Arrangements of Pseudocircles:

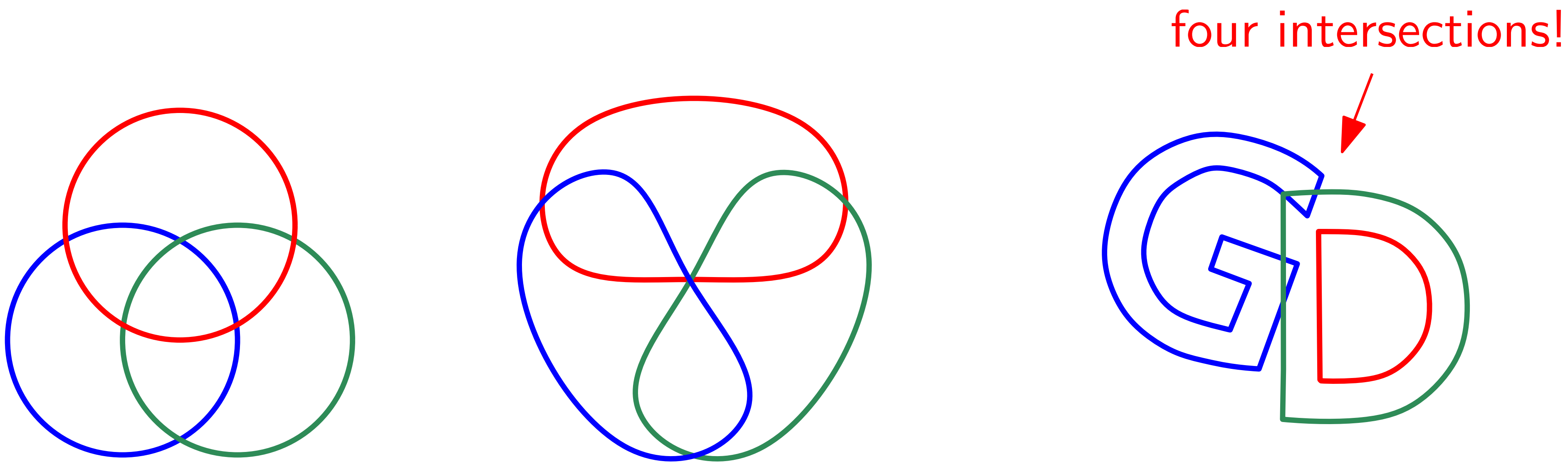
On Circularizability

Stefan Felsner and Manfred Scheucher

Definitions

pseudocircle . . . simple closed curve

arrangement . . . collection of pcs. s.t. intersection of any two pcs. either empty or 2 points where curves cross



arrangement arrangement no arrangement

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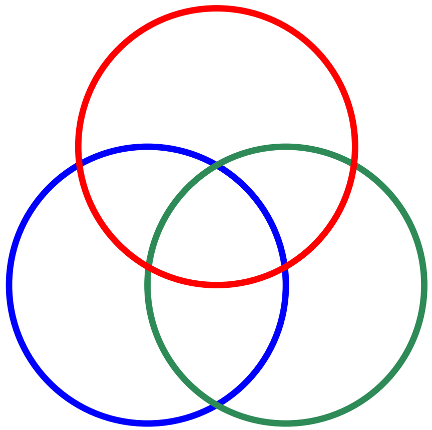
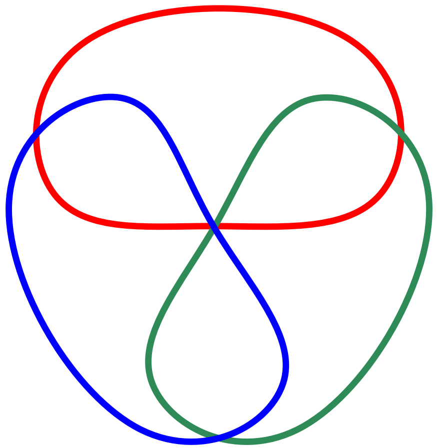


arrangement



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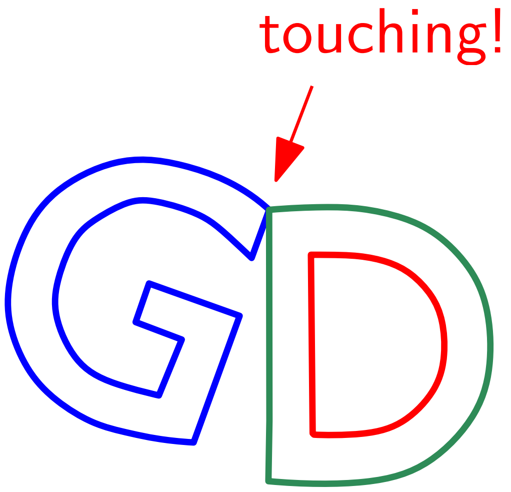

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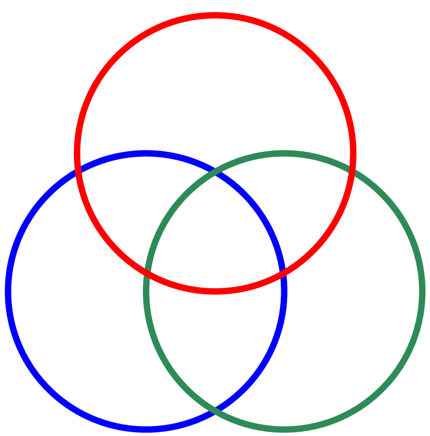

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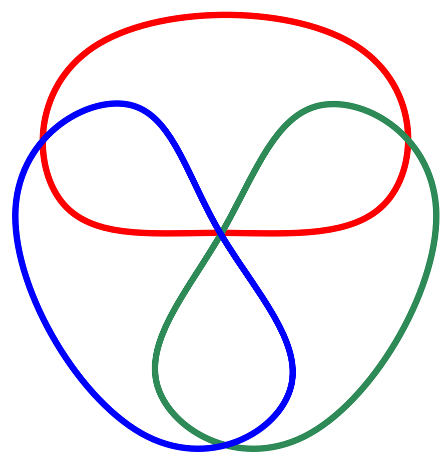

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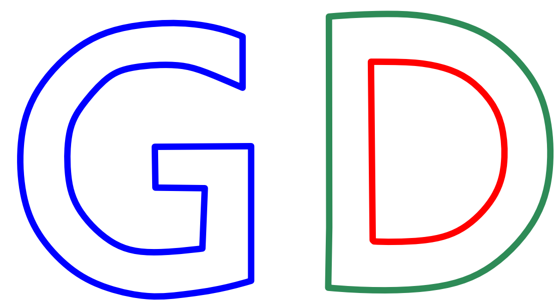
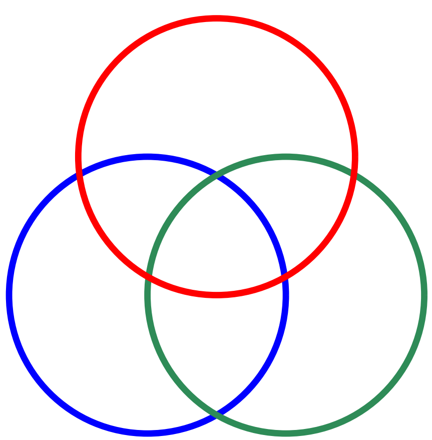


arrangement

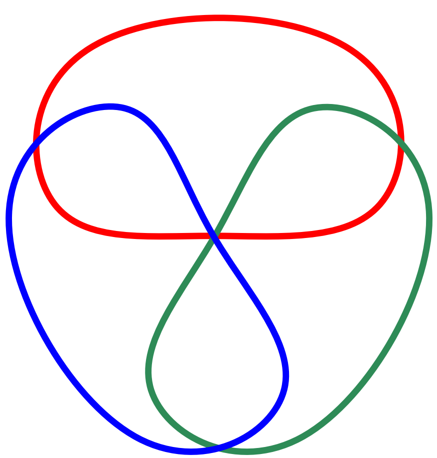
Definitions

simple . . . no 3 pcs. intersect in common point

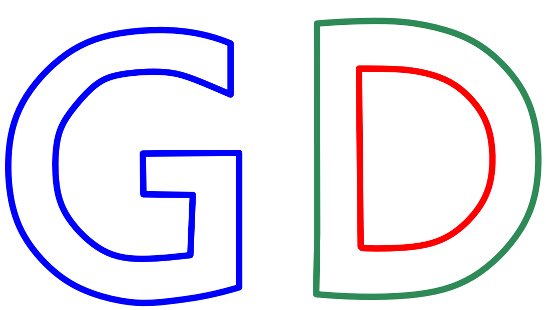
connected . . . intersection graph is connected



simple+connected



not simple



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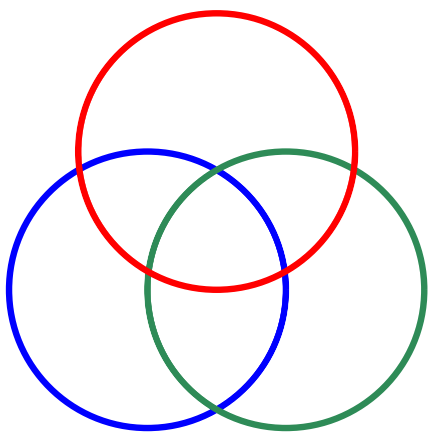

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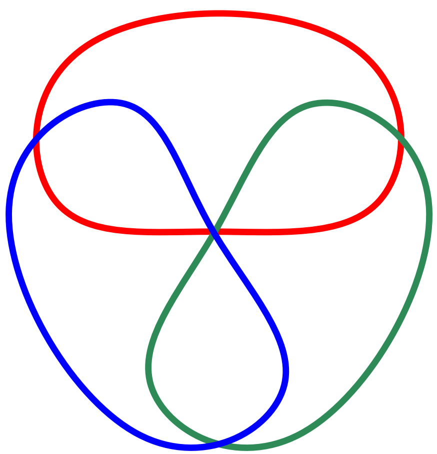

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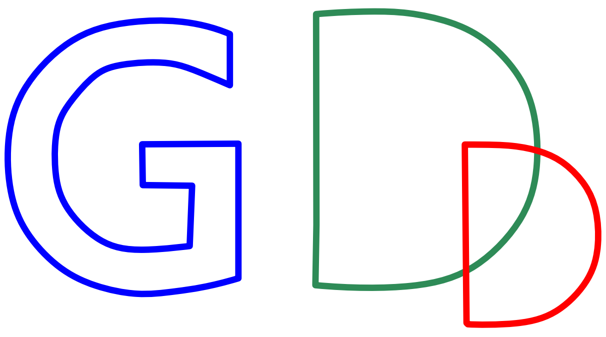

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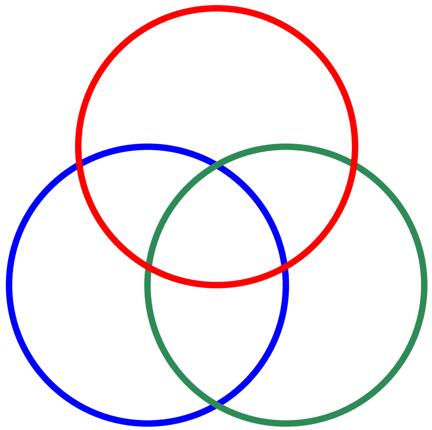
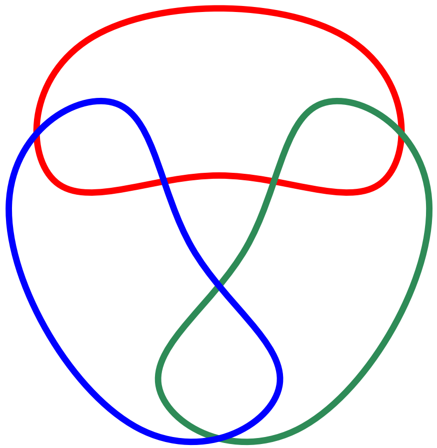
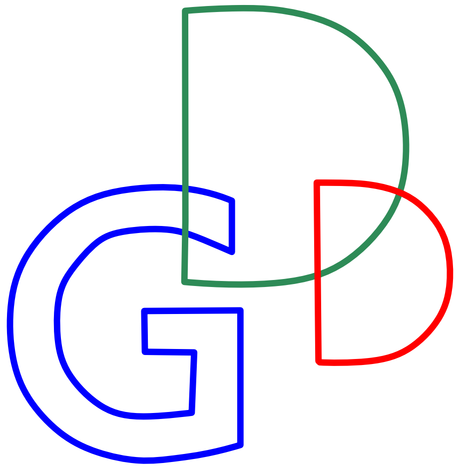
assumptions throughout presentation







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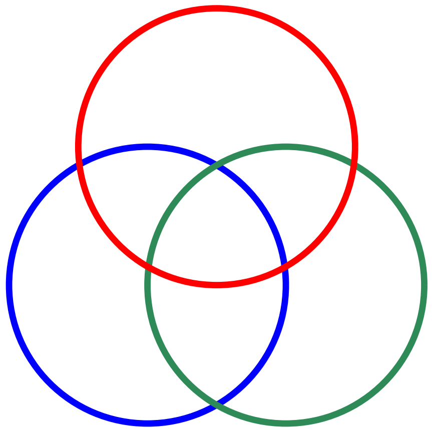
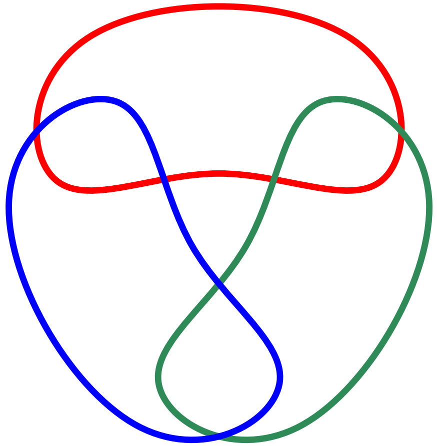
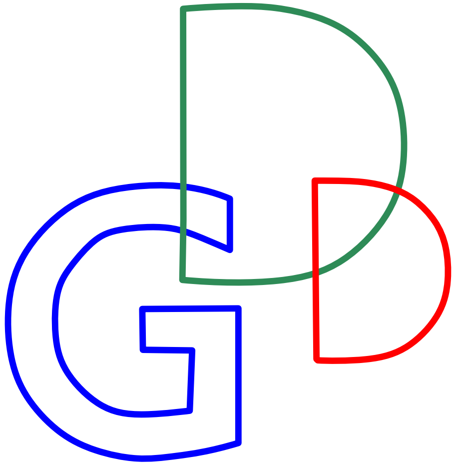


Krupp



NonKrupp



3-Chain

Definitions

simple . . . no 3 pcs. intersect in common point

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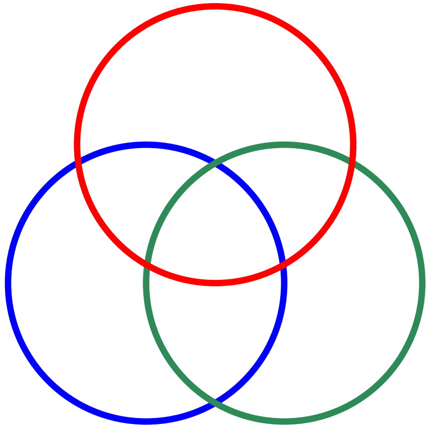
assumptions throughout presentation

circularizable . . . ∃ isomorphic arrangement of circles



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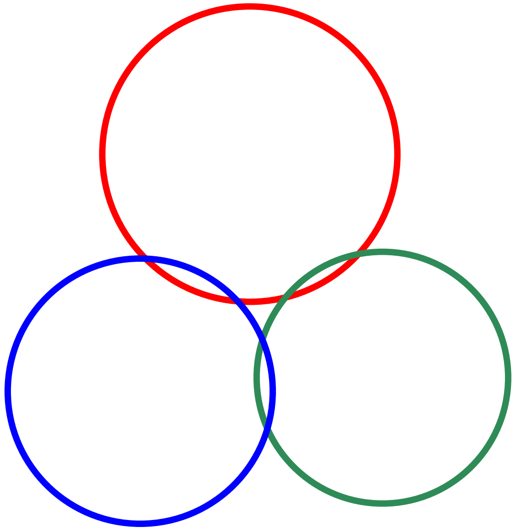

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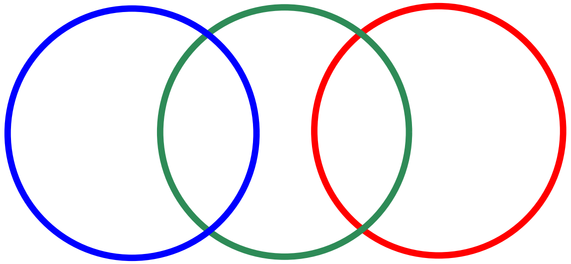


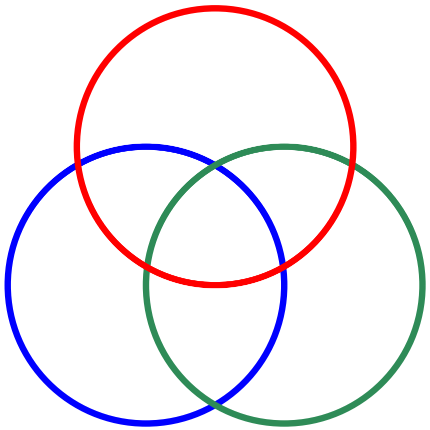
3-Chain

Classes of Arrangements

connected . . . graph of arrangement is connected

↑

intersecting . . . any 2 pseudocircles cross twice



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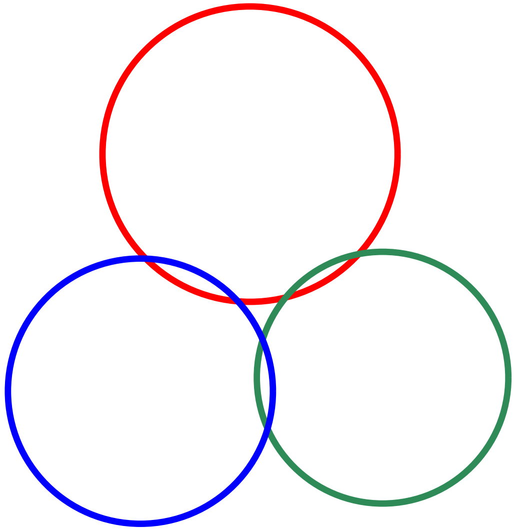

NonKrupp

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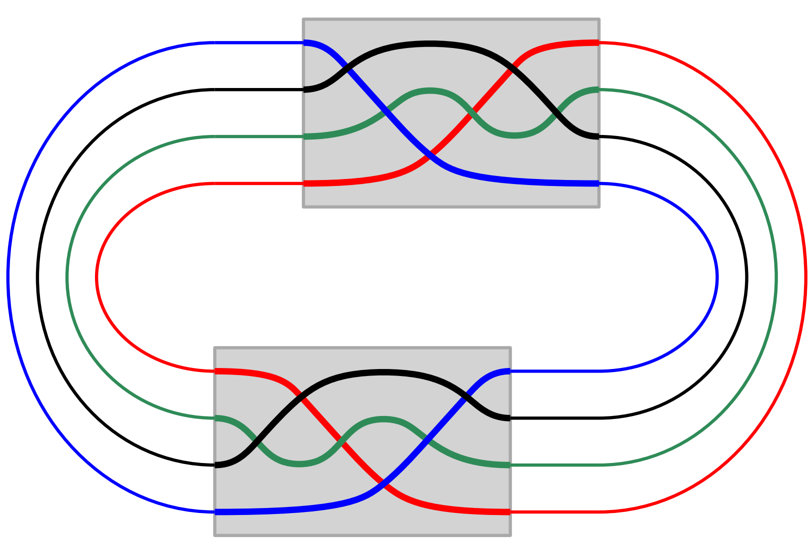

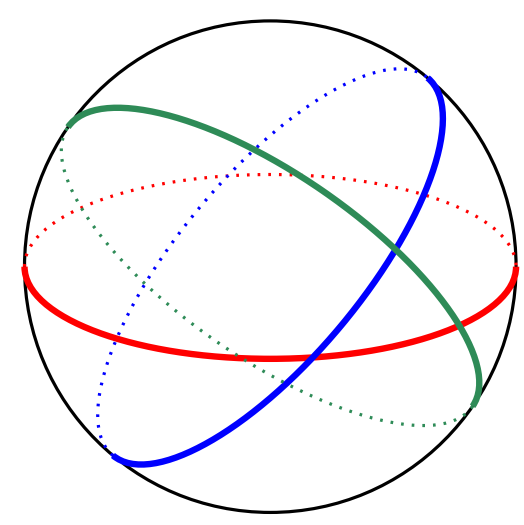

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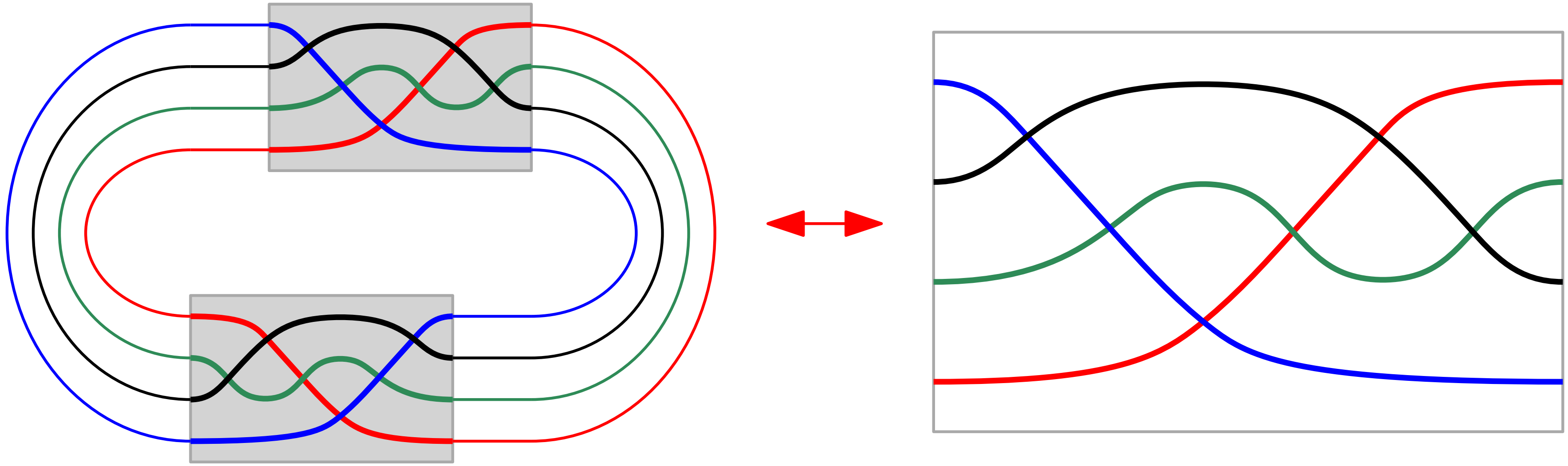

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digon-free . . . no cell bounded by two pcs.

Classes of Arrangements

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cylindrical . . . $\exists$ two cells separated by each of the pcs.

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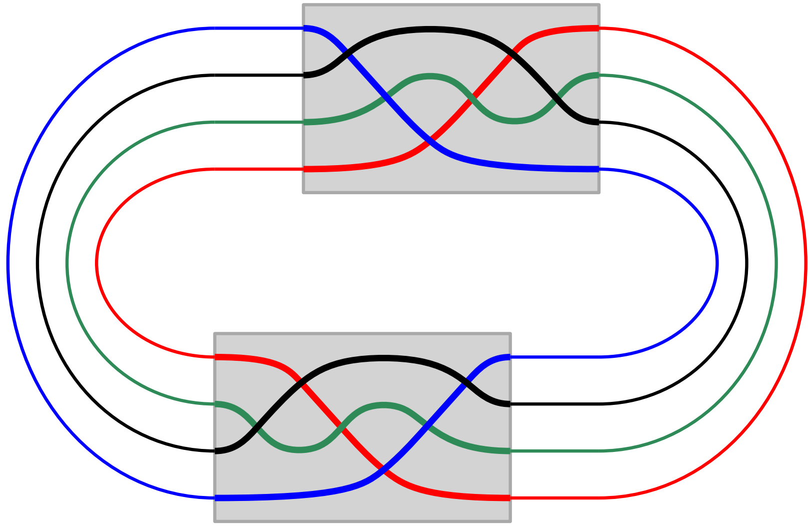

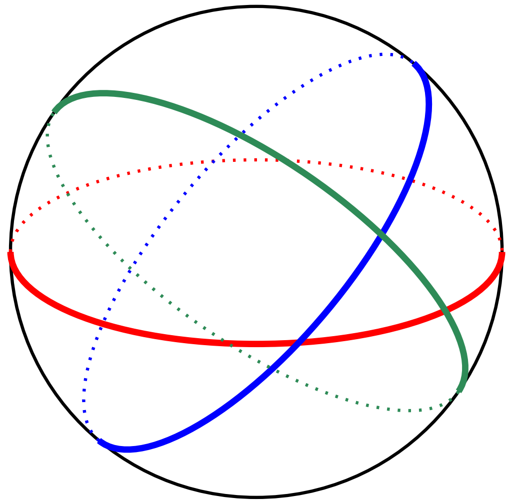

Enumeration of Arrangements

n	3	4	5	6	7
connected	3	21	984	609 423	?
+digon-free	1	3	30	4 509	?
intersecting	2	8	278	145 058	447 905 202
+digon-free	1	2	14	2 131	3 012 972
great-p.c.s	1	1	1	4	11

Table: # of combinatorially different aragements of n pcs.

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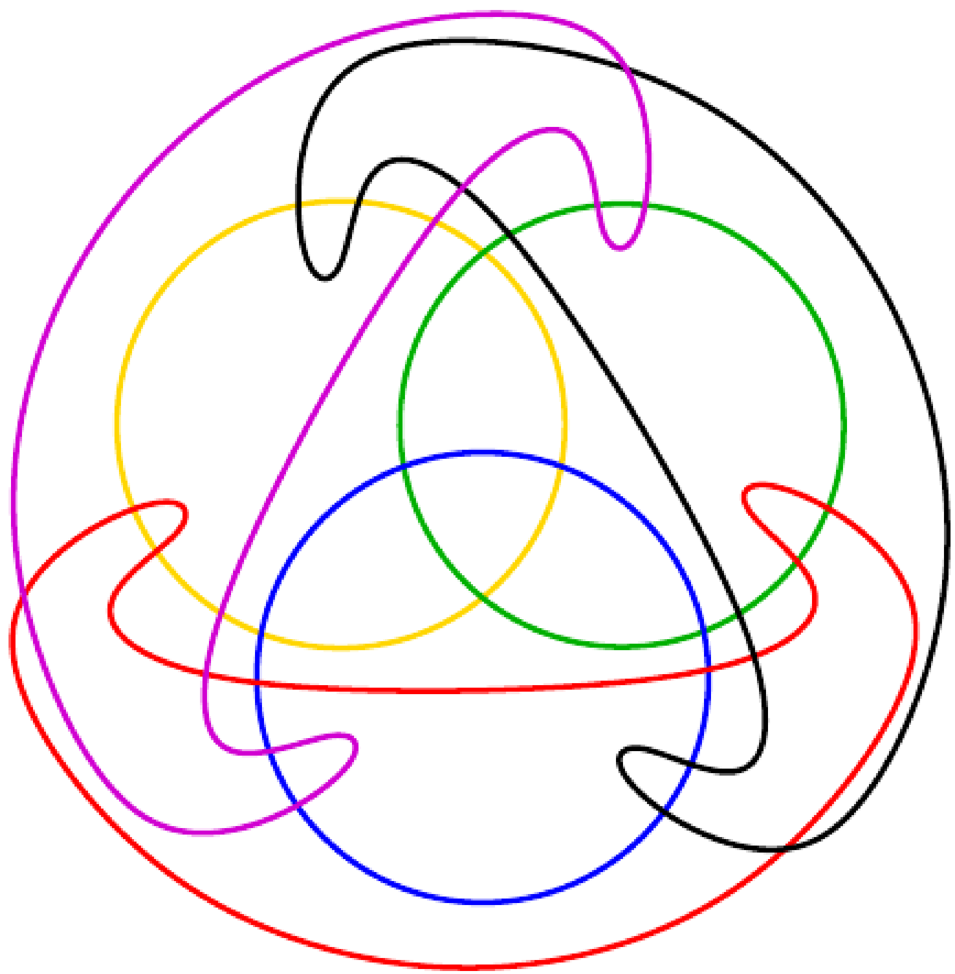
Table: # of combinatorially different arragements of n pcs.

arrangements of pseudocircles: $2^{\Theta(n^2)}$

arrangements of circles: $2^{\Theta(n \log n)}$

Circularizability Results

- non-circularizability of intersecting $n = 6$ arrangement [Edelsbrunner and Ramos '97]



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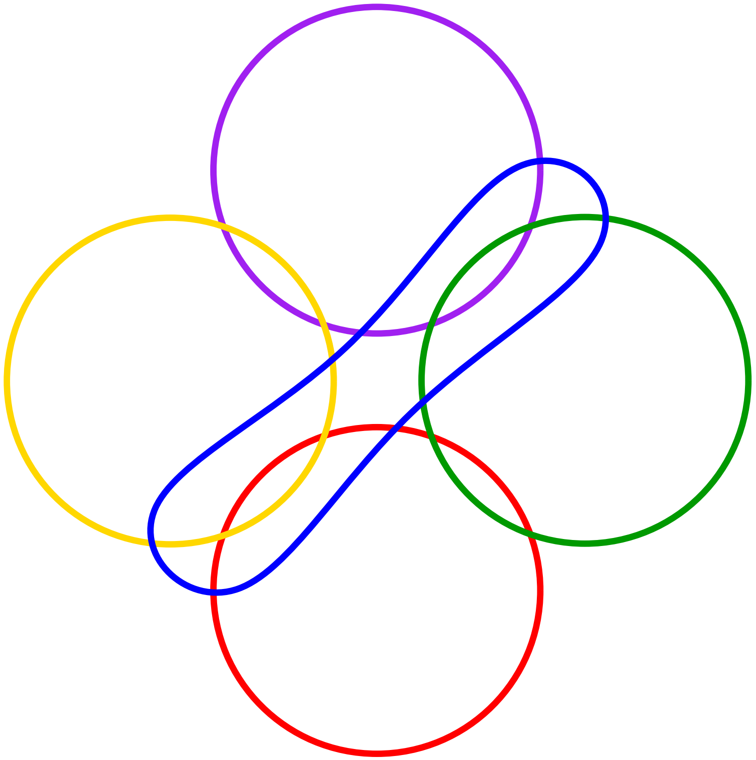

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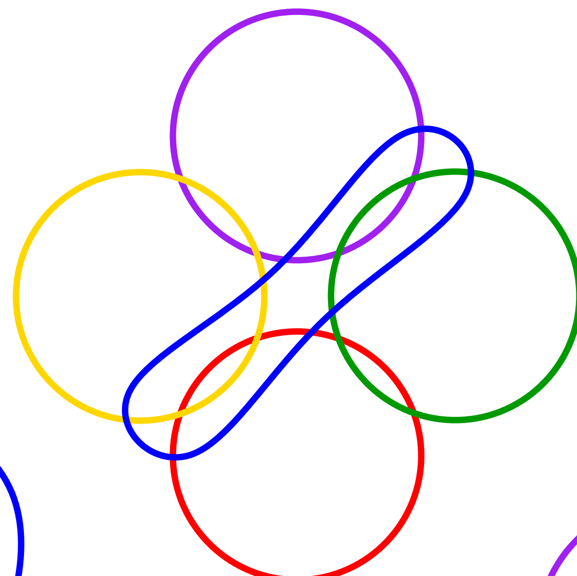
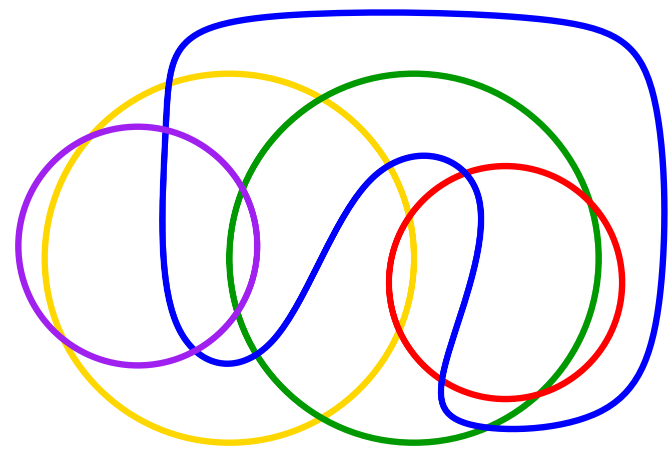
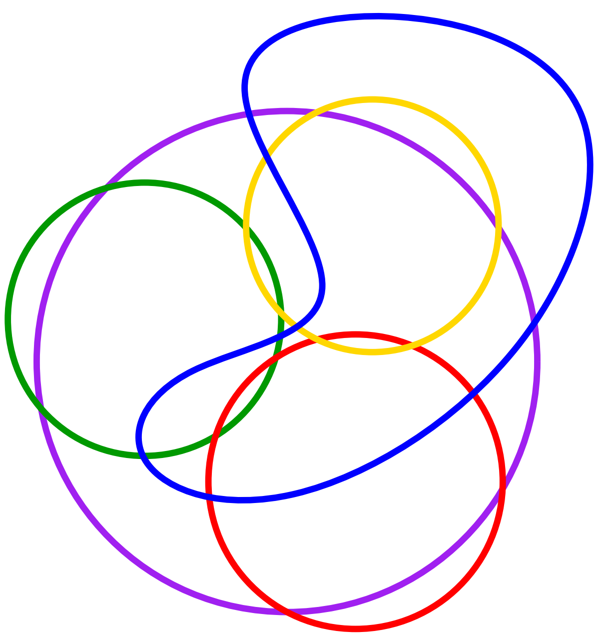
- NP-hardness of circularizability [Kang and Müller '14]

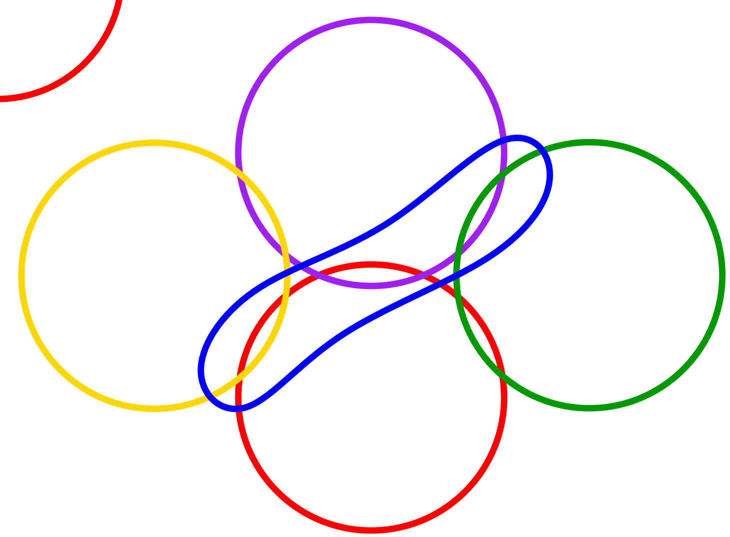
Circularizability Results

Theorem. There are exactly 4 non-circularizable $n = 5$ arrangements (984 classes).









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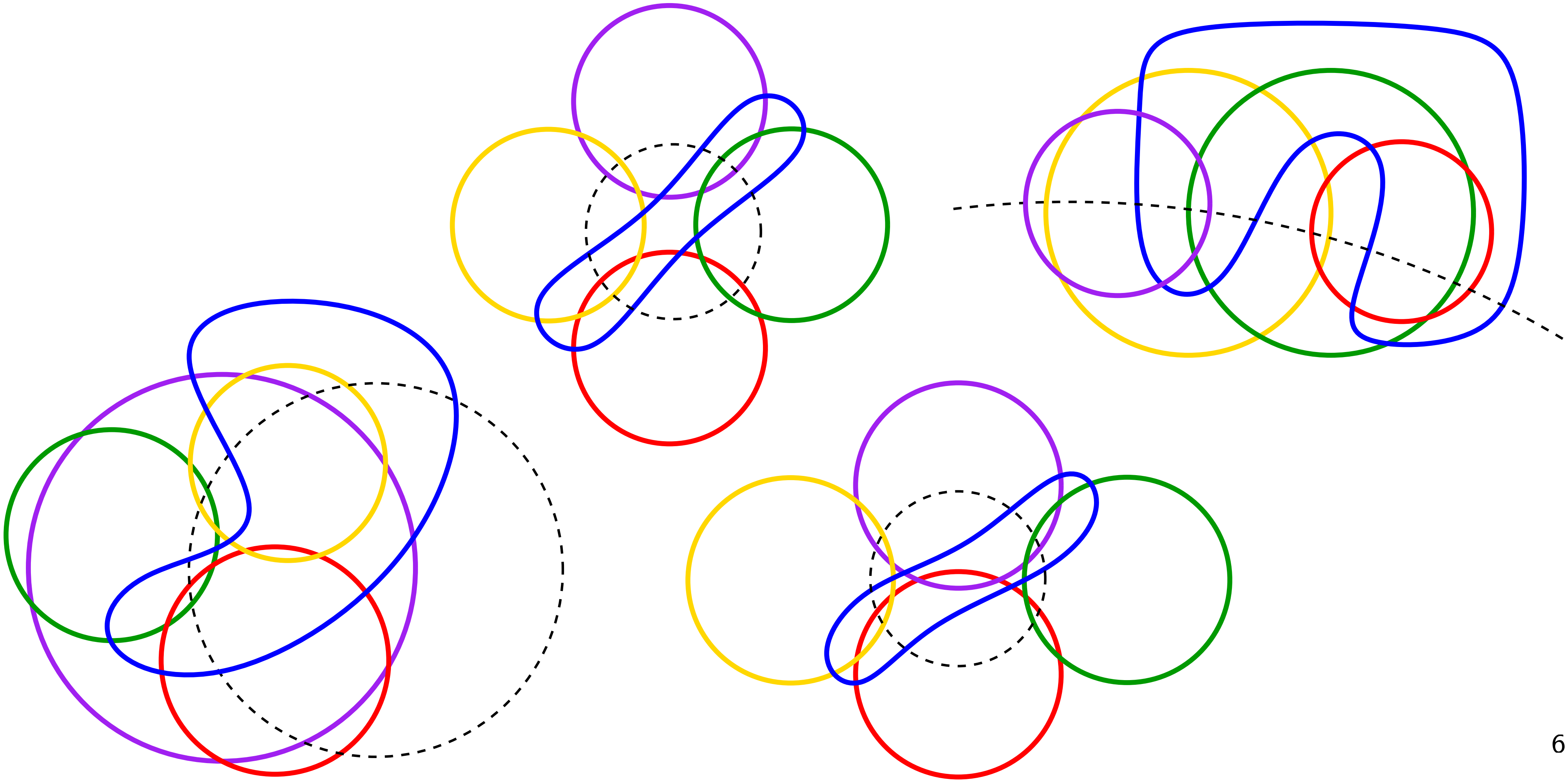
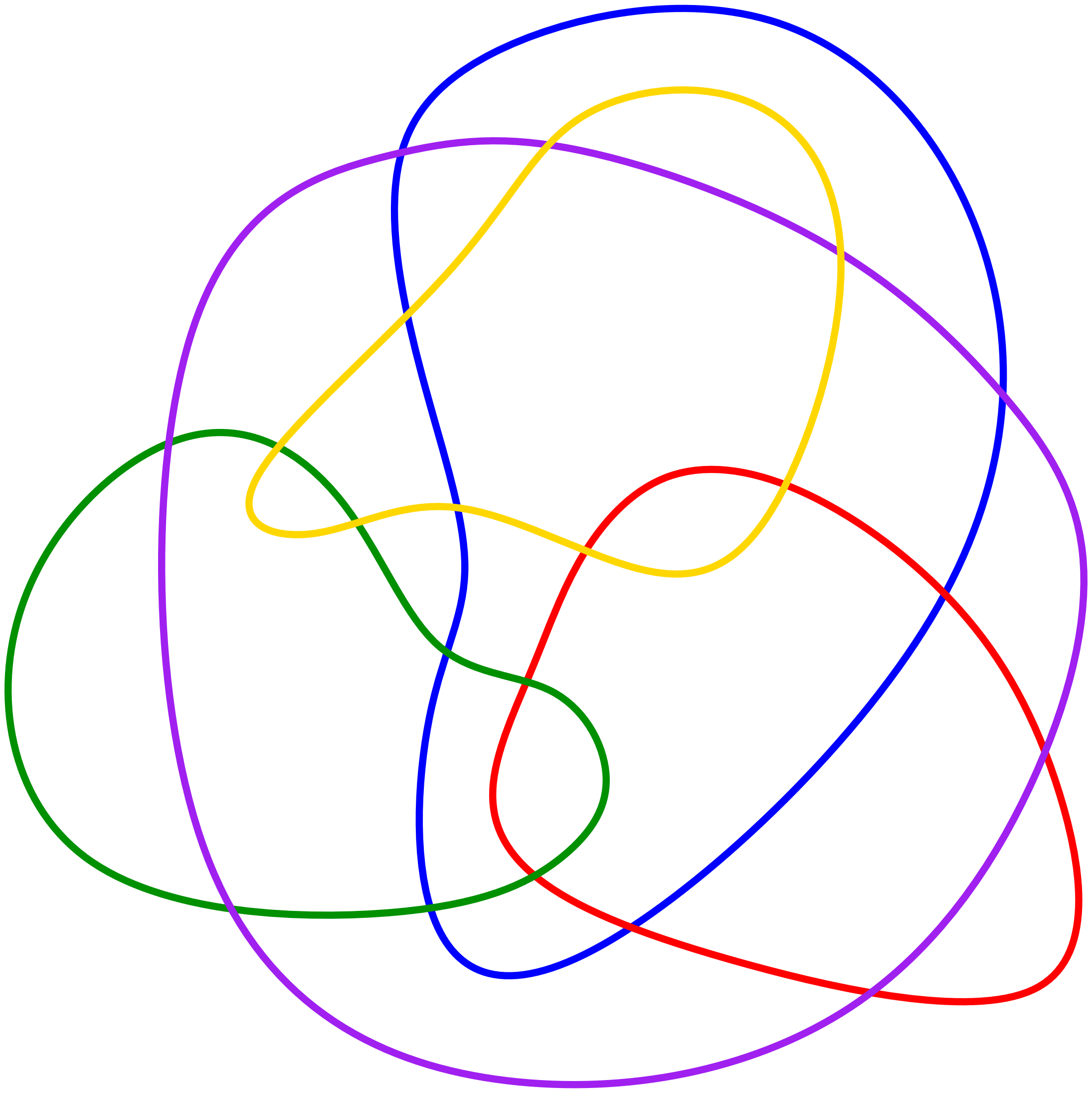
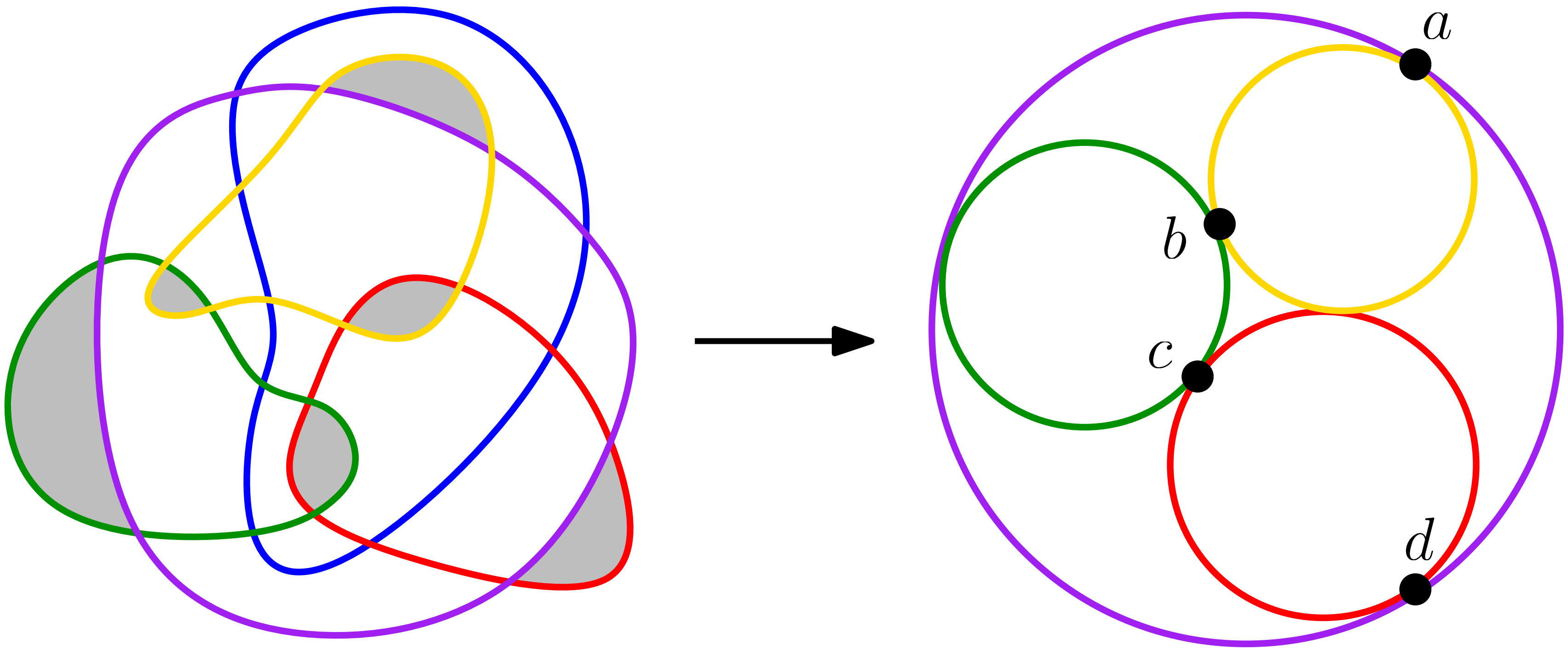


Noncircularizability of $\mathcal{N}_5^1$



Noncircularizability of $\mathcal{N}_5^1$

- assume there is a circle representation of $\mathcal{N}_5^1$
- shrink the yellow, green, and red circle
- cyclic order is preserved (also for blue)



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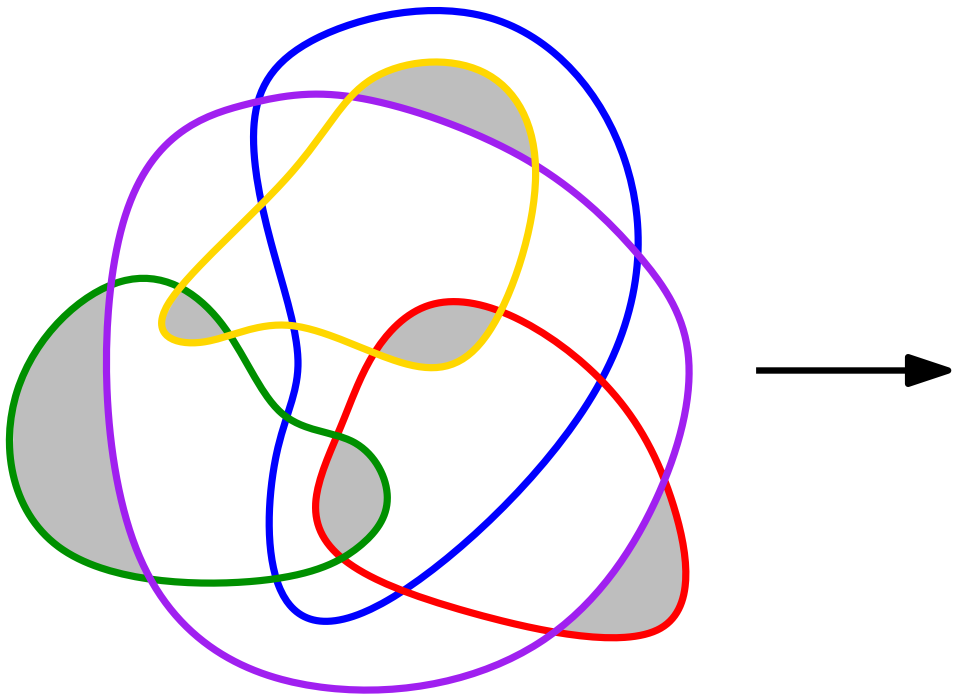

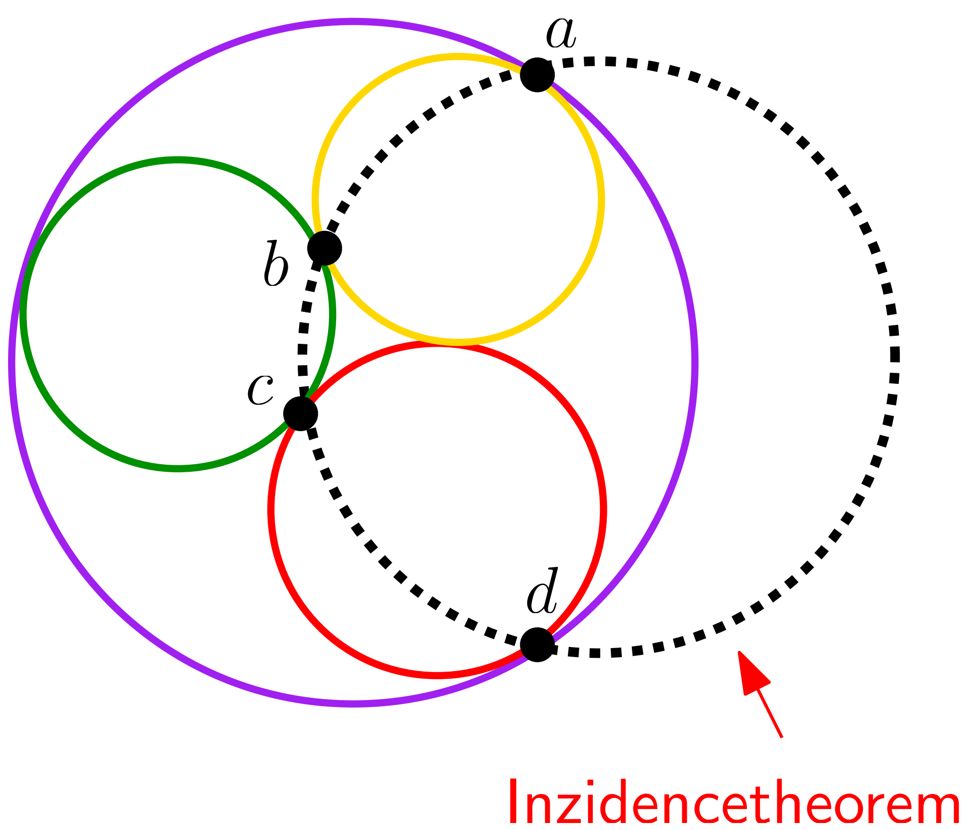

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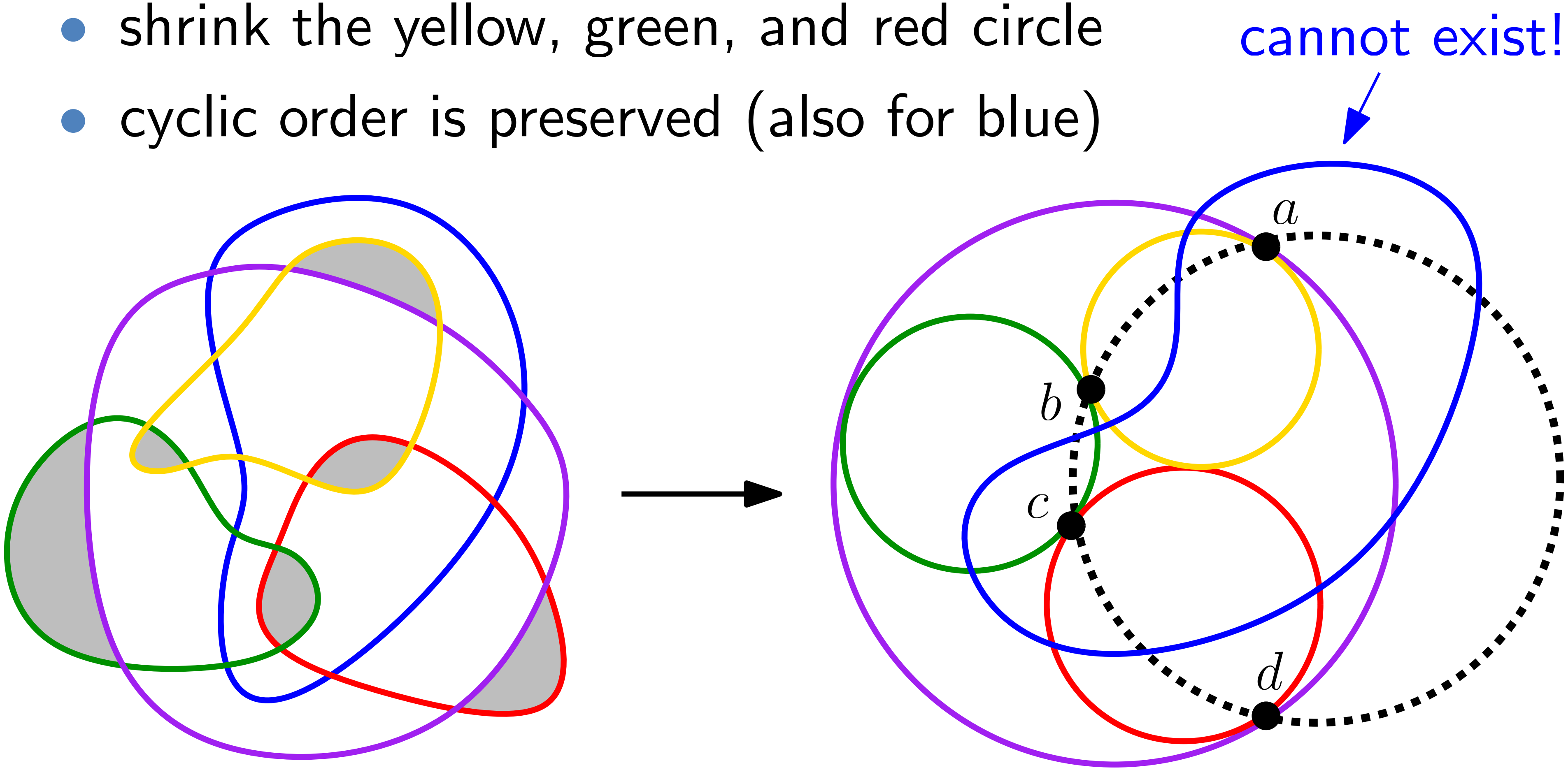
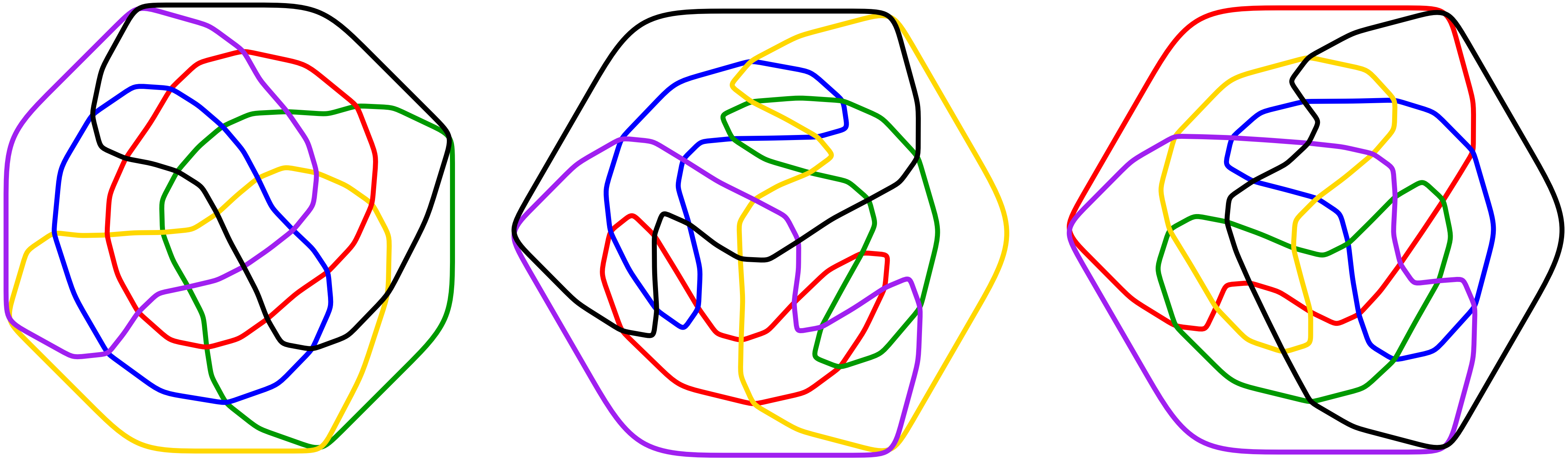


- blue and black: 4 crossings – contradiction

Circularizability Results

Theorem. There are exactly 3 non-circularizable digon-free intersecting $n = 6$ arrangements (2131 classes).



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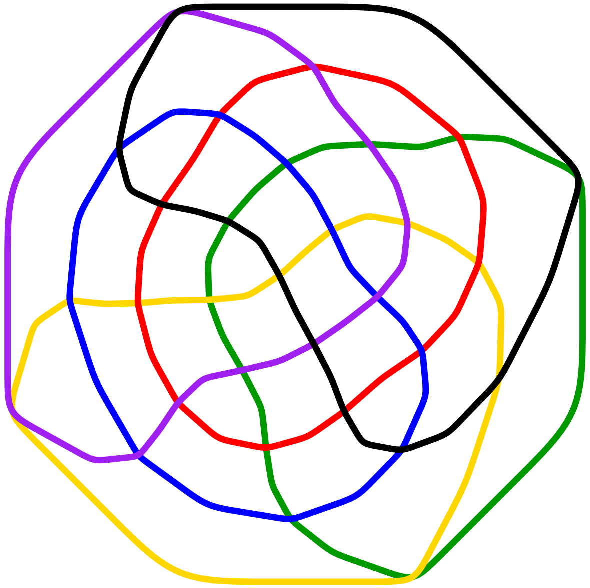

$\mathcal{N}_6^{\triangle}$ is unique digon-free intersecting with 8 triangular cells

Grünbaum Conjecture: $p_3 \geq 2n - 4$

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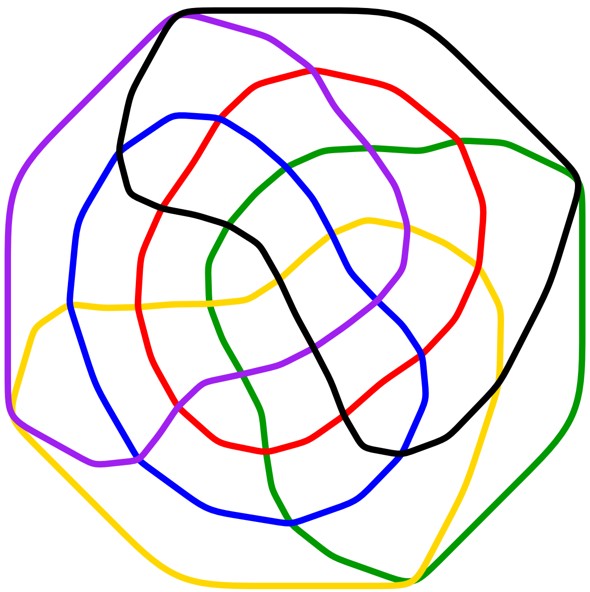

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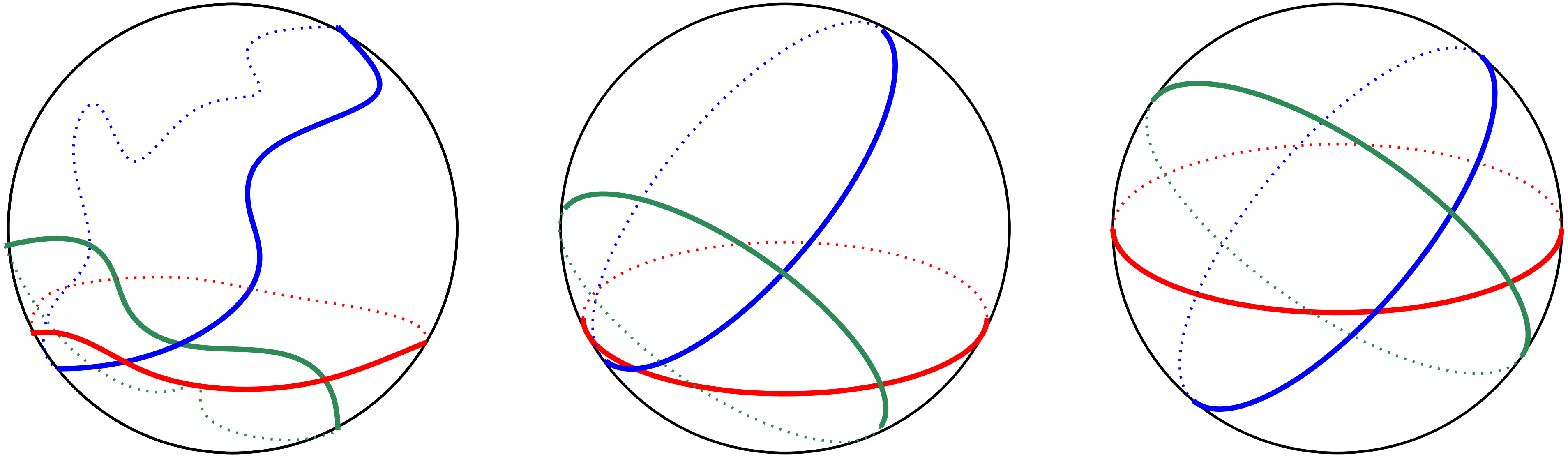
non-circularizability proof based on sweeping argument in 3-D

Great-(Pseudo)Circles

Great-Circle Theorem:
An arr. of great-pcs. is circularizable (i.e., has a circle representation) if and only if it has a great-circle repr.



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Proof.

$C_1, \ldots, C_n$ ... circles on sphere realizing the arrangement

$E_1, \ldots, E_n$ ... planes spanned by $C_1, \ldots, C_n$

for $t \geq 1$, sweep E_i to $\frac{1}{t} E_i$ (towards origin)

□

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all triples are Krupp, thus intersections remain inside sphere during sweep, thus no flip

as $t \to \infty$, we obtain great-circle arrangement □

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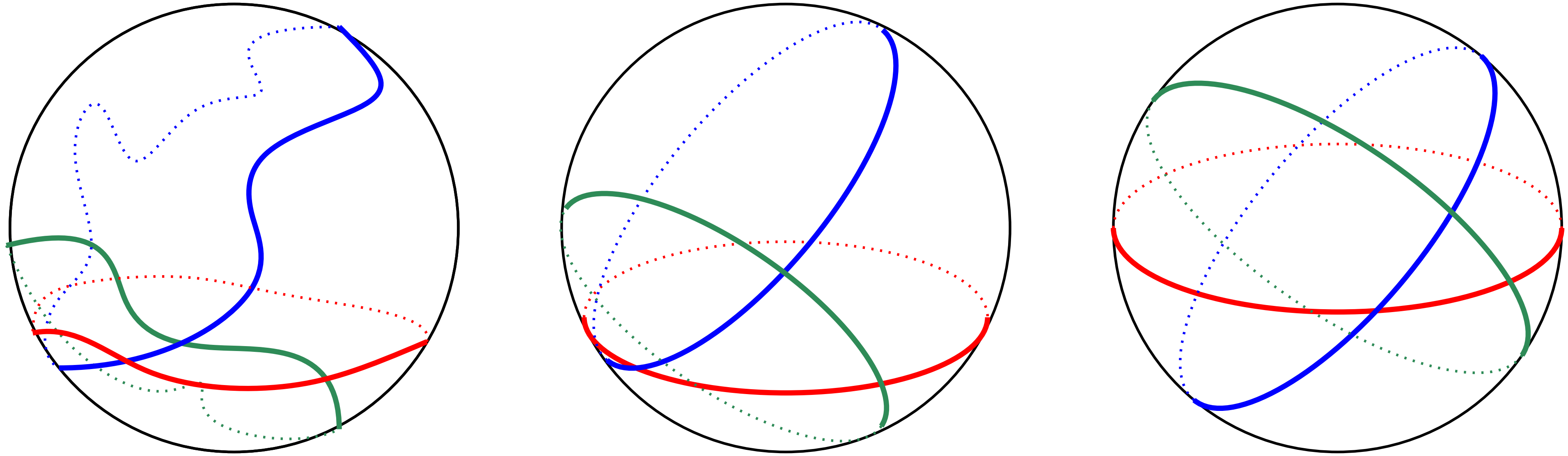

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Corollaries:

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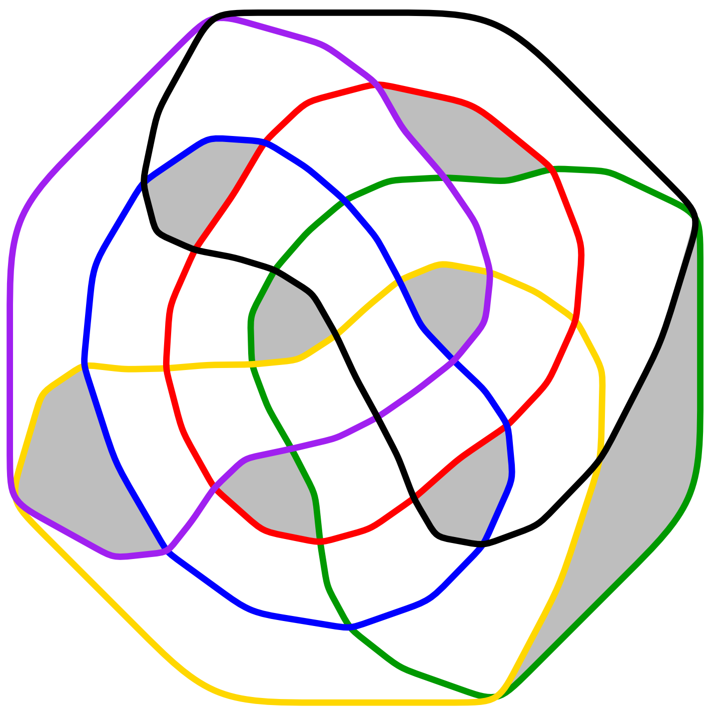
Corollaries:

- Every non-stretchable arr. of pseudolines has a corresponding non-circularizable arr. of pseudocircles
- Deciding circularizability is $\exists\mathbb{R}$-complete
- $\exists$ infinite families of minimal non-circ. arrangements
- $\exists$ circularizable arr with a disconnected realization space
- ...

Non-Circularizability Proof of $\mathcal{N}_6^{\triangle}$



Proof. (similar)

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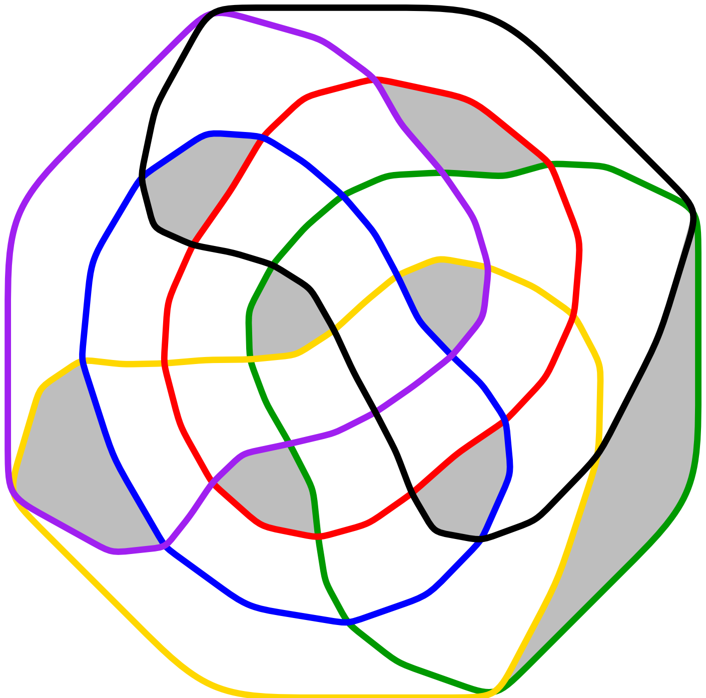

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planes move away from the origin

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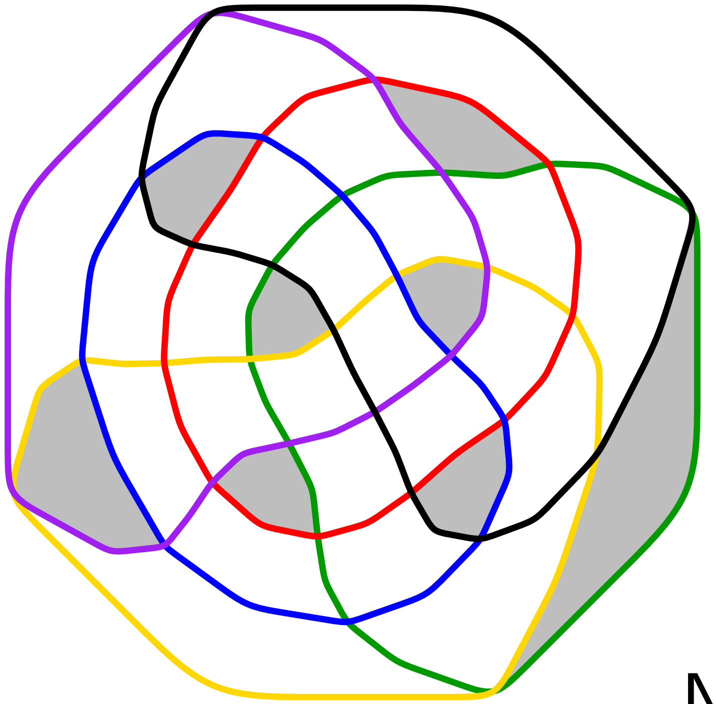

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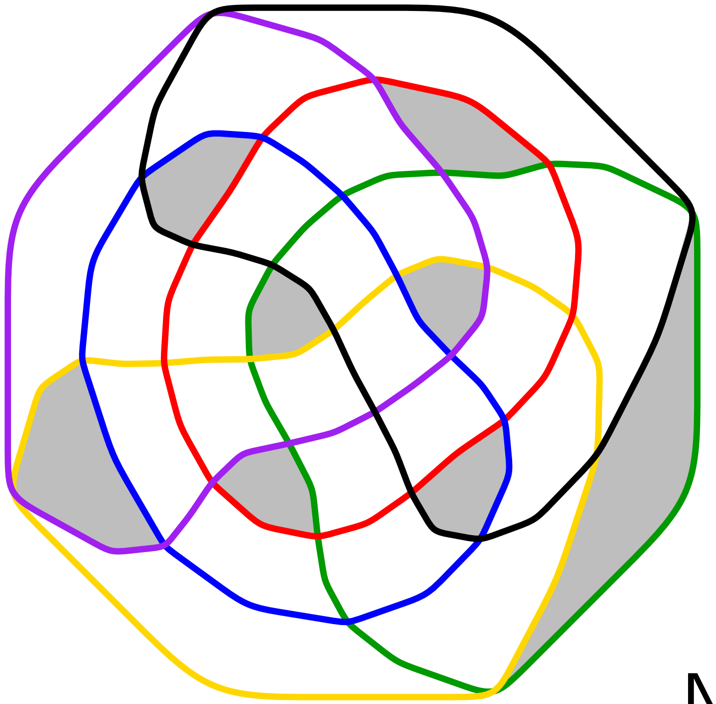
for $t \geq 1$, sweep E_i to $t \cdot E_i$ (to ∞)

No greatcircle arr., thus events occur

planes move away from the origin

not all planes contain the origin

Non-Circularizability Proof of $\mathcal{N}_6^{\triangle}$



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for $t \geq 1$, sweep E_i to $t \cdot E_i$ (to ∞)

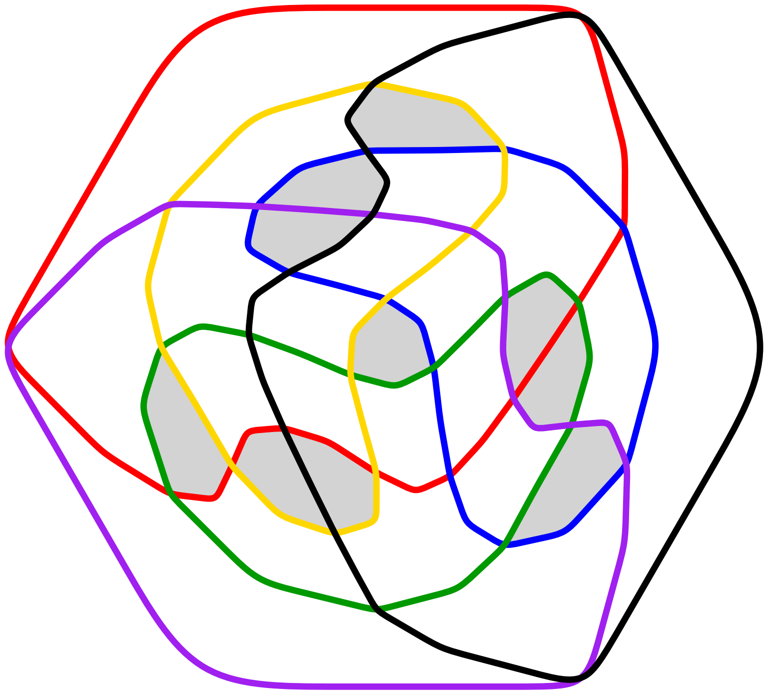
No greatcircle arr., thus events occur

first event is triangle flip (no digons)

but triangle flip not possible because all triangles in NonKrupp. Contradiction.

□

Non-Circularizability Proof of $\mathcal{N}_6^2$



Proof. (also similar)

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$E_1, \ldots, E_6$ ... planes

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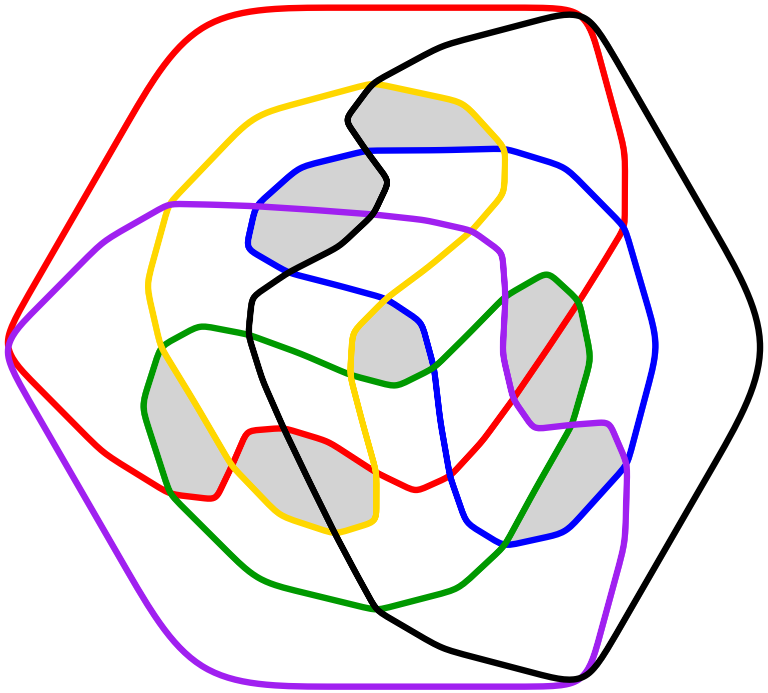

Proof. (also similar)

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for $t \geq 1$, sweep E_i to $1/t \cdot E_i$

planes move towards origin

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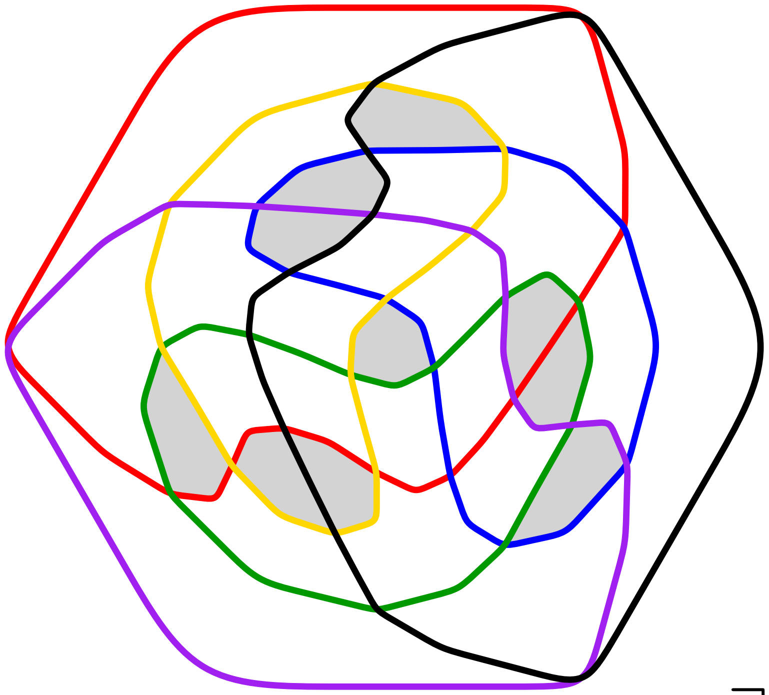

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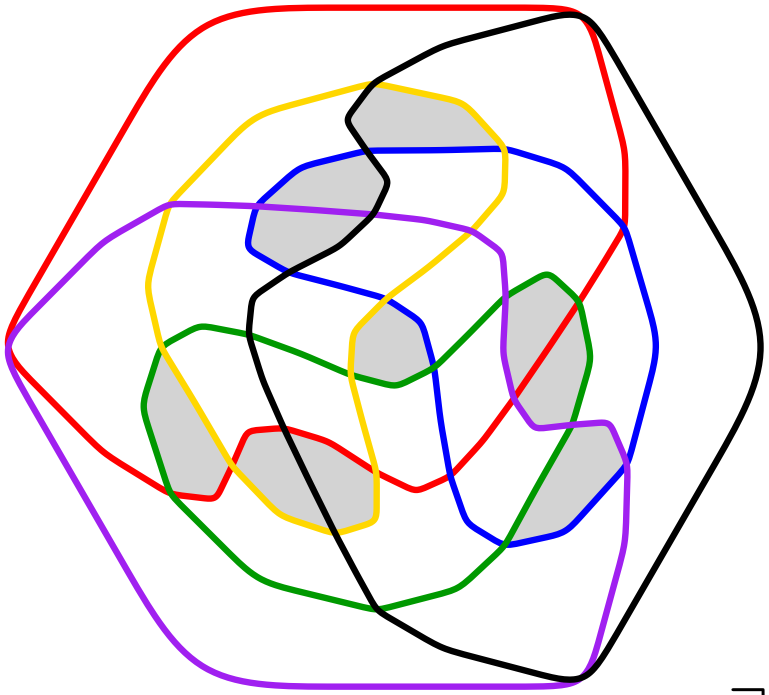
planes move towards origin

$\exists$ NonKrupp subarr. $\Rightarrow$ events occur

$\exists$ point of intersection outside the unit-sphere (will move inside)

Non-Circularizability Proof of $\mathcal{N}_6^2$



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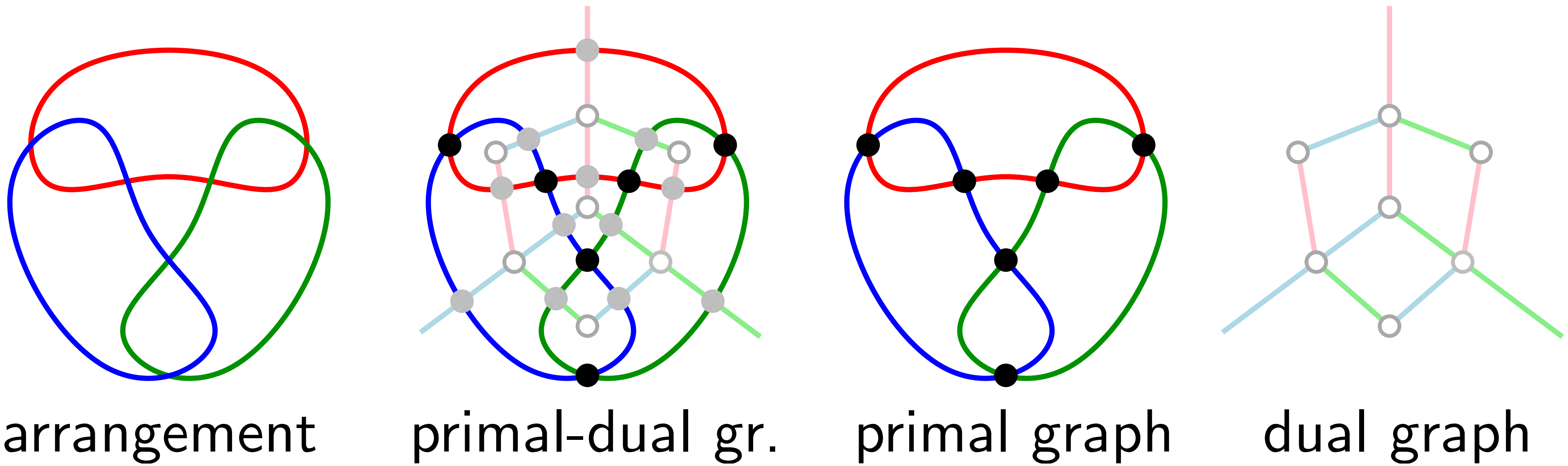
$\square$

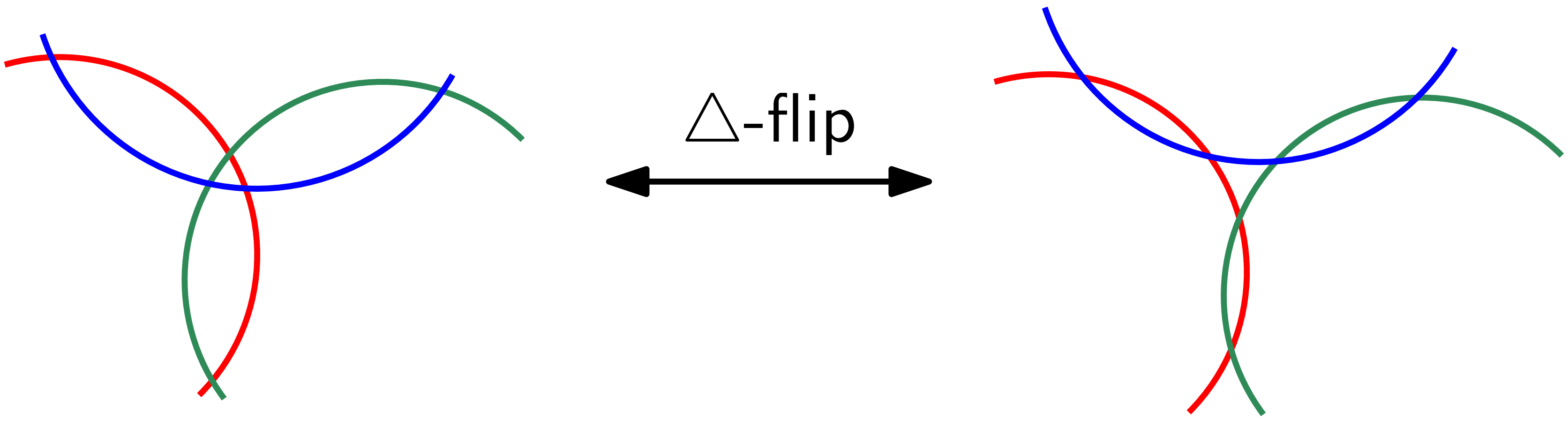
Computational Part

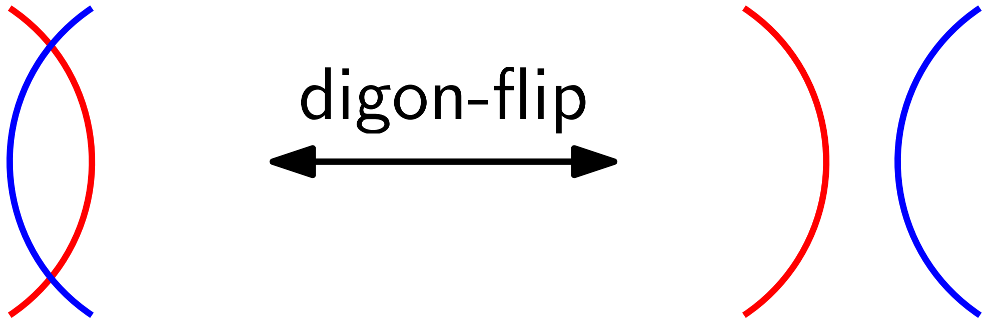
- connected arrangements encoded via primal-dual graph
- intersecting arrangements encoded via dual graph



Computational Part

- enumeration via recursive search on flip graph





Computational Part

- circle representations heuristically
- hard instances by hand

Further Results (Full Version)

- alternative proofs for $\mathcal{N}_6^{\triangle}$ and $\mathcal{N}_6^2$
- 8 additional arrangements of 6 pseudocircles (group of symmetries ≥ 4)

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- ≈130 000 intersecting arrangements of 6 circles (≈90%), and
- ≈2M intersecting digon-free arrangements of 7 circles (≈66%).

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- ≈2M intersecting digon-free arrangements of 7 circles (≈66%).
- some results on the flipgraph
- $2^{\Theta(n^2)}$and $2^{\Theta(n \log n)}$ bound ($\#$ of arrangements)

Thank you for your attention!