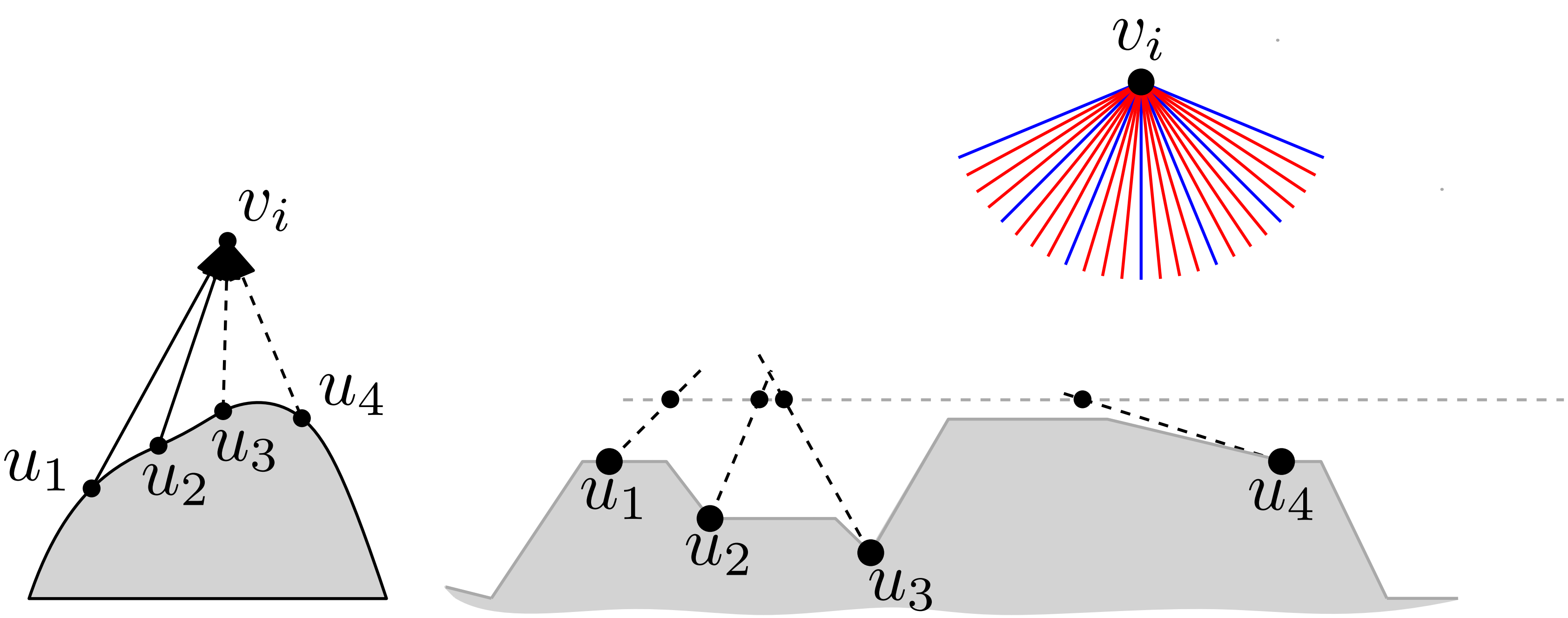
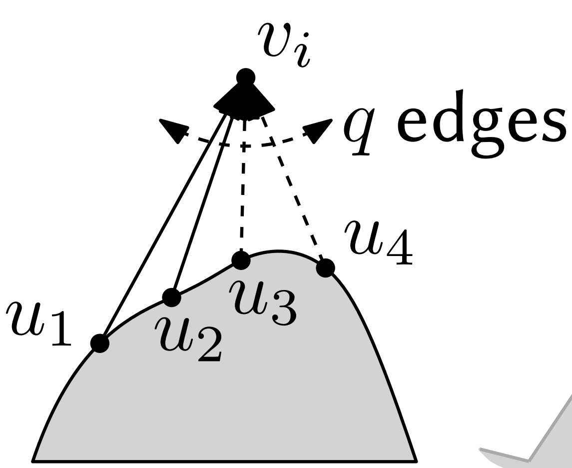
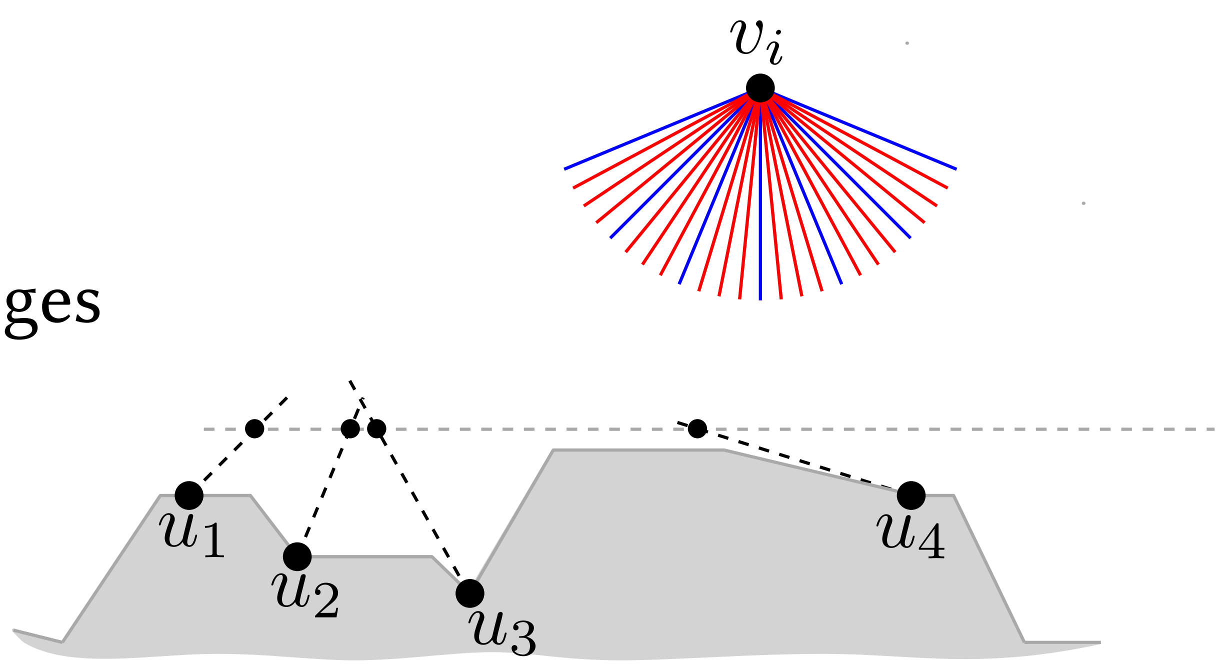


Construction of Γ

Addition of v_i ($2 < i < n$)

Arbitrarily choose $q - 2$ slopes
(real or dummy as needed)

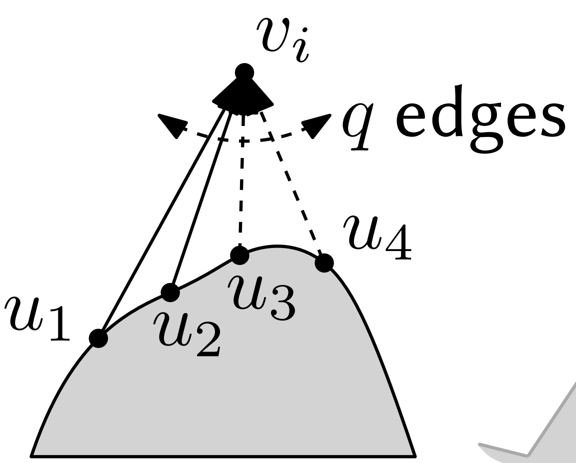


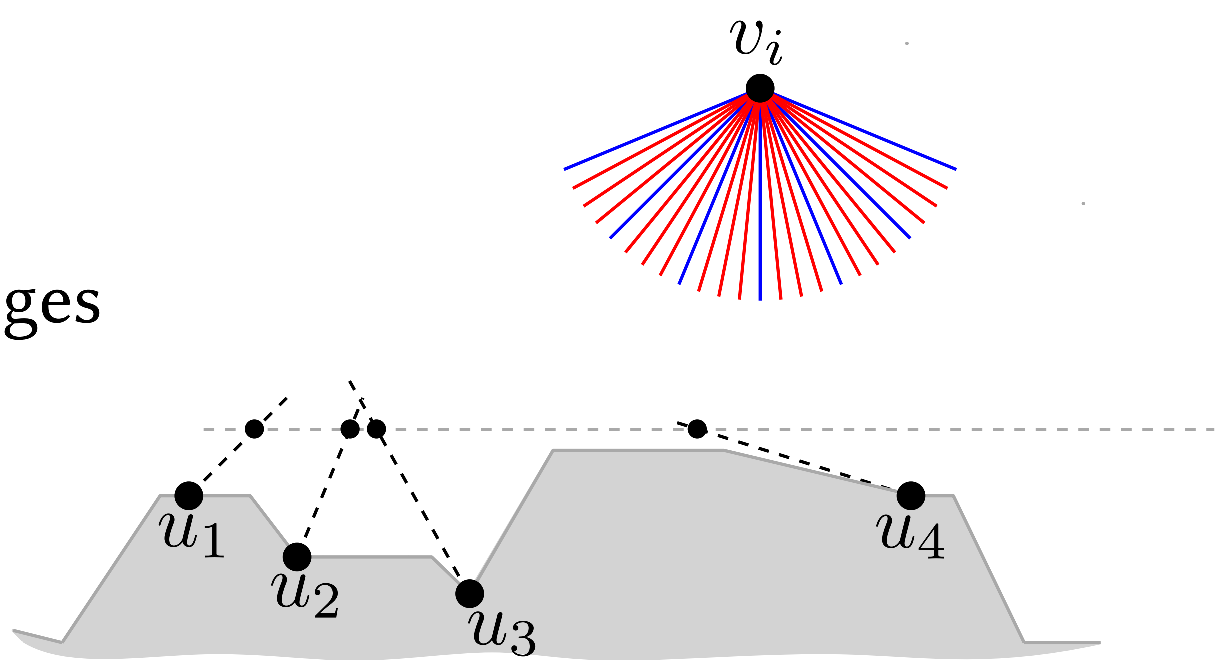


Construction of Γ

Addition of v_i $(2 < i < n)$

Arbitrarily choose $q-2$ slopes
(real or dummy as needed)
(The number of real edges is at most $\Delta - 1$ *and we have* $\Delta - 1$ *real slopes below* v_i*)*





Construction of Γ

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Arbitrarily choose $q - 2$ slopes
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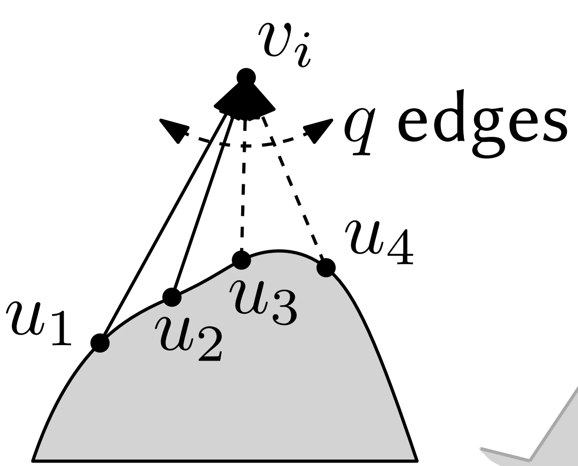
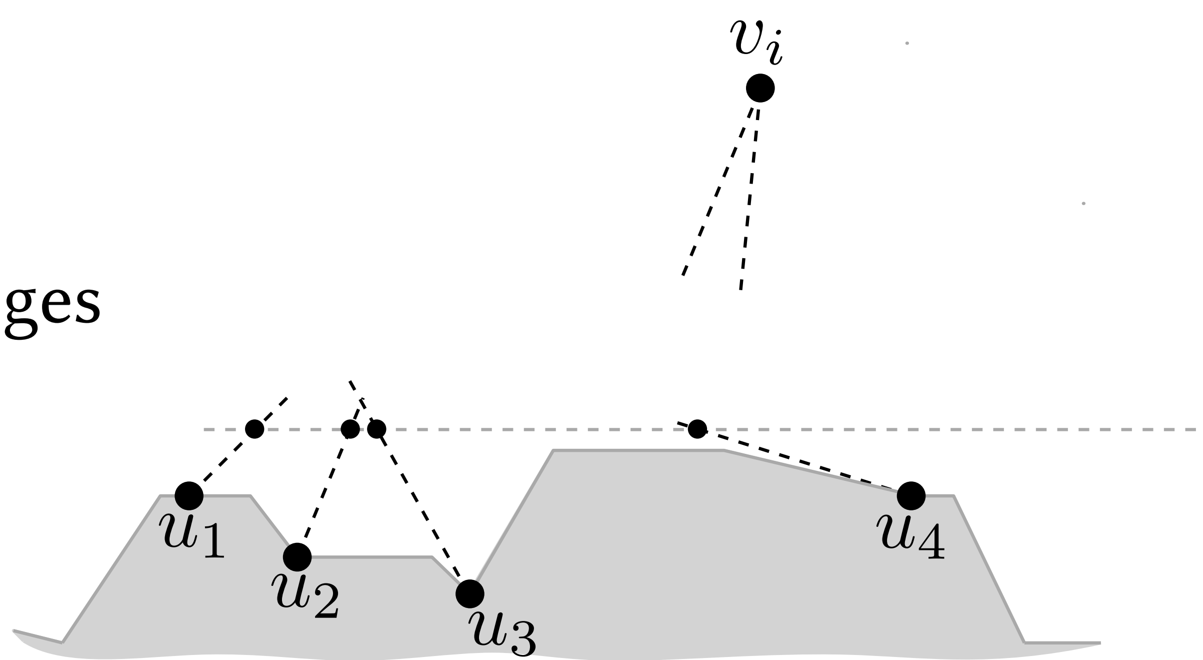
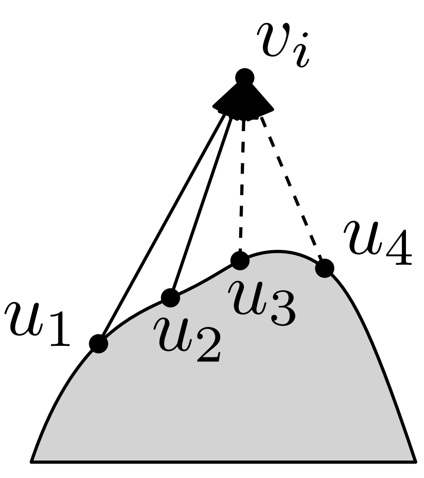
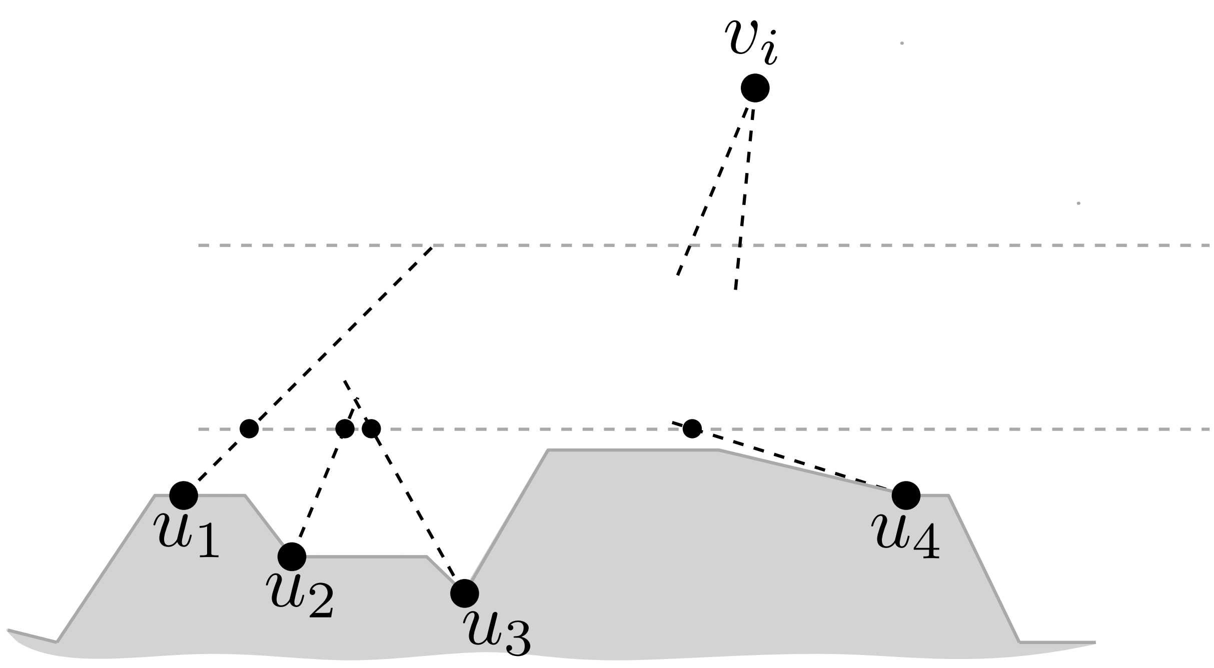




Construction of Γ

Addition of v_i $(2 < i < n)$





Construction of Γ

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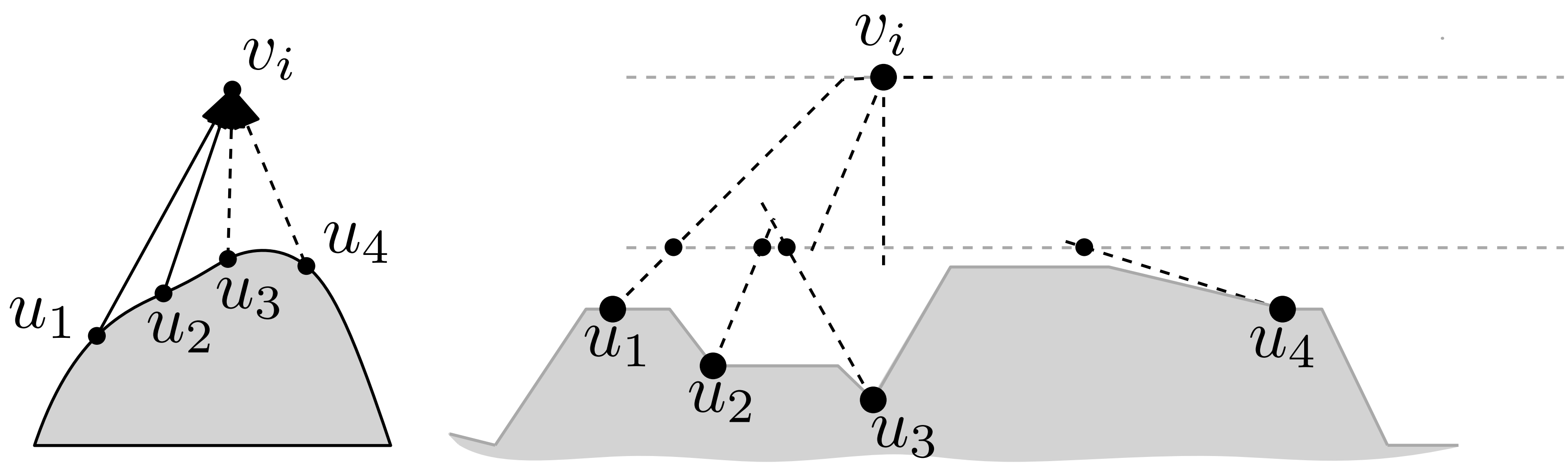

Construction of Γ

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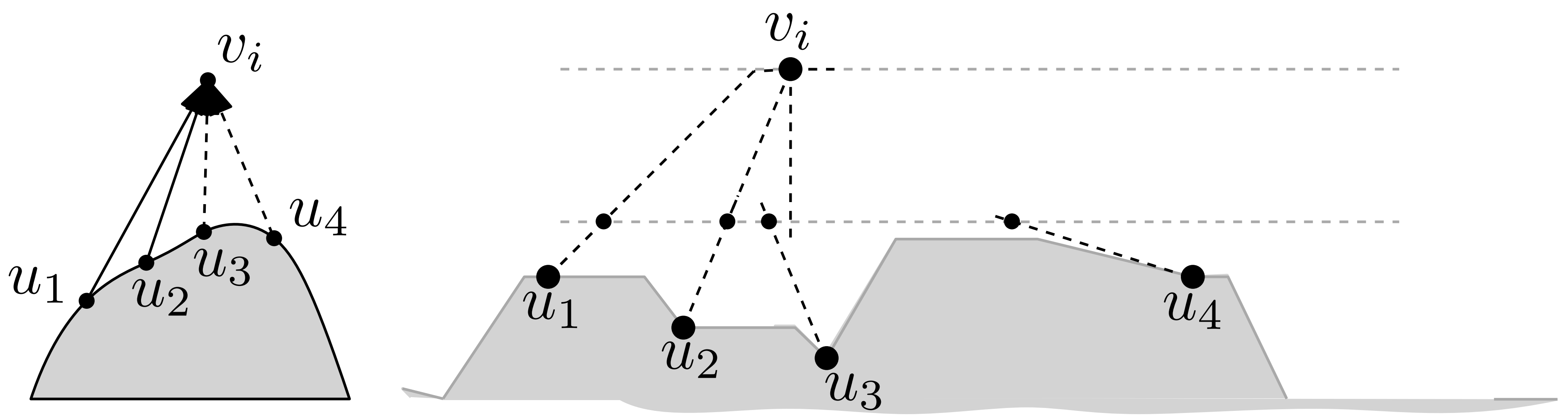

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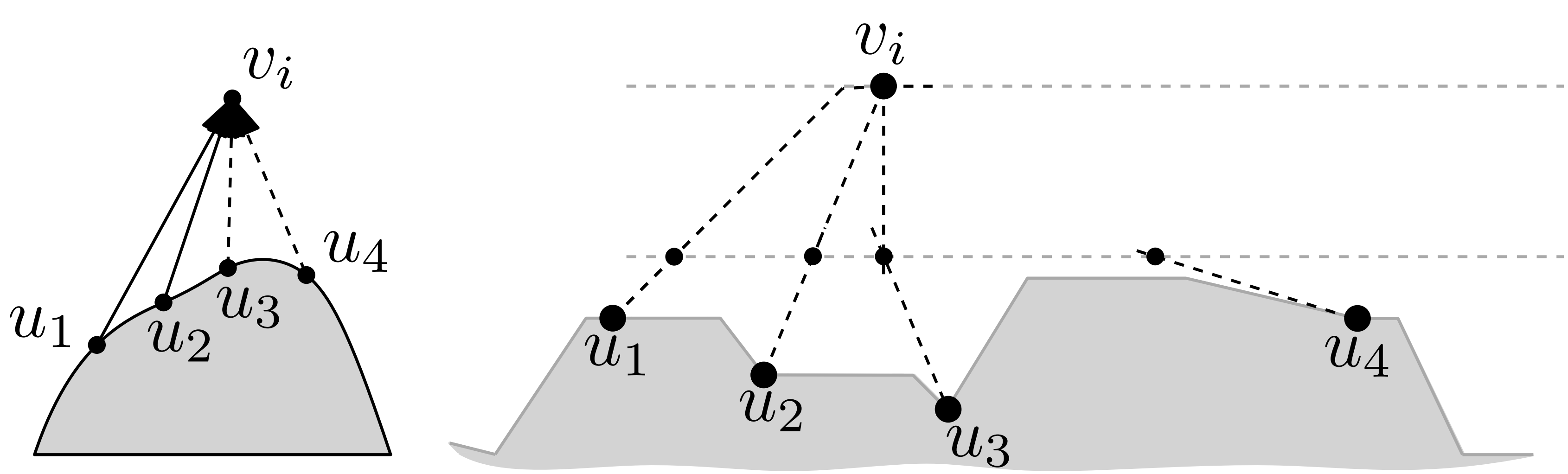

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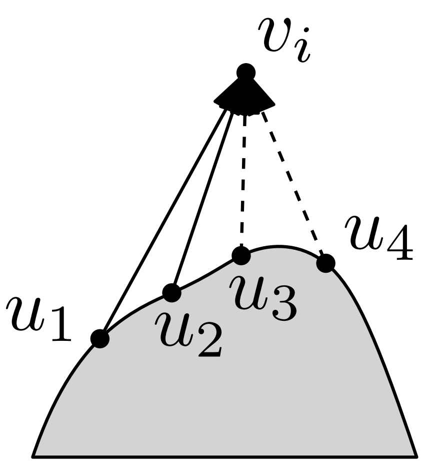

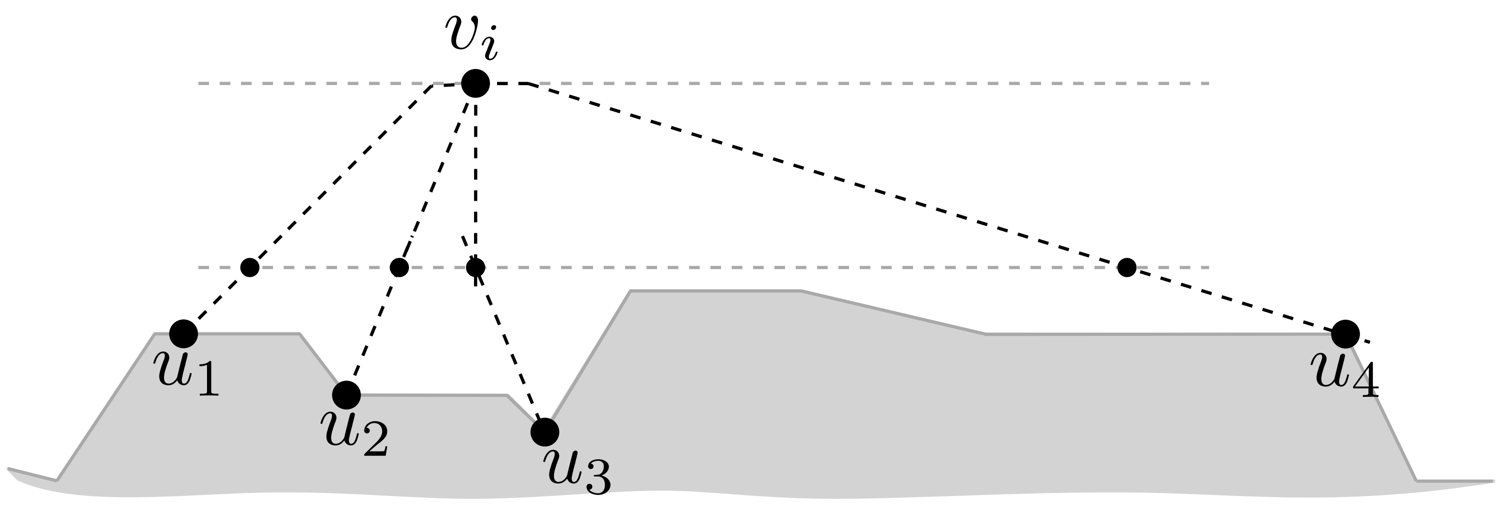

Construction of Γ

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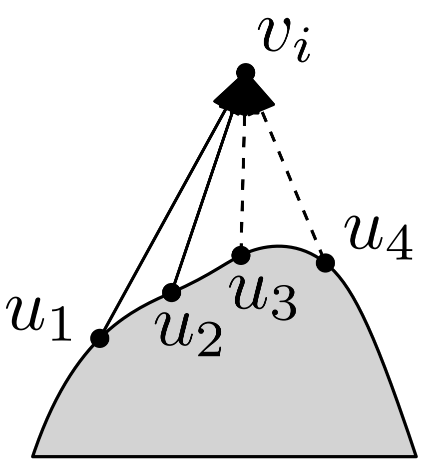
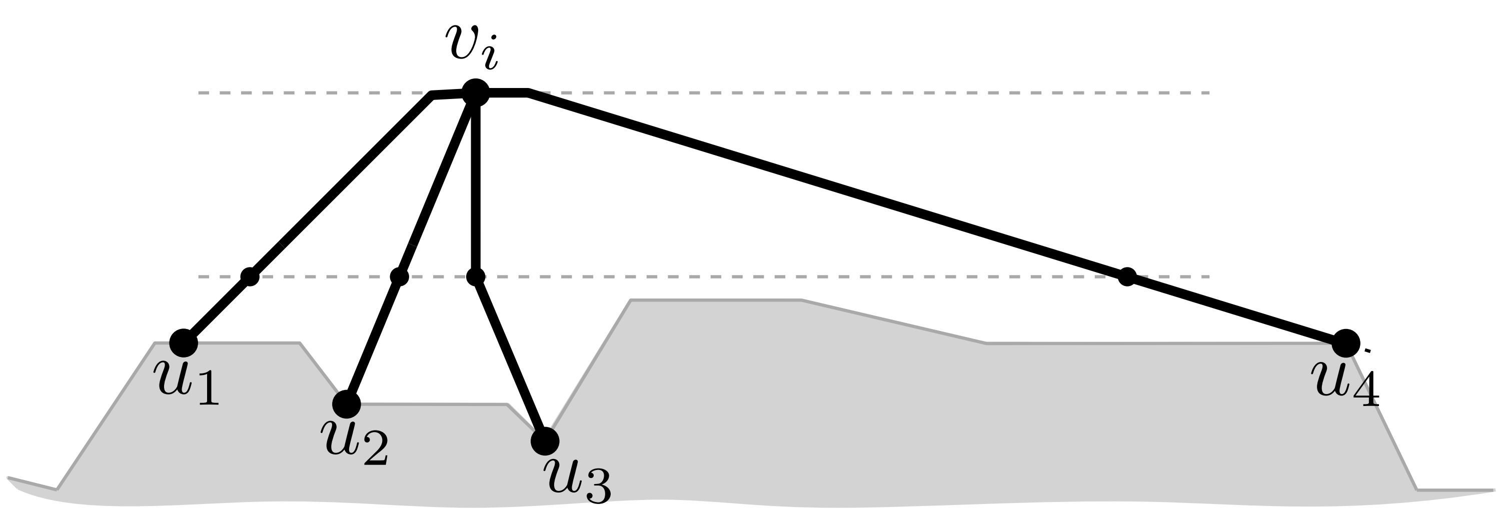
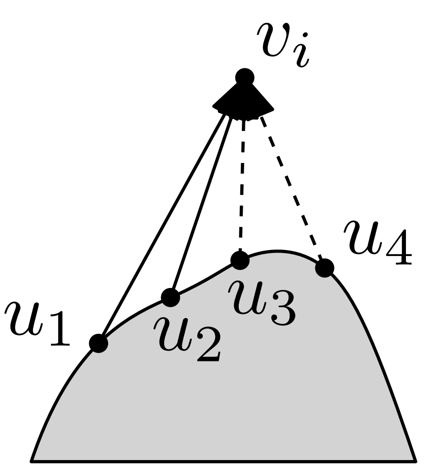


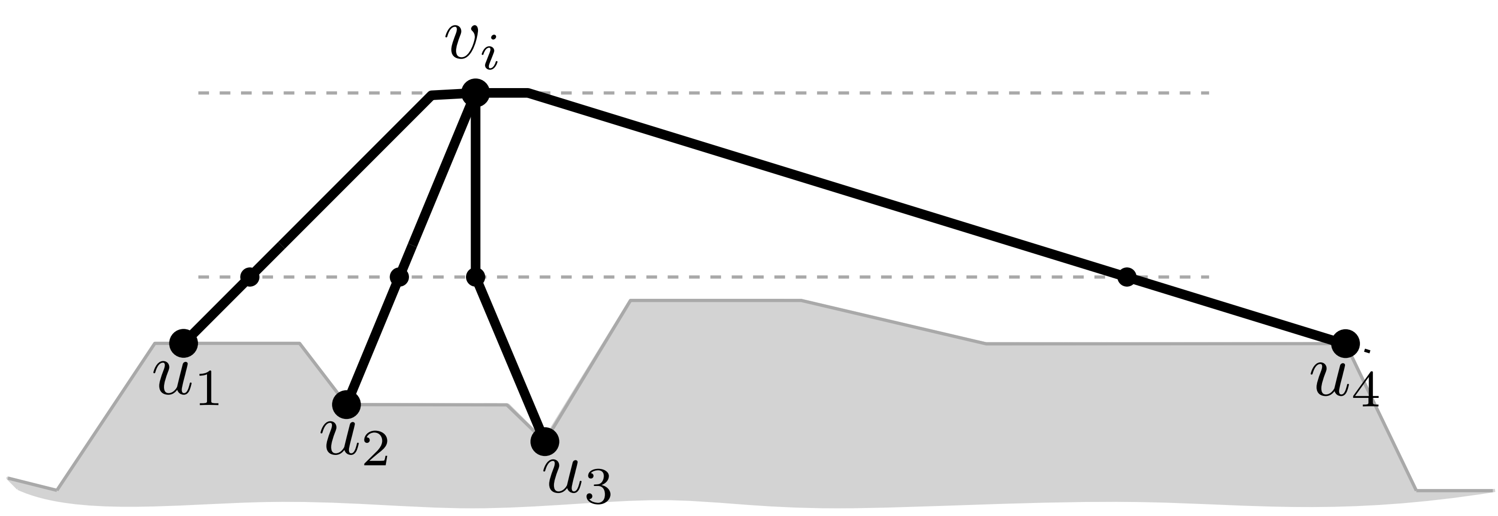


Construction of Γ

Addition of v_i $(2 < i < n)$

The computed drawing satisfies **I1-I4**





Construction of Γ

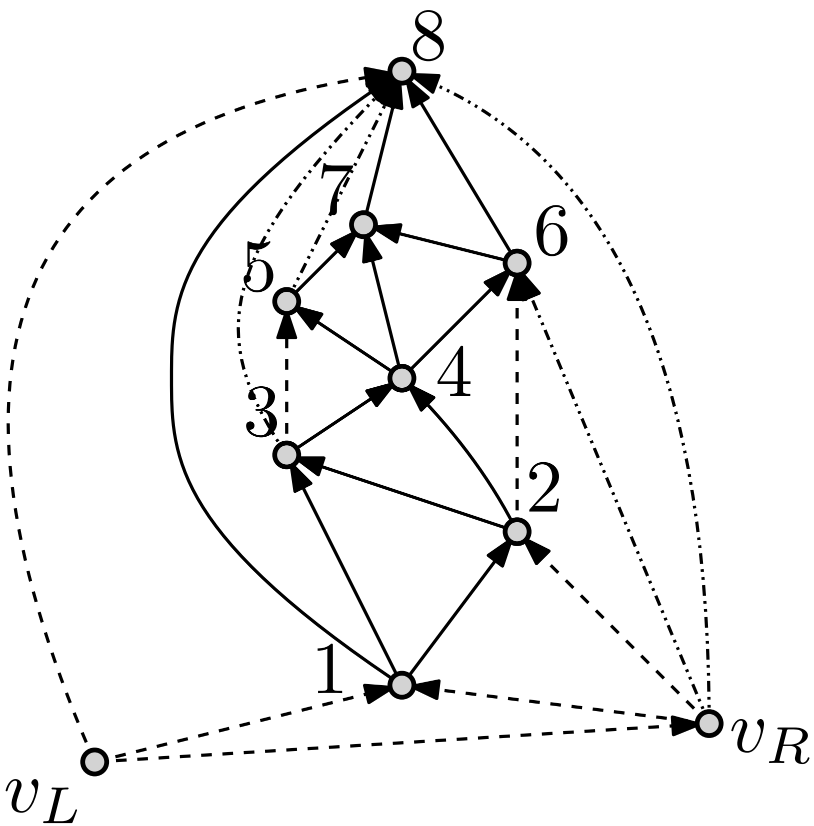
Vertex v_n can be added similarly, but in this case the number of real edges to be drawn can be up to Δ

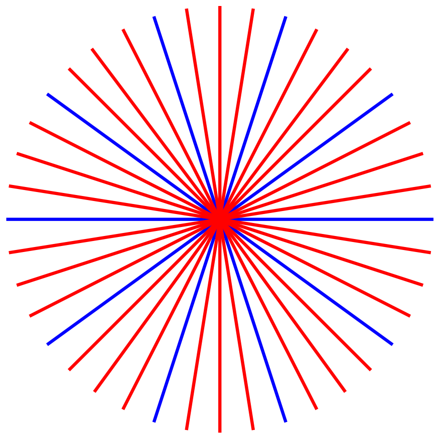
The first and the last edge are dummy, so they are supposed to use the horizontal slope

Thus there are only $\Delta - 1$ real slopes to host the Δ real edges

We modify the technique so that one real edge uses the horizontal and some dummy edges are not drawn at all

An example

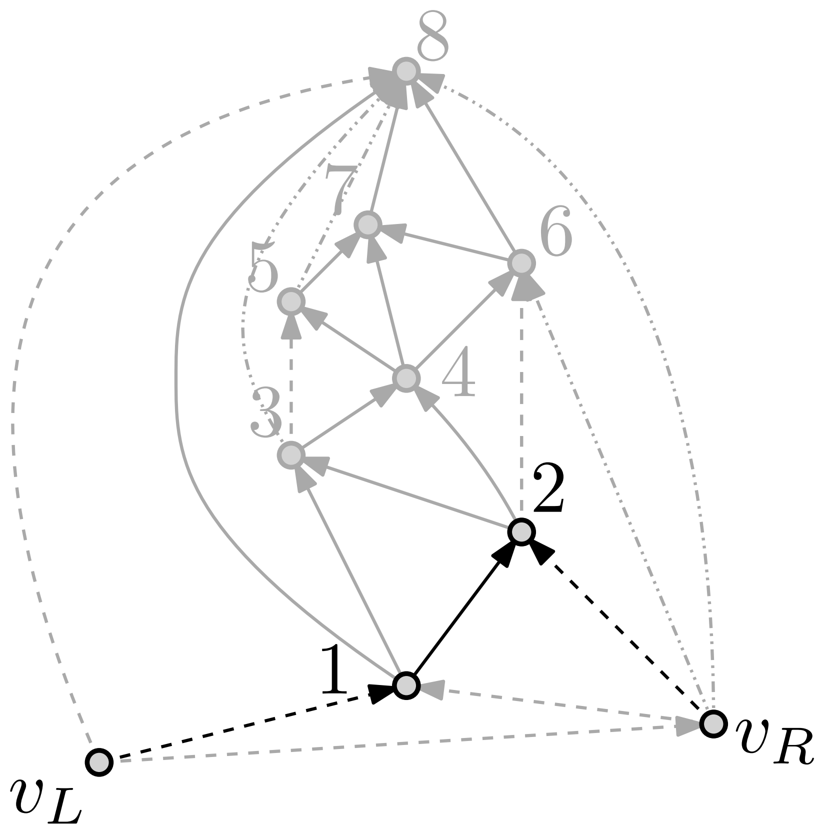


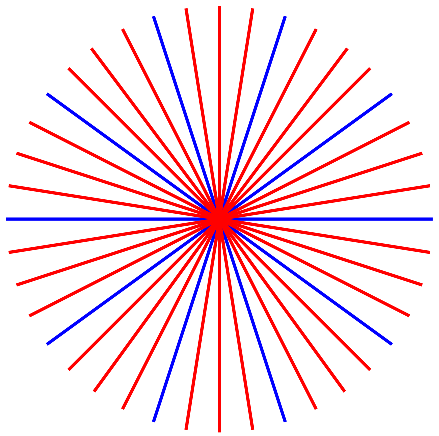


$\Delta = 5$

$\Delta^* = 3$

An example





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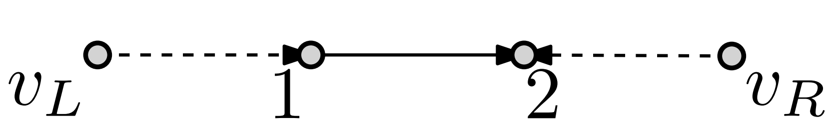
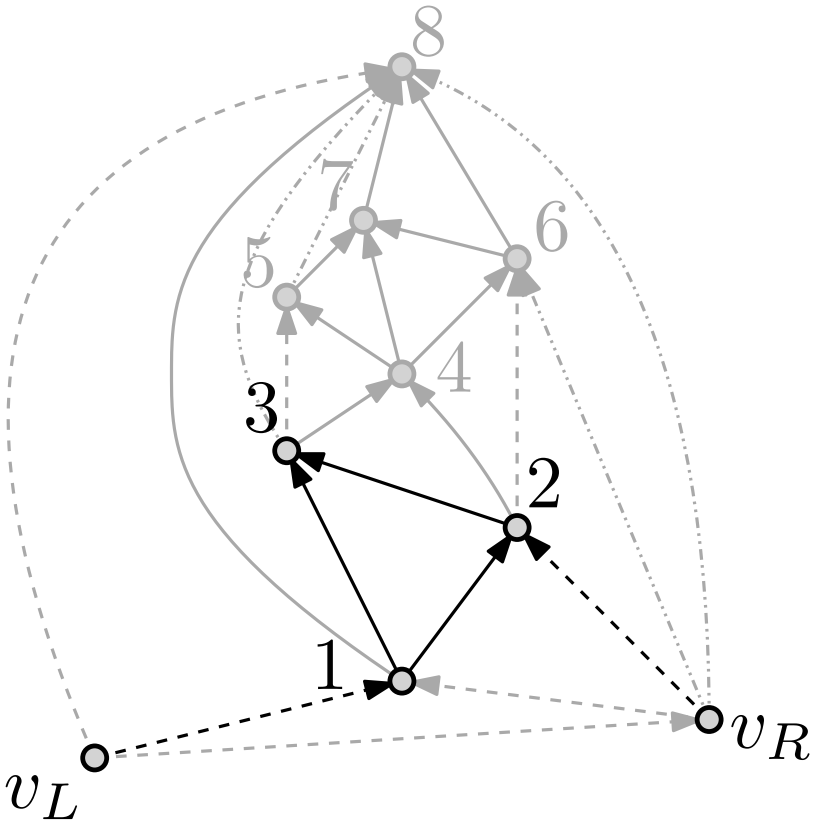
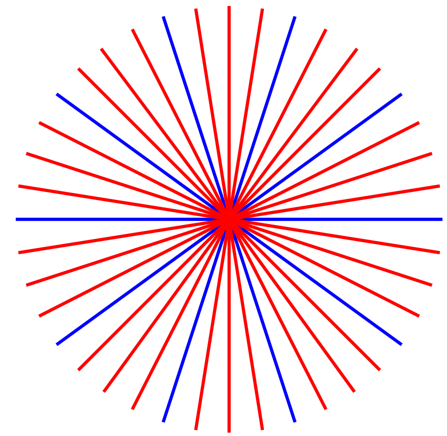


An example





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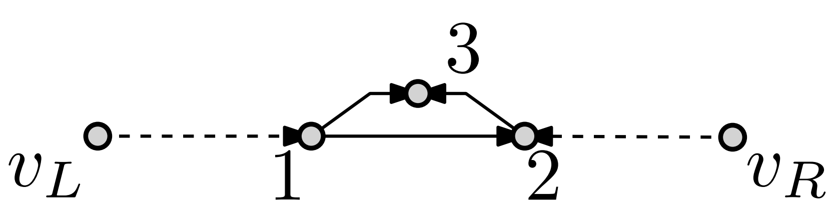

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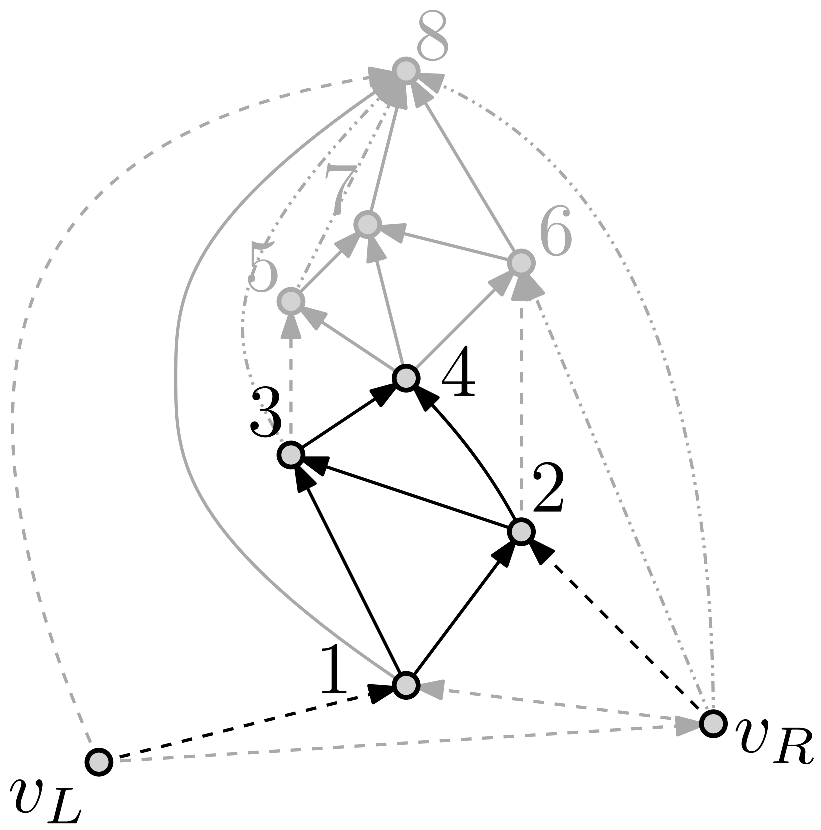

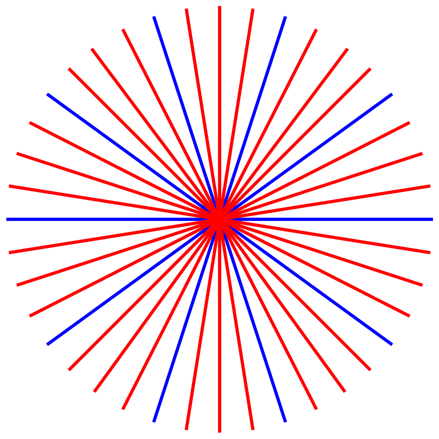

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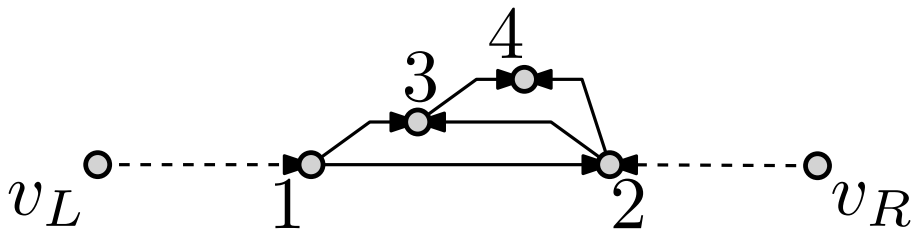

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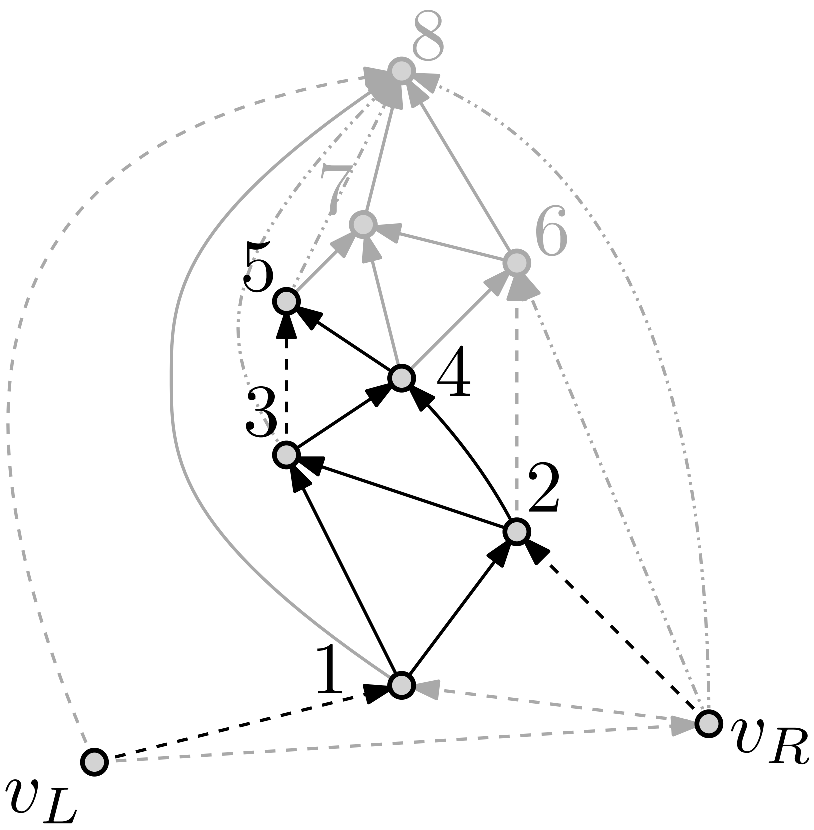

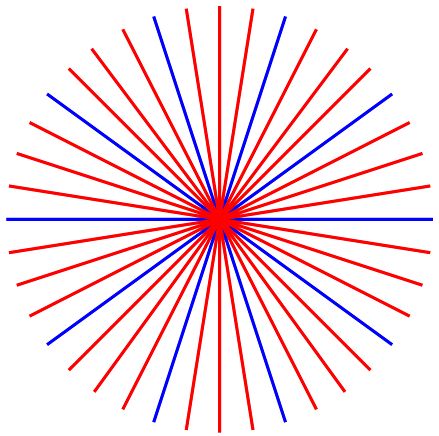

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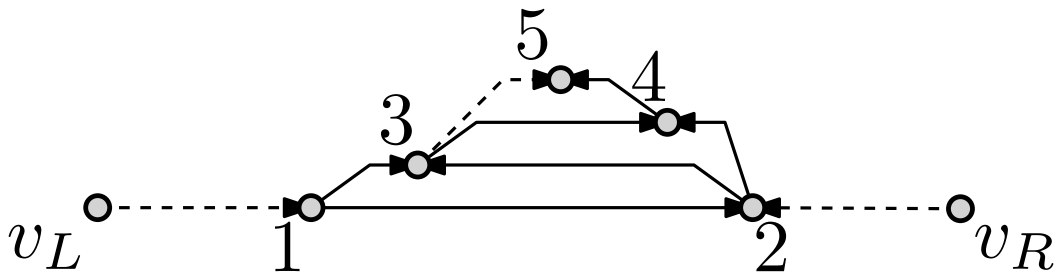

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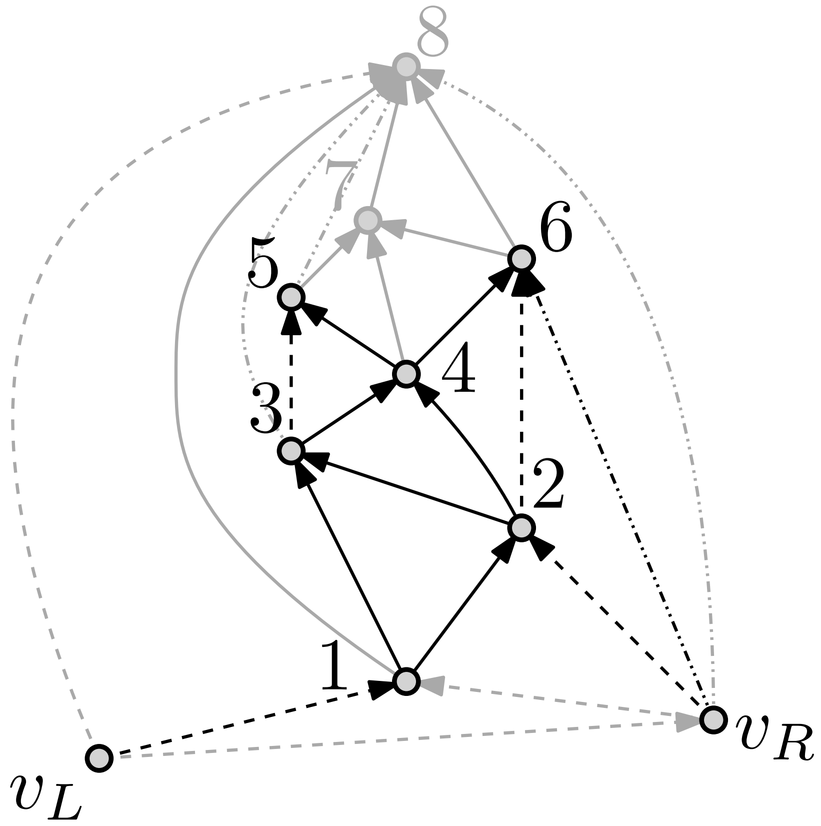

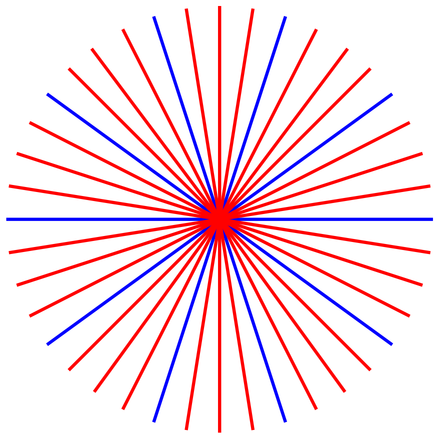

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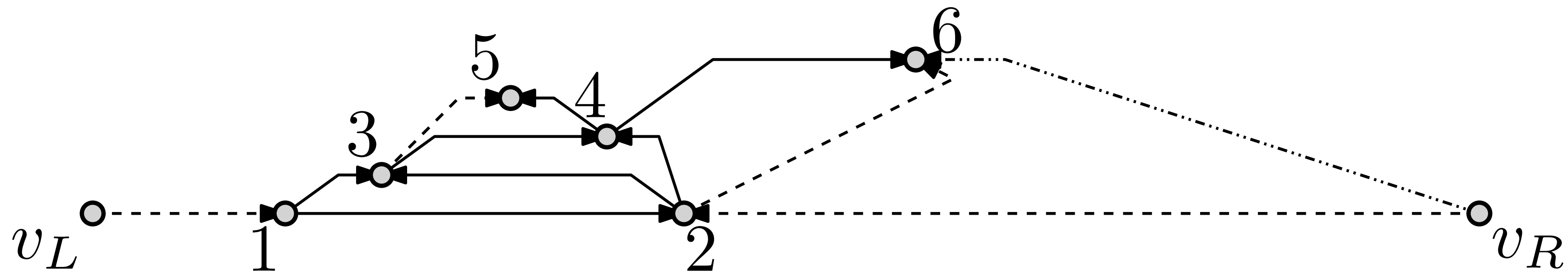
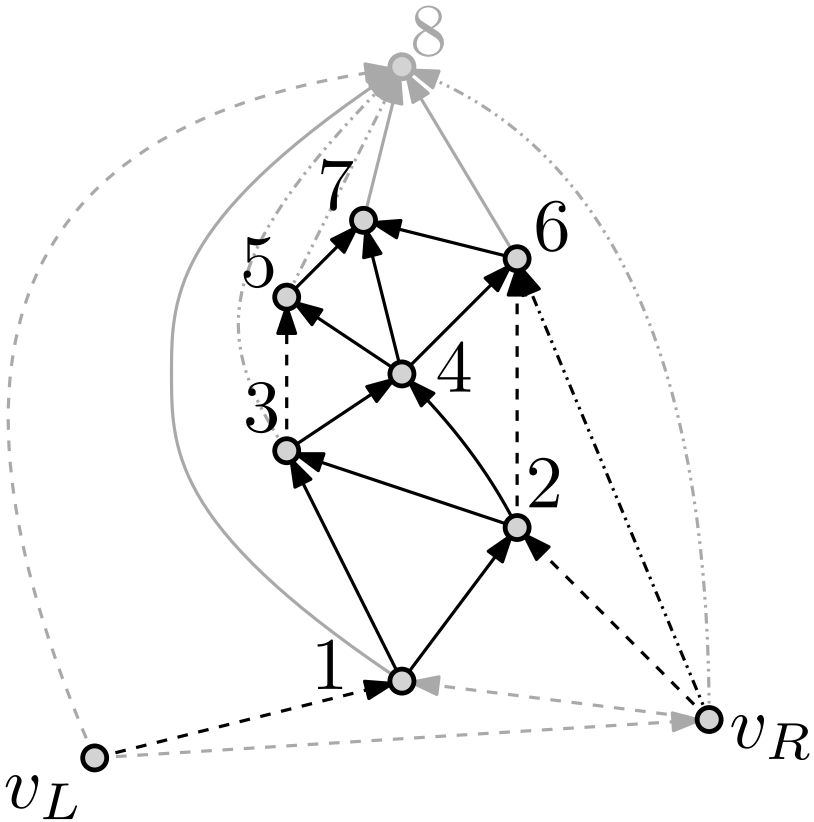
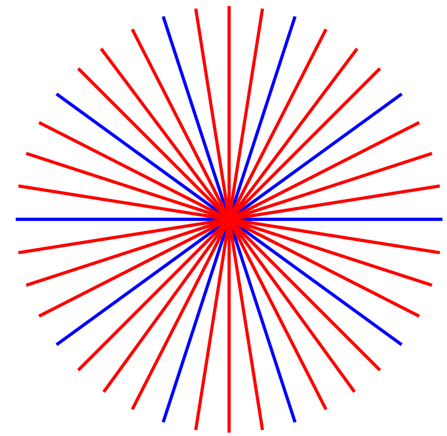


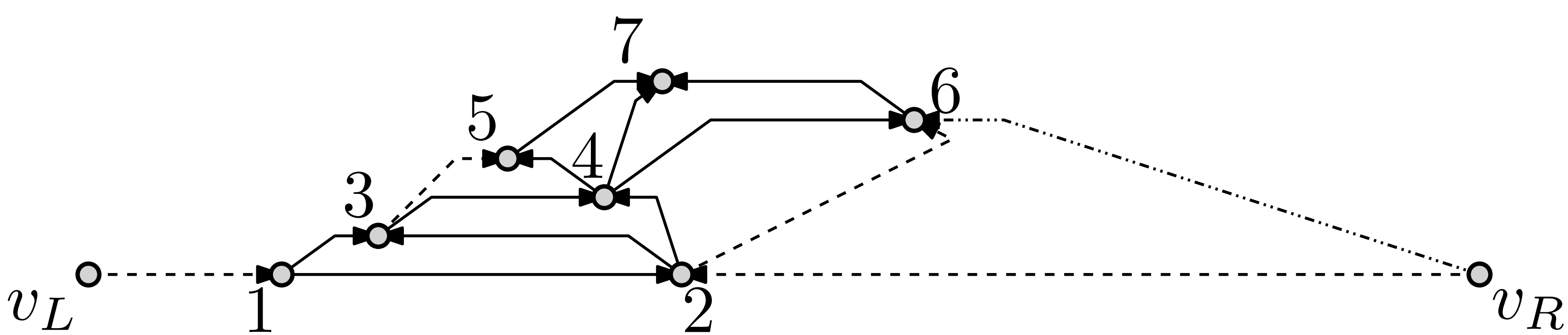
An example





$\Delta = 5$

$\Delta^* = 3$



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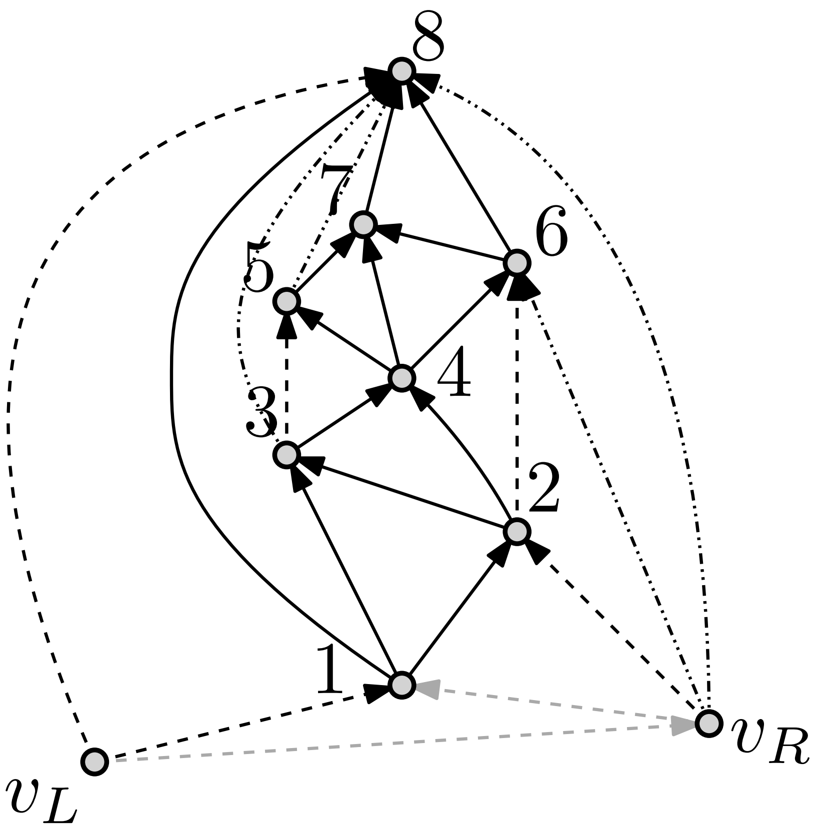

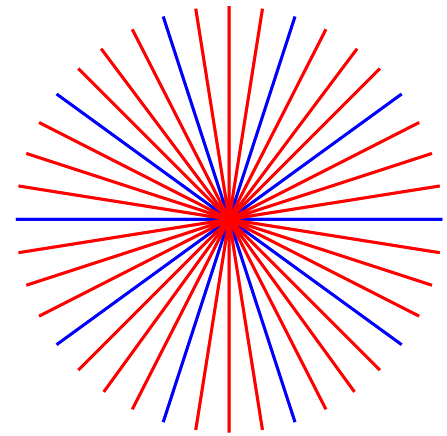

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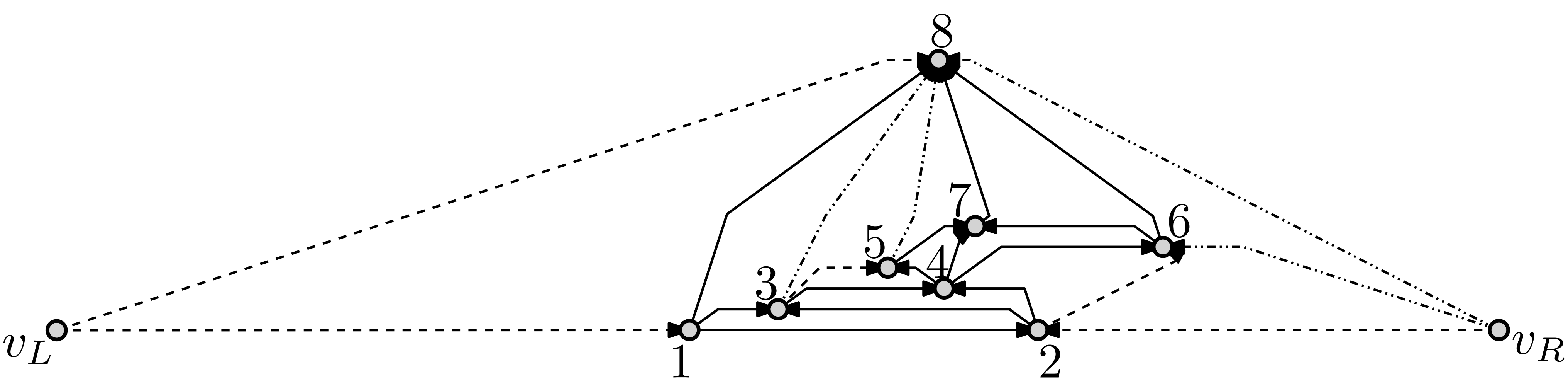

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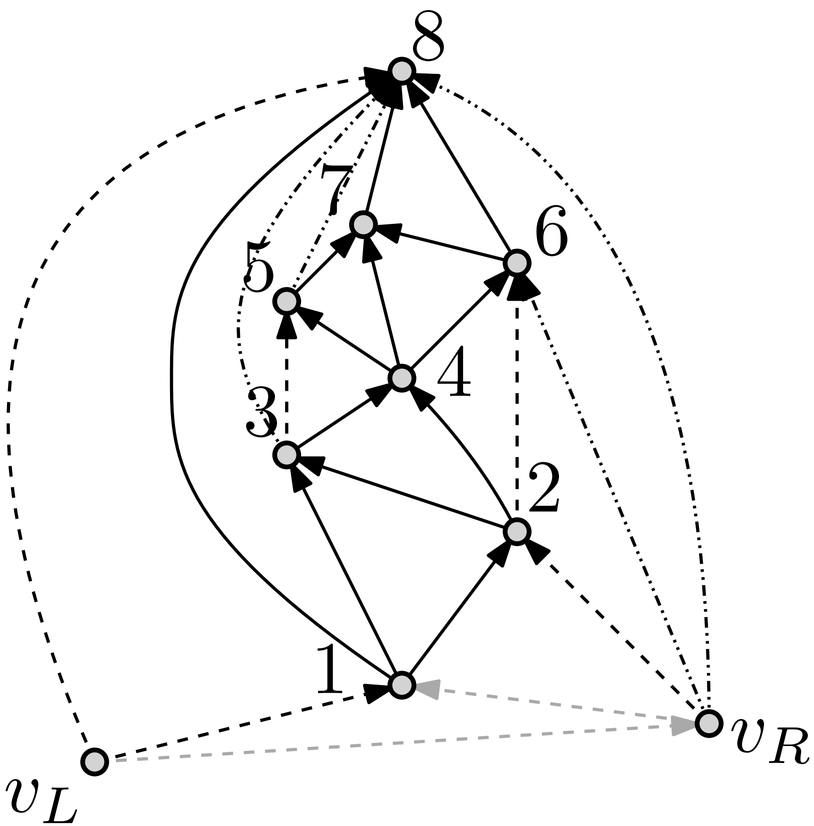

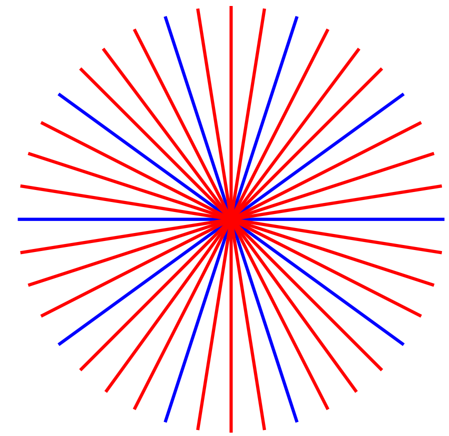

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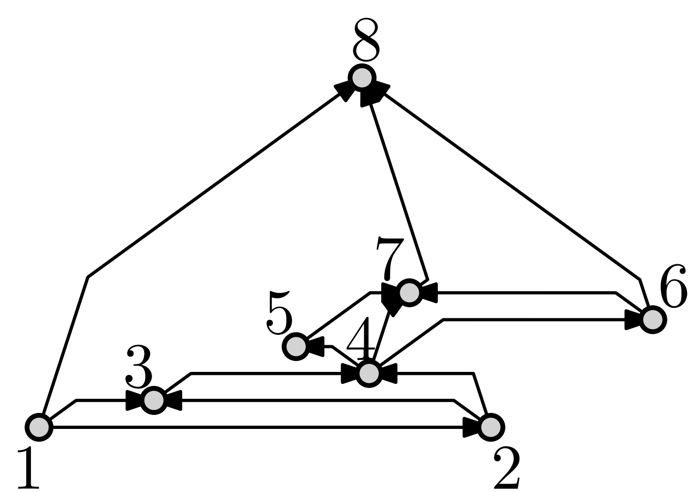

Open problems

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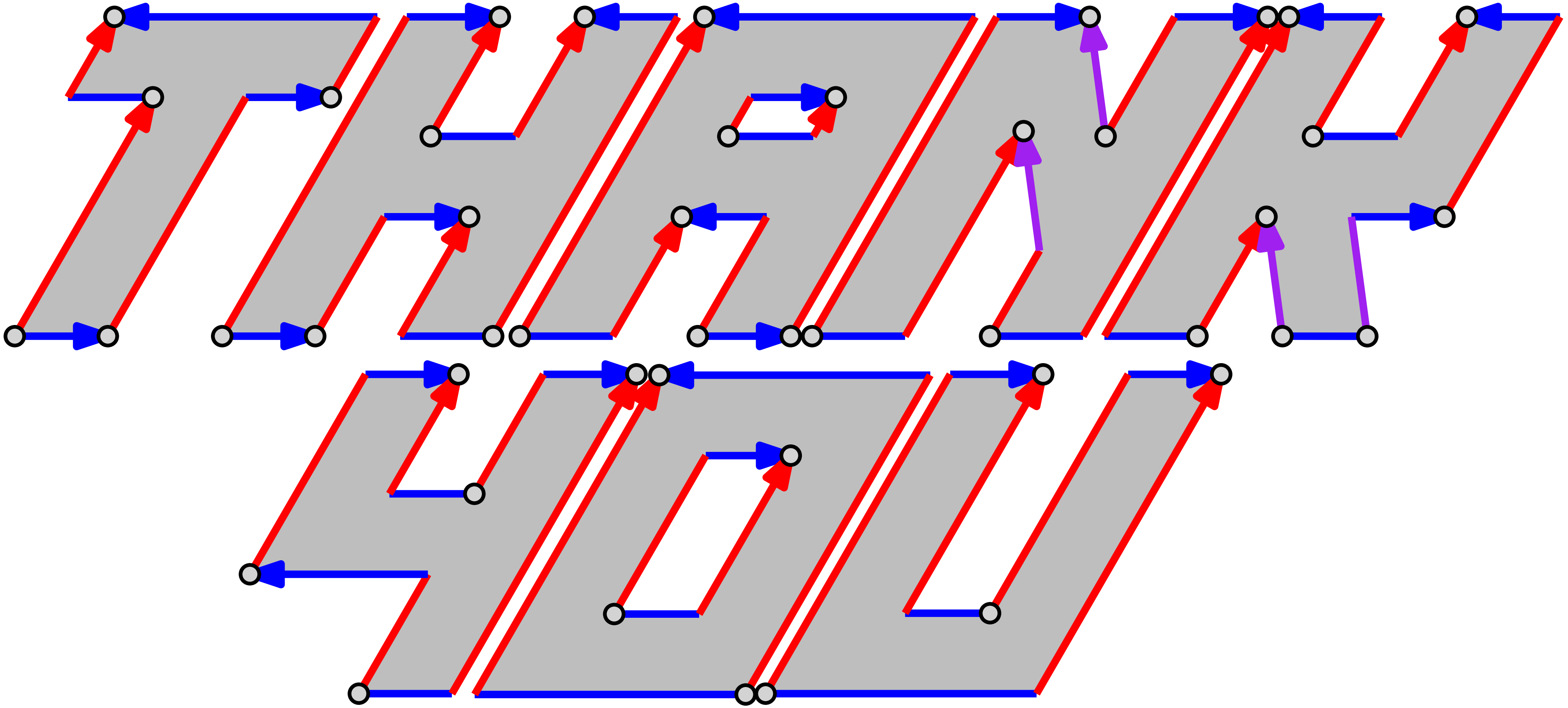
For upward planar digraphs we proved an upper bound of Δ for 2-bend drawings.

- What is the straight-line upward planar slope number of upward planar digraphs?



$$\mathrm{upsn}_1(\text{Thank you}) \leq 3$$