# Ortho-polygon Visibility Representations of $3$-connected $1$-plane Graphs

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Ortho-polygon Visibility representations (OPVRs) of embedded graphs

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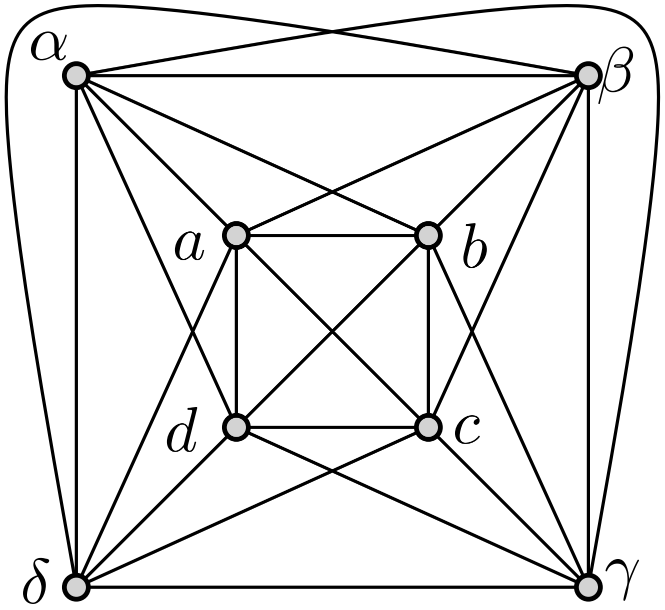

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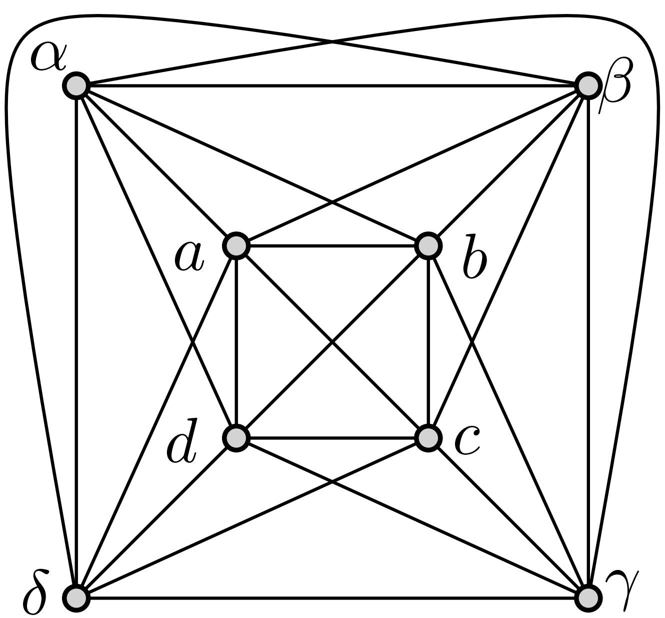
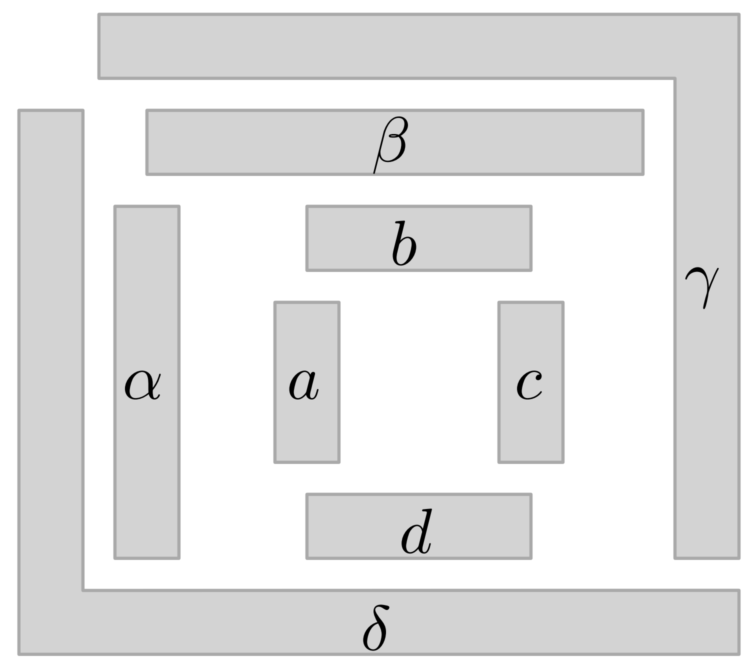
- Output: OPVR

Vertices $\rightarrow$ Orthogonal polygons



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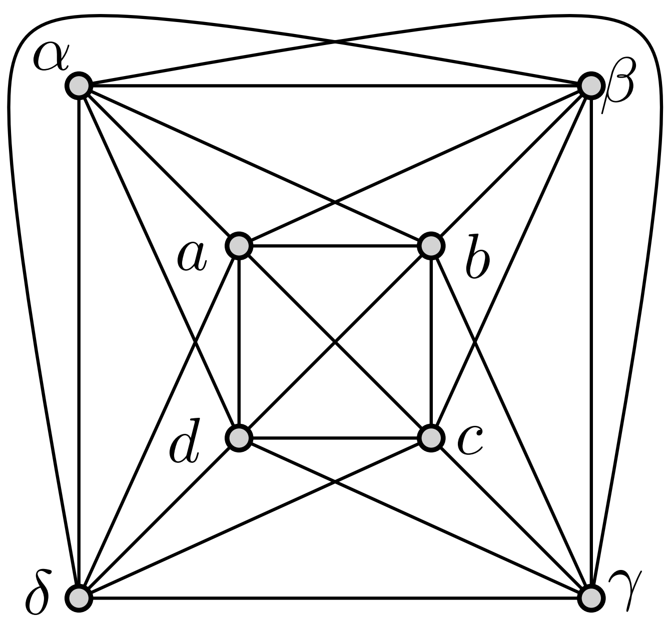
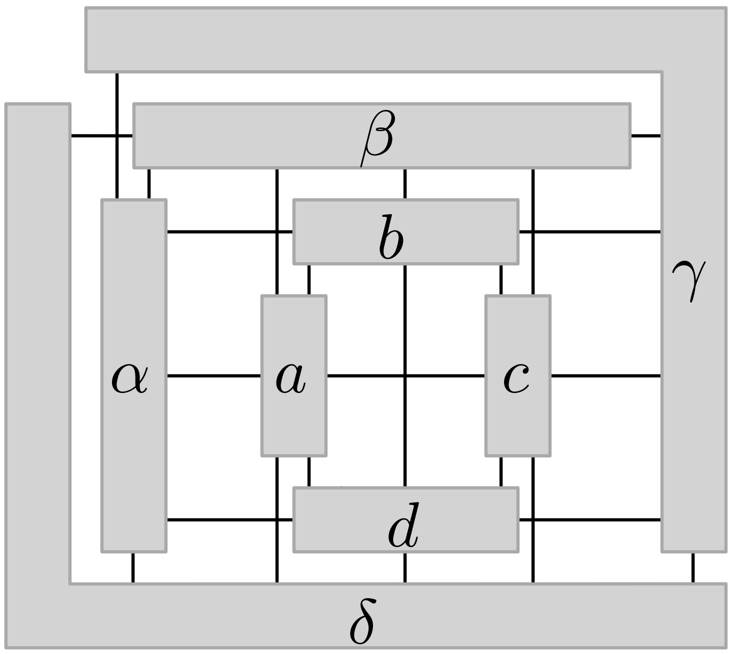
Vertices $\rightarrow$ Orthogonal polygons

Edges $\rightarrow$ Horizontal/Vertical visibilities



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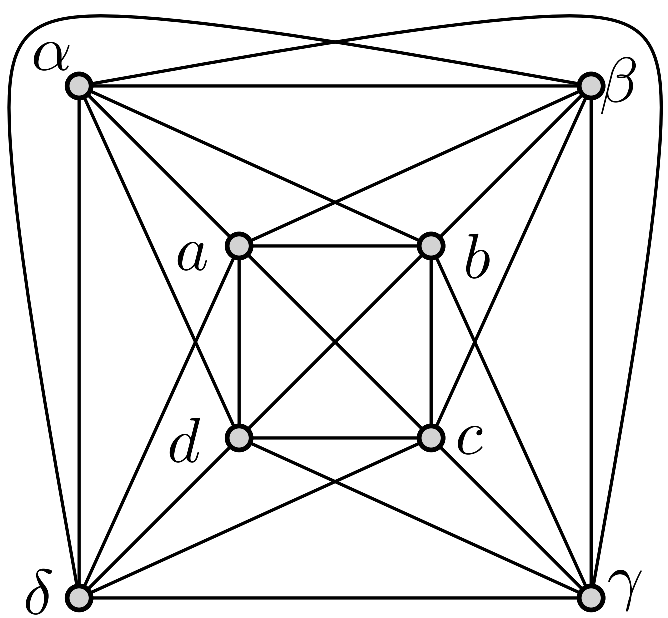
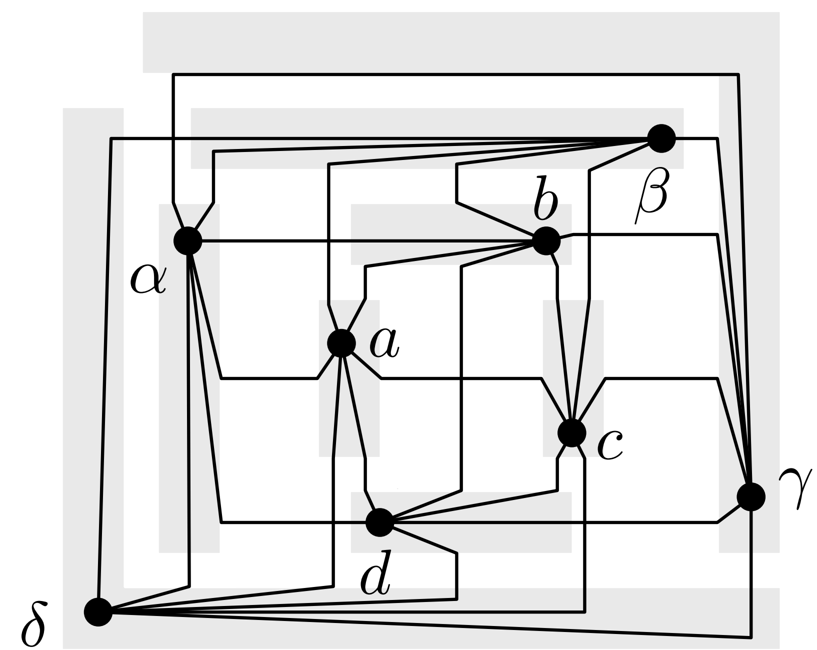
- Output: OPVR

  Vertices → Orthogonal polygons

  Edges → Horizontal/Vertical visibilities

  Embedding preserved



# Rectangle Visibility Representations

OPVRS introduced by Di Giacomo et al. as a generalization of rectangle visibility representations [Di Giacomo, Didimo, Evans, Liotta, Mejer, M., Wismath 2016]

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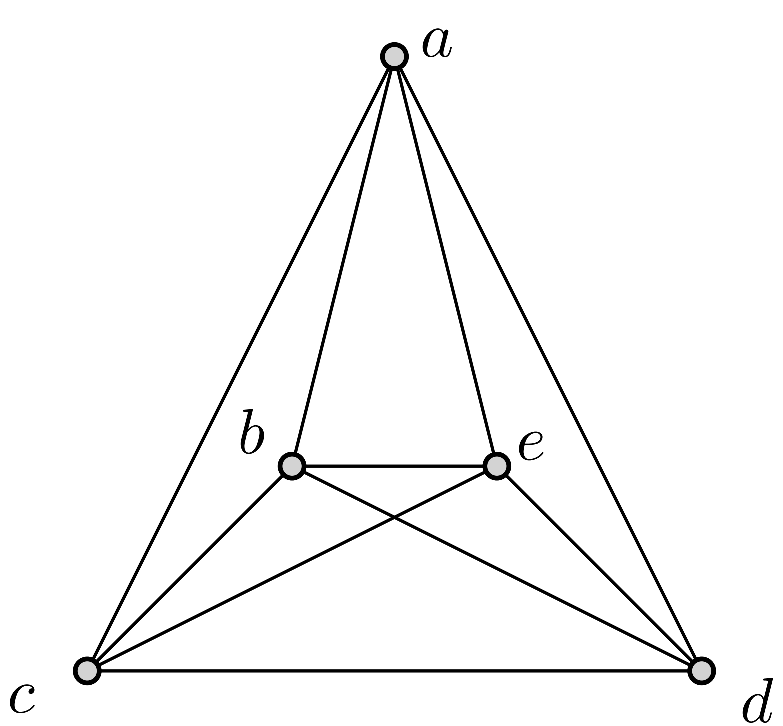
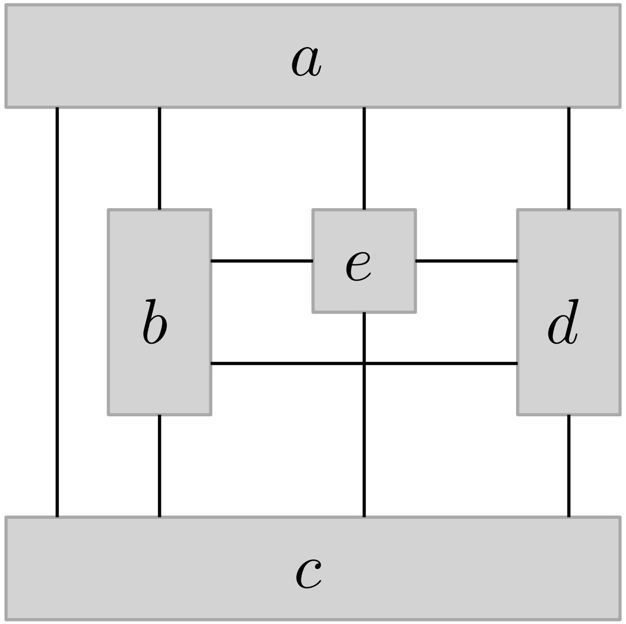
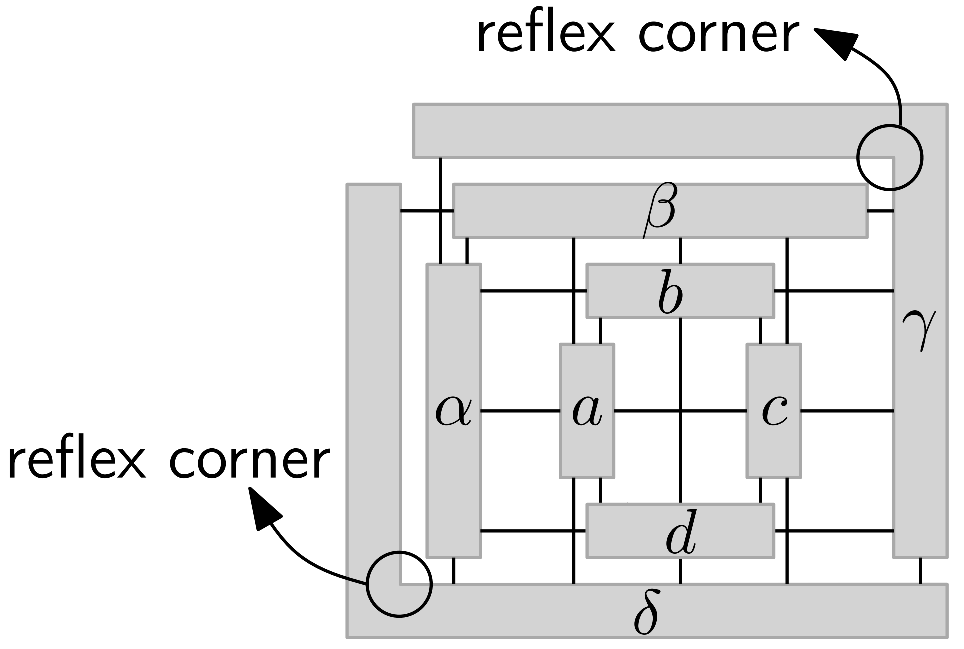
Rectangle Visibility representation (RVR) of embedded graphs:
Vertices $\rightarrow$ Axis-aligned rectangles
Edges $\rightarrow$ Horizontal/Vertical visibilities



# Vertex complexity




vertex complexity $= 1$

Vertex complexity of an OPVR $\Gamma$ = maximum number of reflex corners of a polygon in $\Gamma$

$\rightarrow$ RVR = OPVR with vertex complexity $0$

# Related work and motivation

- Deciding if a graph has an embedding that can be drawn as a RVR is NP-complete [Shermer 1996]

VARIABLE EMBEDDING

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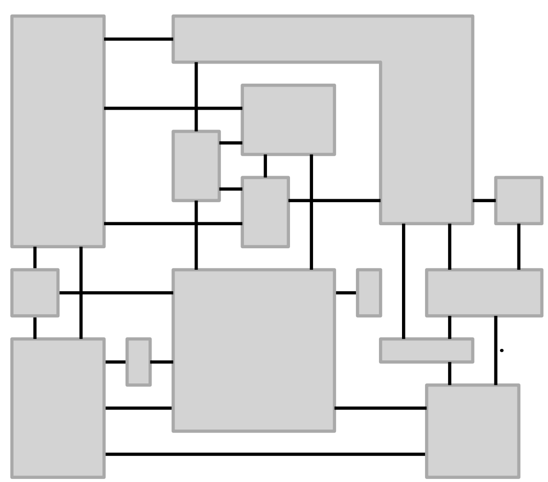
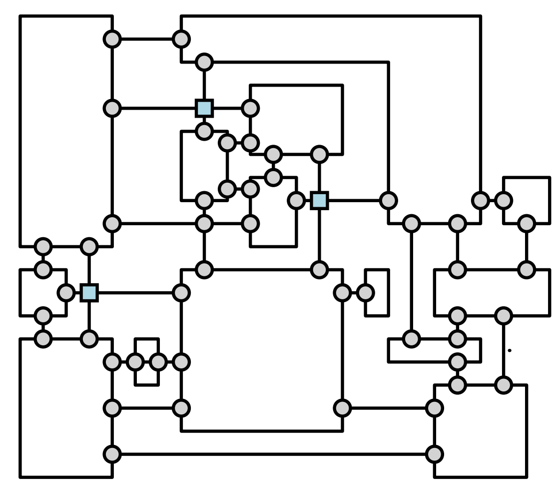
VARIABLE EMBEDDING

FIXED EMBEDDING

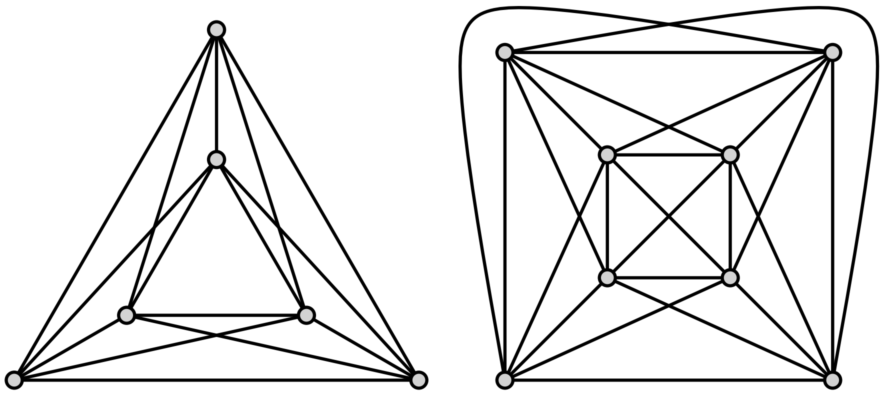
- Deciding if an embedded graph has a RVR is polynomial [Biedl, Liotta, M. 2016]
- Deciding if an embedded graph has an OPVR is polynomial [Di Giacomo et al. 2016]
- Algorithm to compute an OPVR with minimum vertex complexity in $O(n^{\frac{5}{2}} \log^{\frac{3}{2}} n)$ time [Di Giacomo et al. 2016]





# Related work and motivation

An embedded graph is 1-plane if it has at most one crossing per edge.



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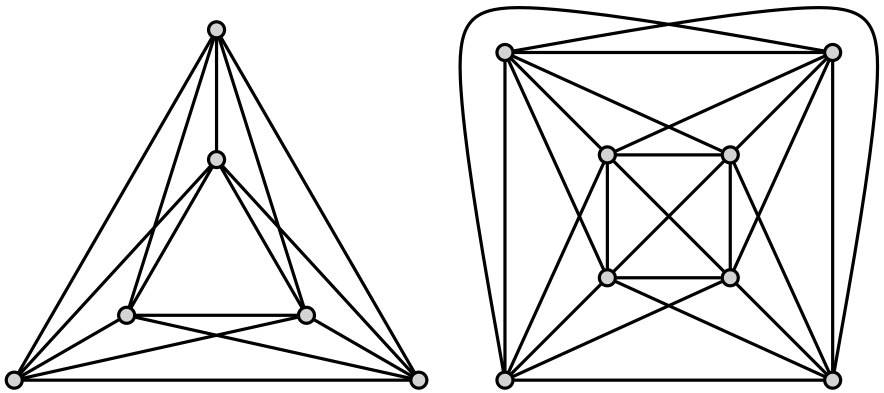
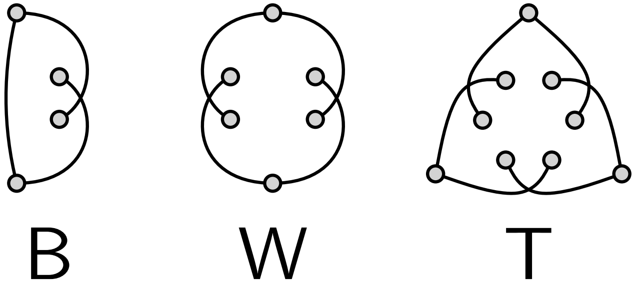


FORBIDDEN CONFIGURATIONS




- A $1$-plane graph admits a RVR if and only if it does not contain any B-, W-, and T-configuration as a subgrpah

[Biedl, Liotta, M. 2016]

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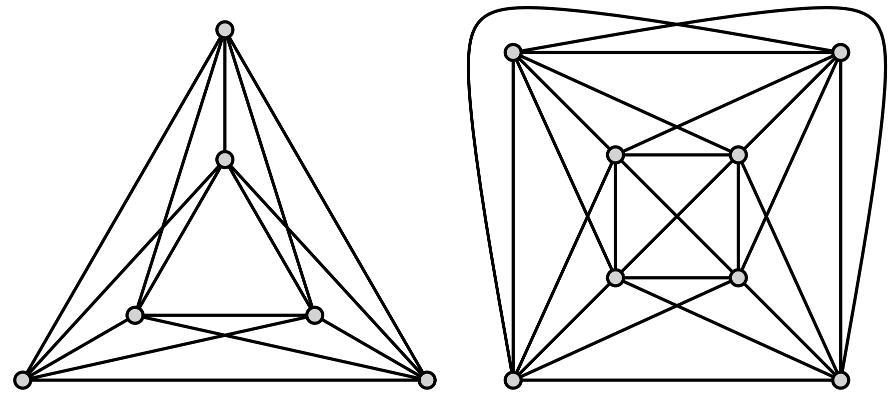

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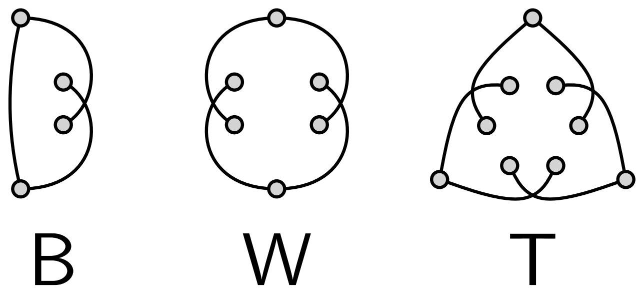


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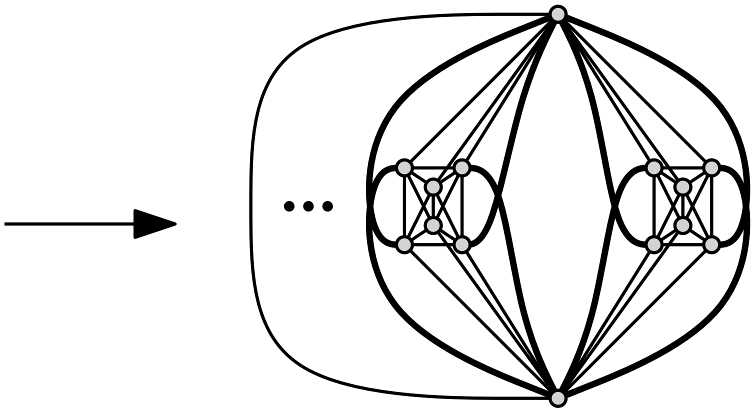
[Biedl, Liotta, M. 2016]

• Every 1-plane graph $G$ has an OPVR. An OPVR of $G$ with minimum vertex complexity can be computed in $O(n^{\frac{7}{4}} \log \sqrt{n})$ time
[Di Giacomo, Didimo, Evans, Liotta, Mejer, M., Wismath 2016]

# Related work and motivation

• There are $2$-connected $1$-plane graphs such that any OPVR has vertex complexity $\Omega(n)$ [Di Giacomo et al. 2016]



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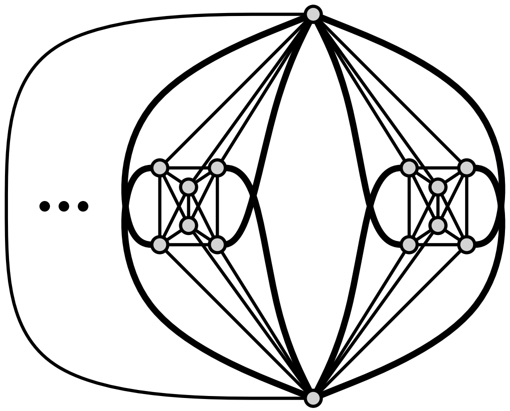


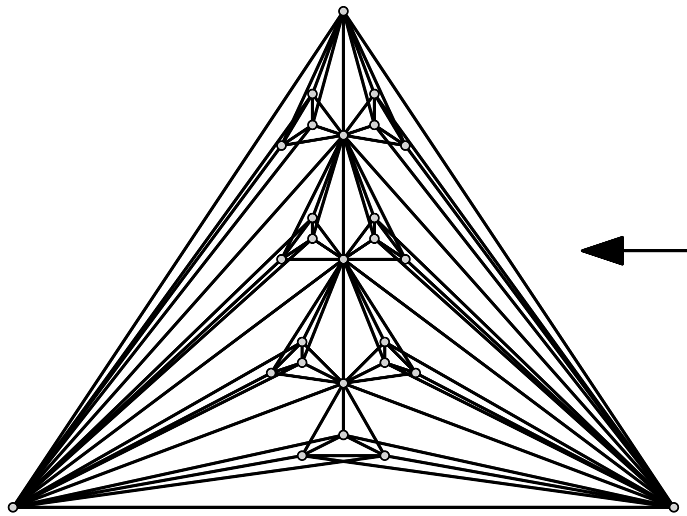

• There are $3$-connected $1$-plane graphs such that any OPVR has vertex complexity at least 2 [Di Giacomo et al. 2016]

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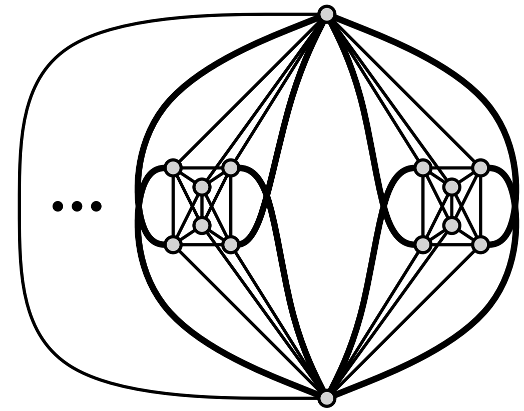
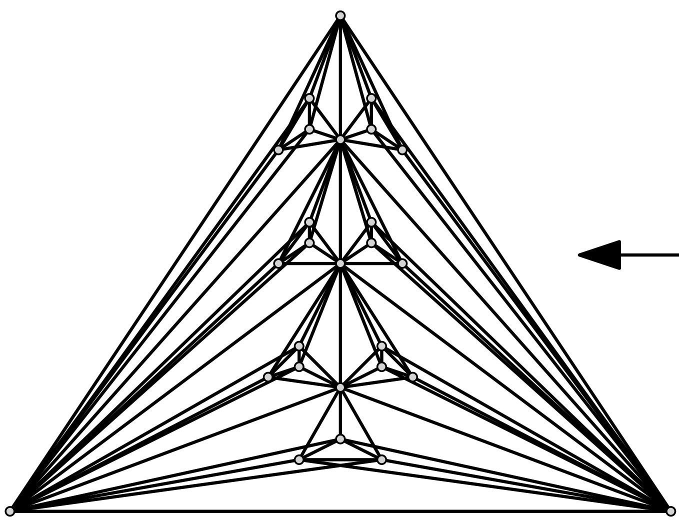
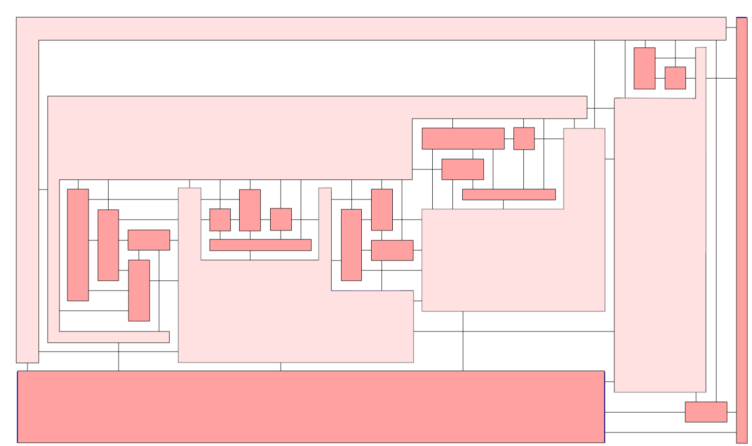


- There are $3$-connected $1$-plane graphs such that any OPVR has vertex complexity at least 2 [Di Giacomo et al. 2016]

- Every $3$-connected $1$-plane graph has an OPVR with vertex complexity at most $12$ [Di Giacomo et al. 2016]



# Contribution

- There are $3$-connected $1$-plane graphs such that any OPVR has vertex complexity at least 4
  - Lower bound increased from $2$ to $4$

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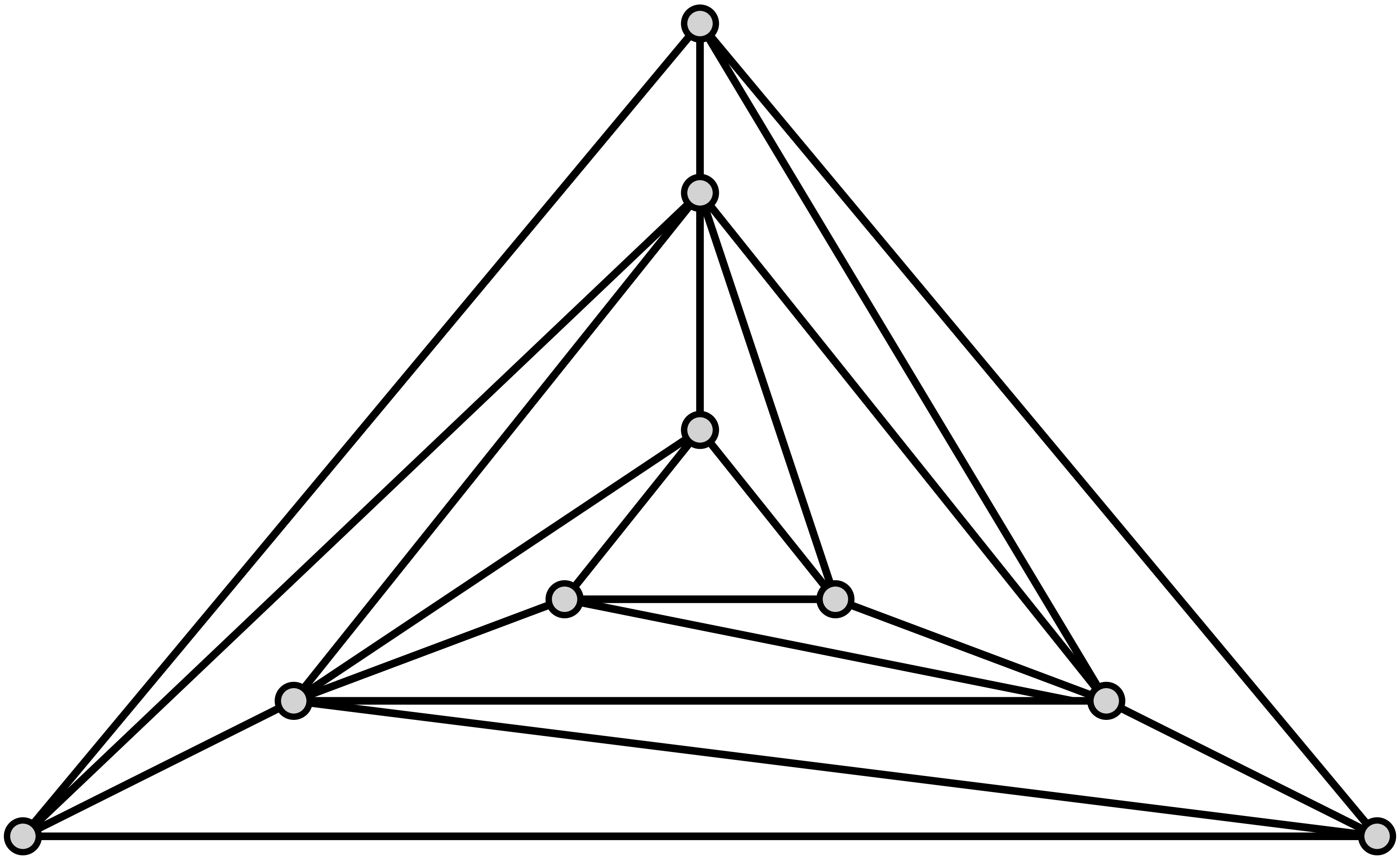
- There are $3$-connected $1$-plane graphs such that any OPVR has vertex complexity at least 4
    - Lower bound increased from $2$ to $4$

- Every $3$-connected $1$-plane graph has an OPVR with vertex complexity at most $5$, which can be computed in $\tilde{O}(n^{\frac{10}{7}})$ time
    - Upper bound reduced from $12$ to $5$
    - Running time reduced by using recent results on the min-cost flow problem (not in this talk)

# Proving the Lower Bound

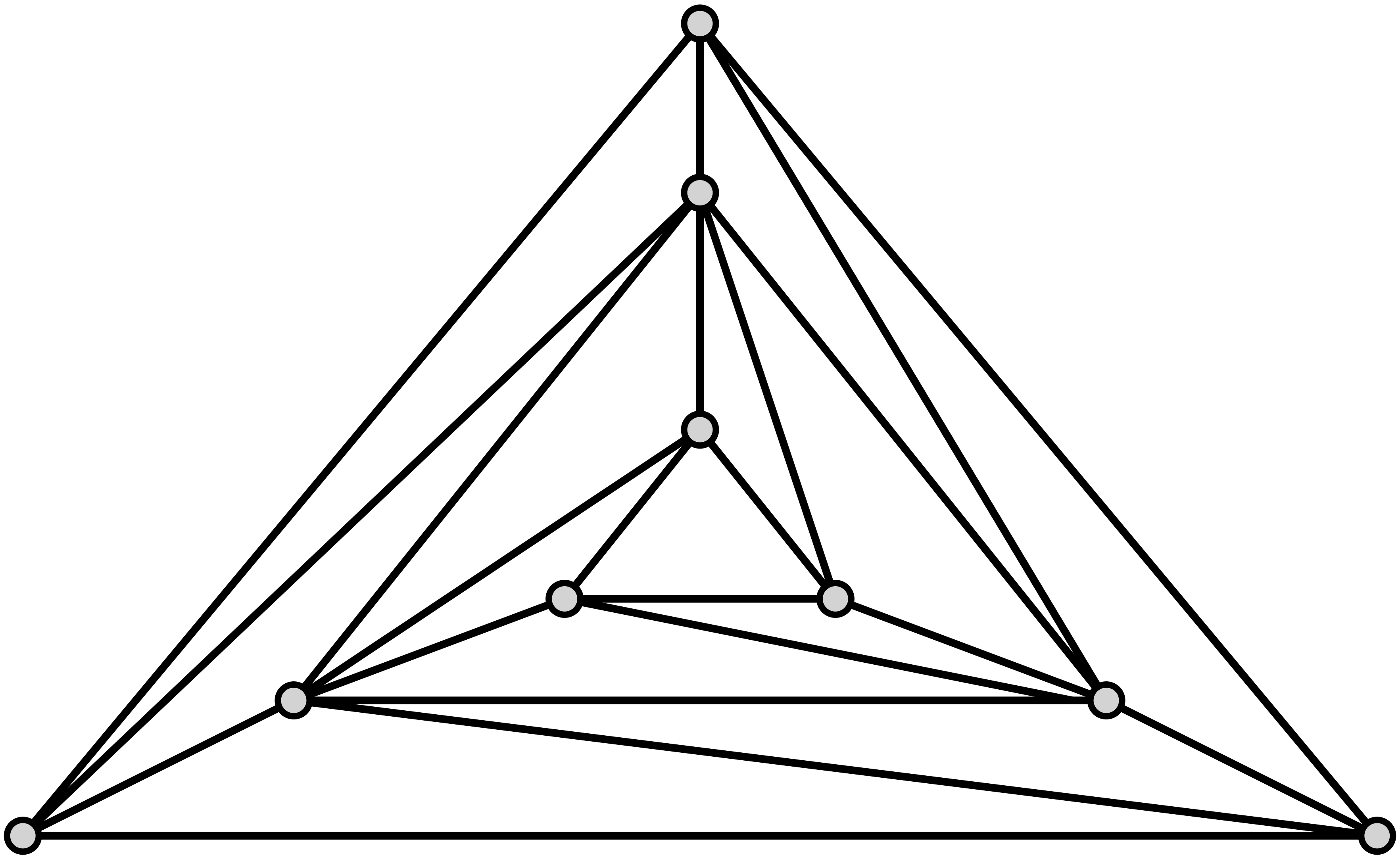
**Theorem 1** *There exists an infinite family $\mathcal{G}$ of $3$-connected $1$-plane graphs, such that for any $G \in \mathcal{G}$ and for any OPVR $\Gamma$ of $G$, the vertex complexity of $\Gamma$ is at least $4$.*

# Proving the lower bound



Start with a nested triangle graph with $n$ vertices

# Proving the lower bound



Color blue $n-2$ faces s.t. no 2 of them share an edge

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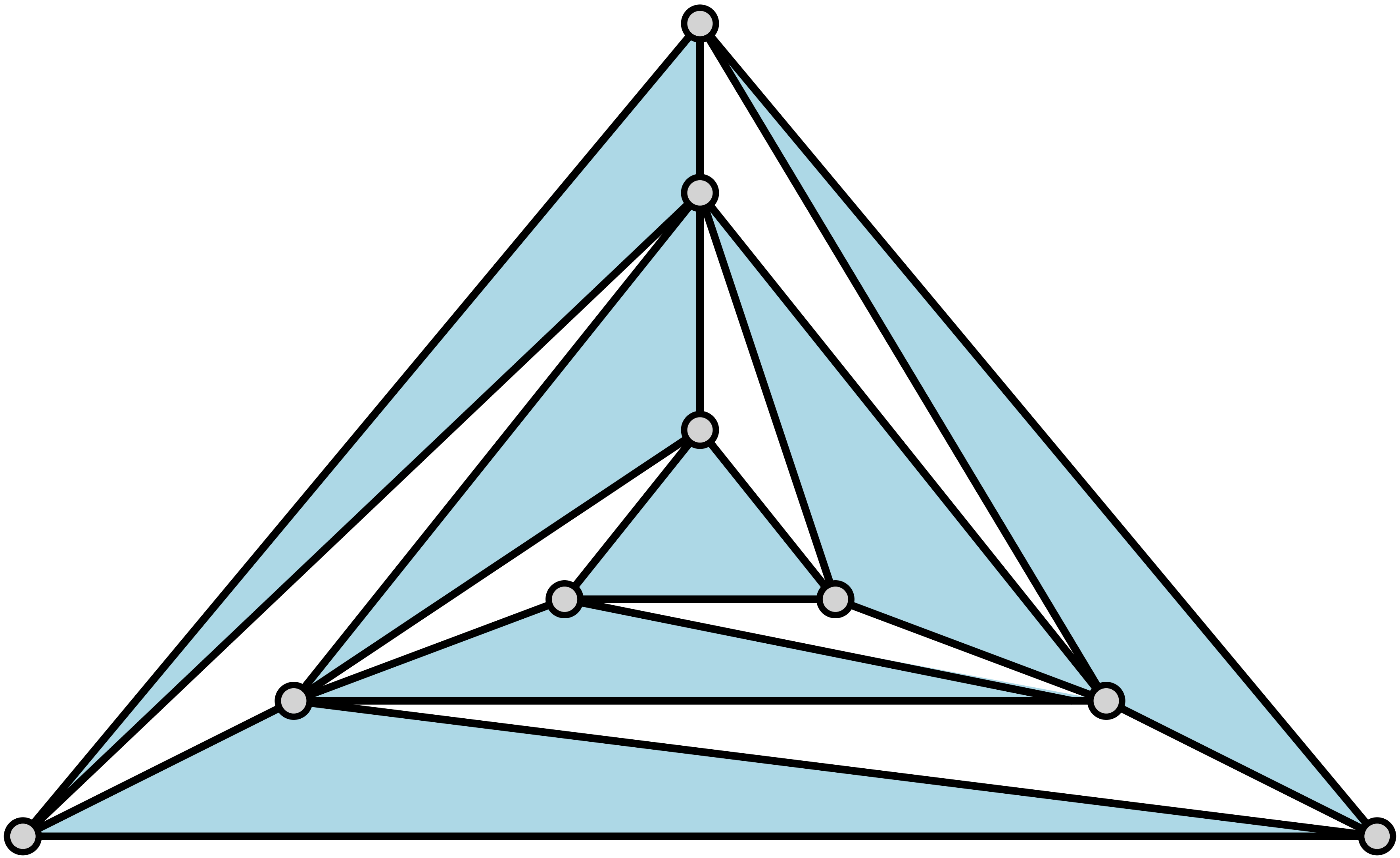

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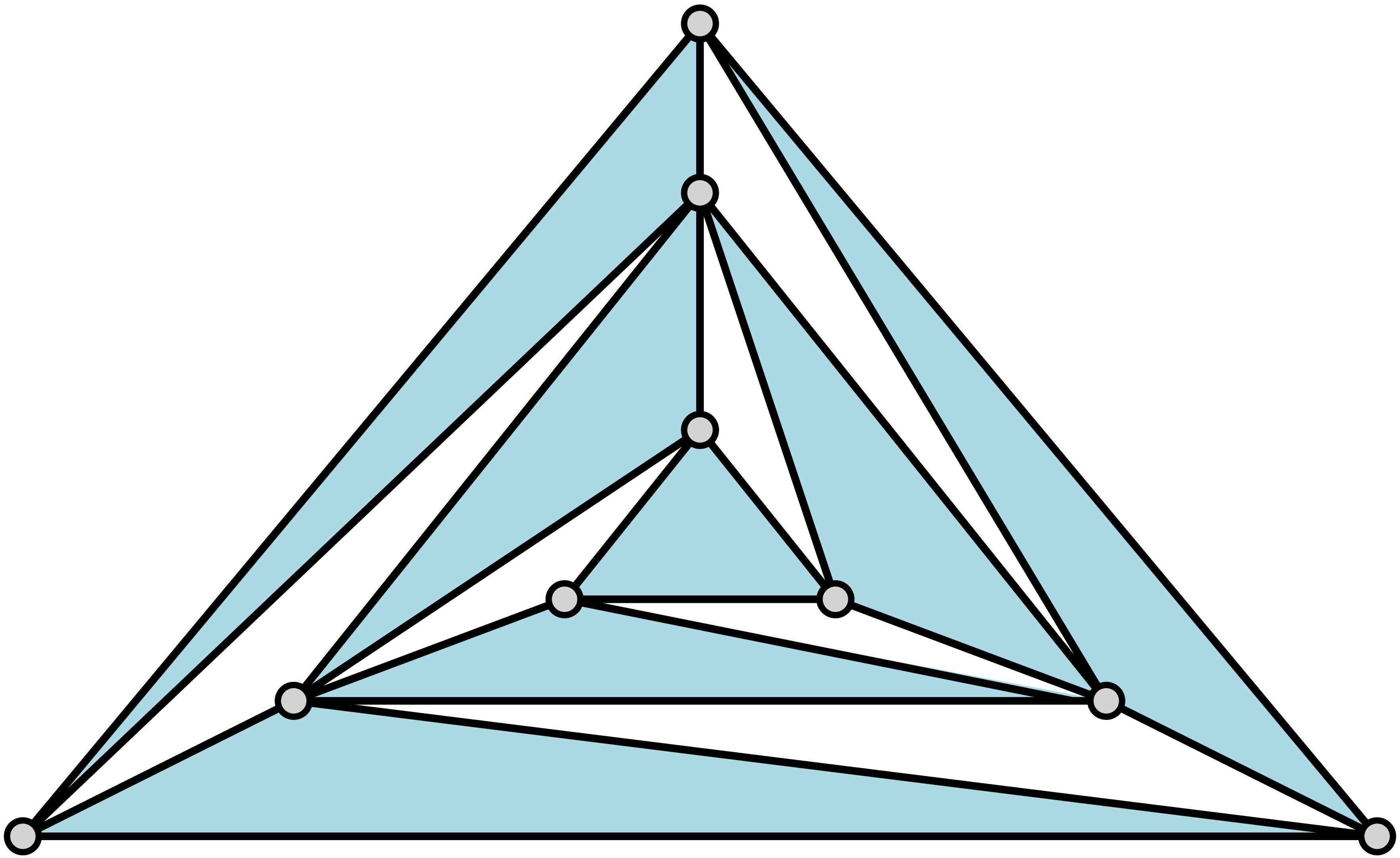

Insert a T-configuration in each blue face and a B-configuration along each edge

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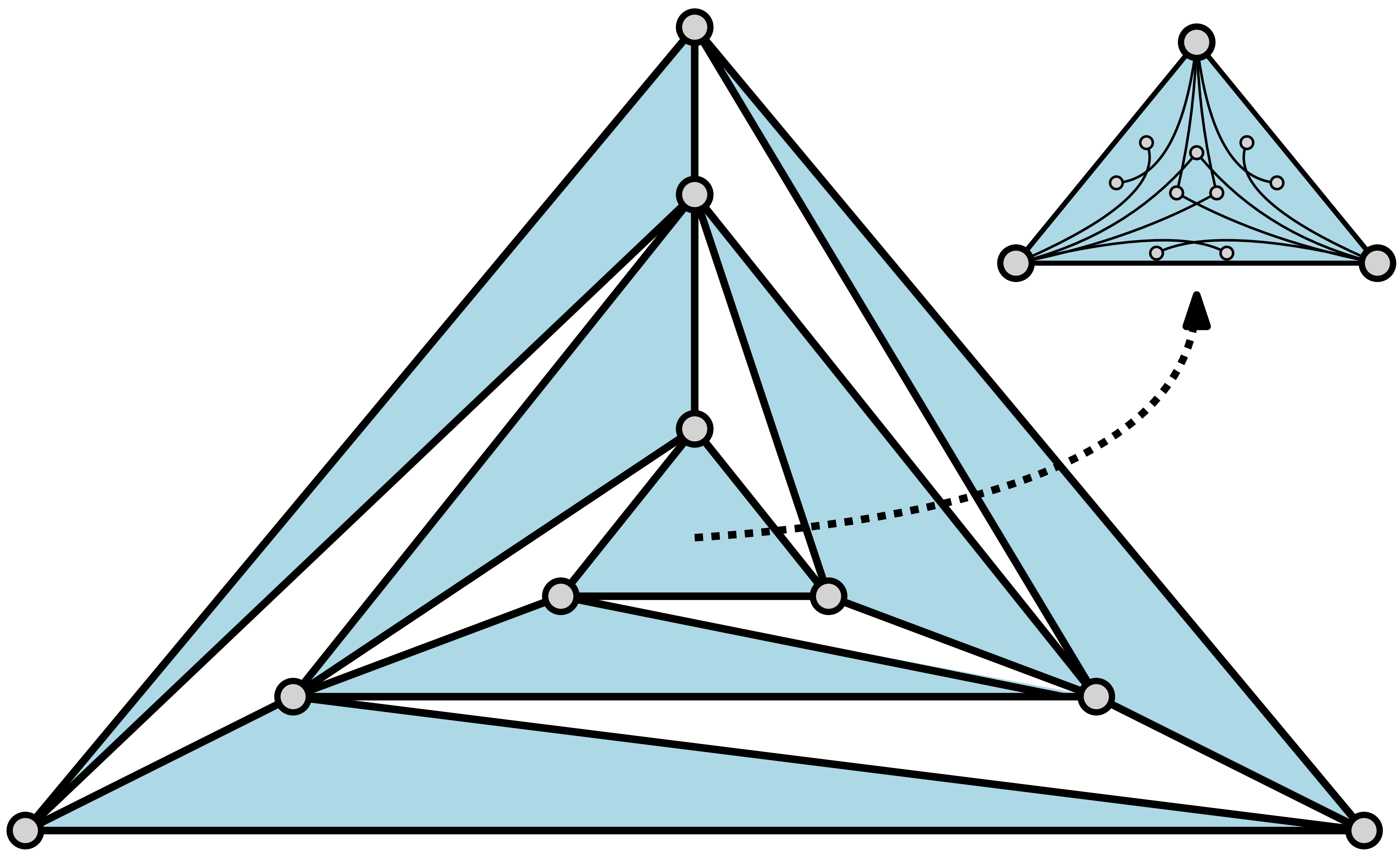

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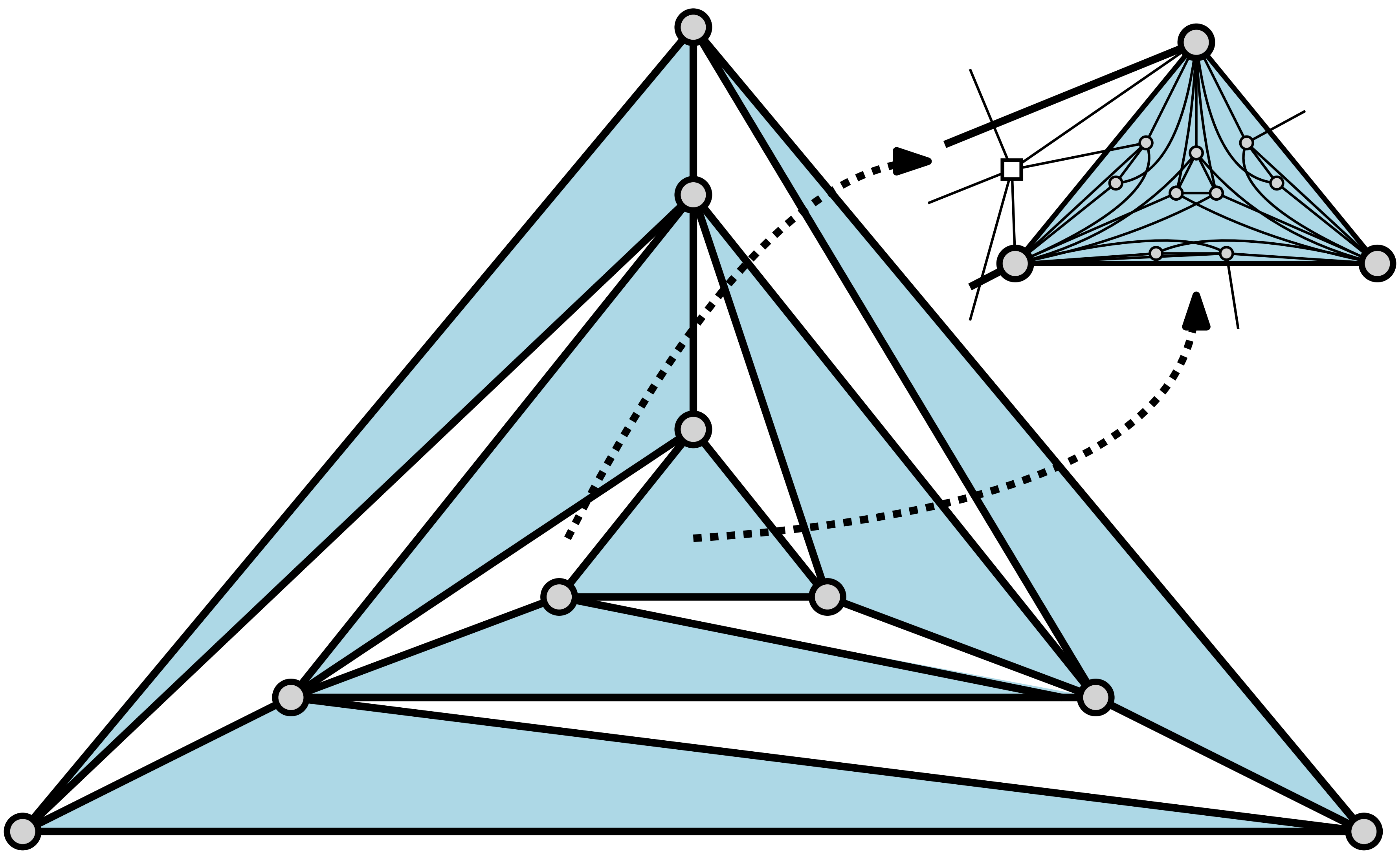
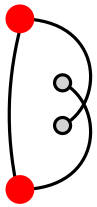
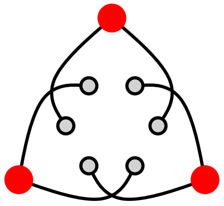


Insert a vertex in each white face and add dummy edges to make the graph $3$-connected (and still $1$-planar)

# Proving the lower bound

- The resulting graph $G$ has:
  - $3n-6$ B-configurations 
  - $n-2$ T-configurations 

The red vertices are the poles of the configurations

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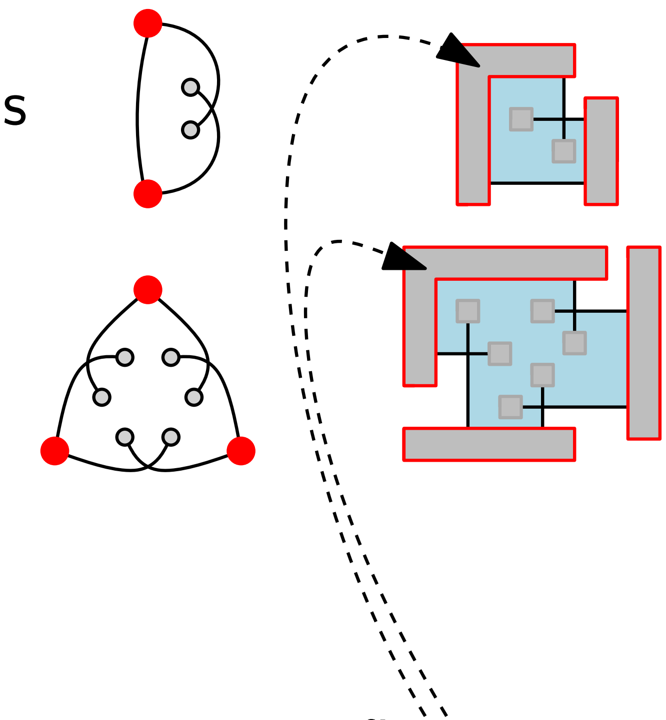

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- Each B-/T-configuration requires a reflex corner on one of its poles and in its interior region (light blue background) [Biedl, Liotta, M. 2016]

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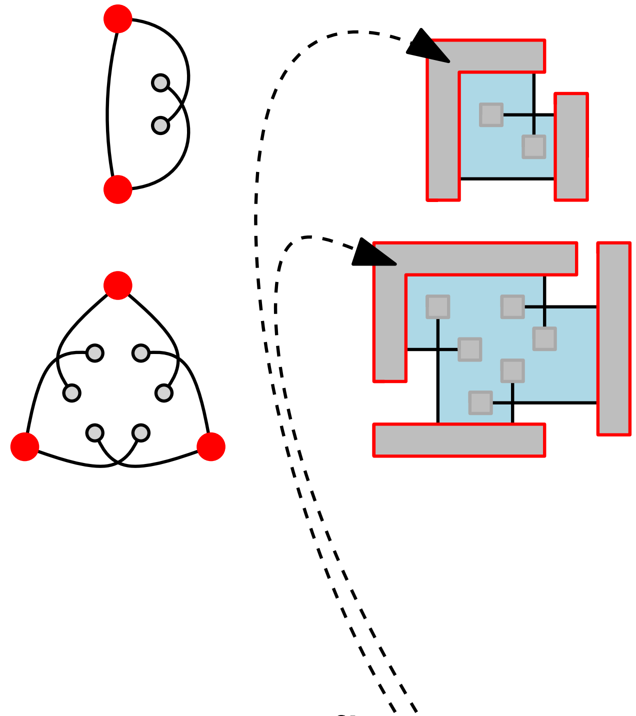

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- Each B-/T-configuration requires a reflex corner on one of its poles and in its interior region (light blue background) [Biedl, Liotta, M. 2016]
- $G$ contains $4n-8$ configurations with $n$ poles and whose interior regions are pairwise disjoint
  - At least one pole has at least $4$ reflex corners (if $n>8$) $\square$

# Proving the Upper Bound

**Theorem 2** *Every $3$-connected $1$-plane graph has an OPVR with vertex complexity at most $5$.*

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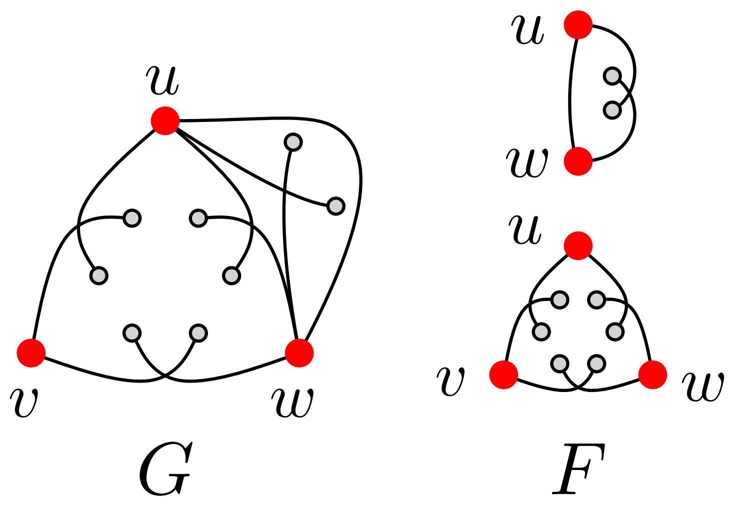

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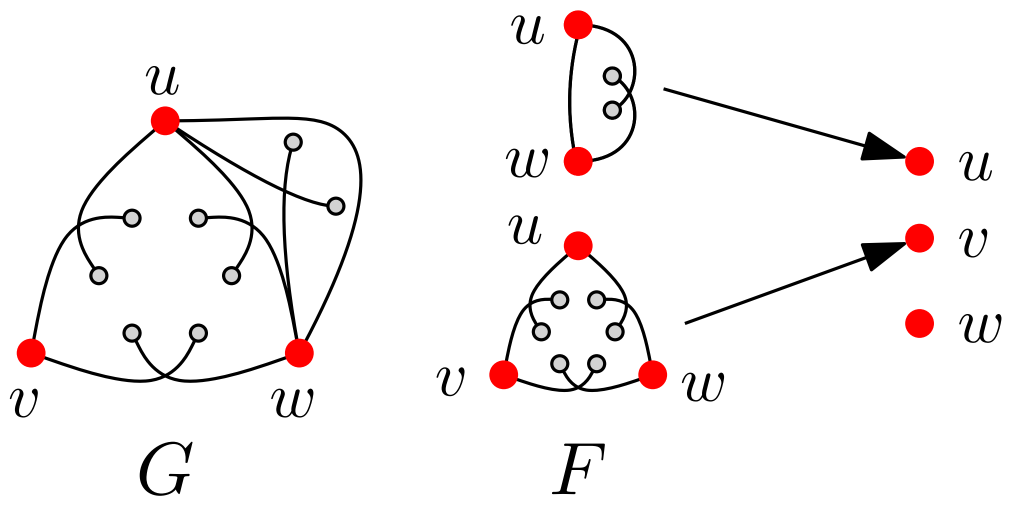
- Assign each configuration of this set to one of its poles such that each pole is assigned with at most $5$ configurations



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- Compute a special set $F$ of B-/W-/T-configurations from $G$
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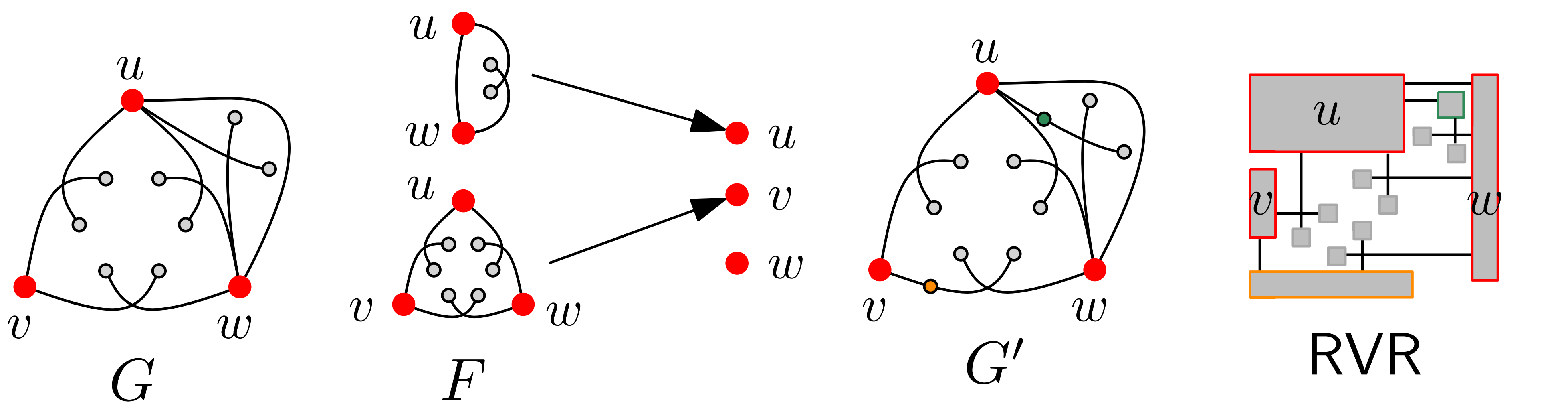
- Remove the configurations and compute a RVR



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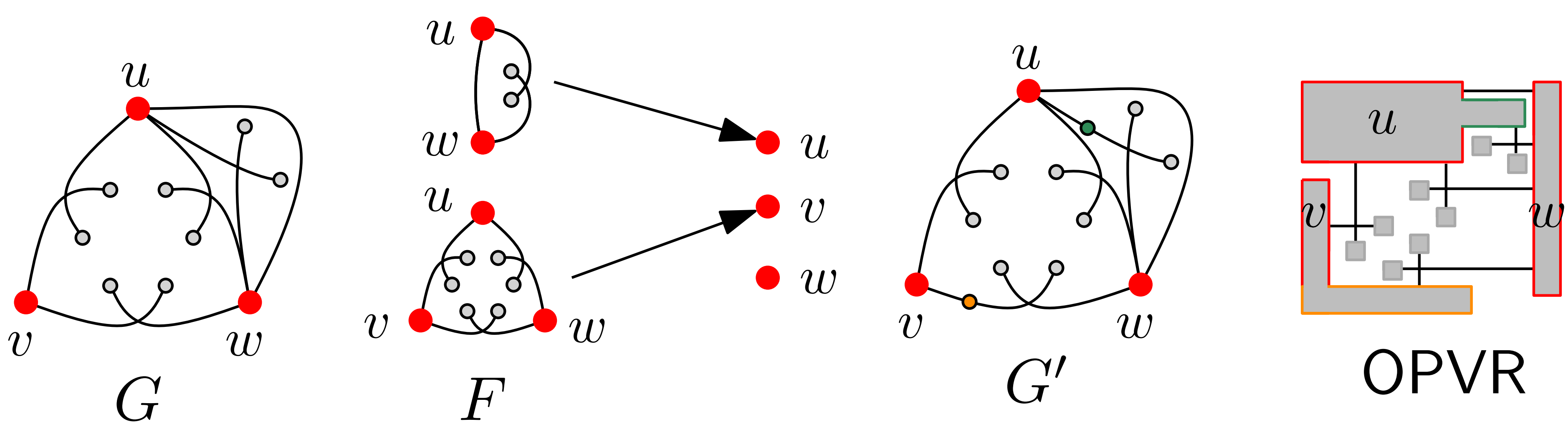
- Reinsert the configurations by attaching at most 5 spokes to the rectangles representing the matched poles



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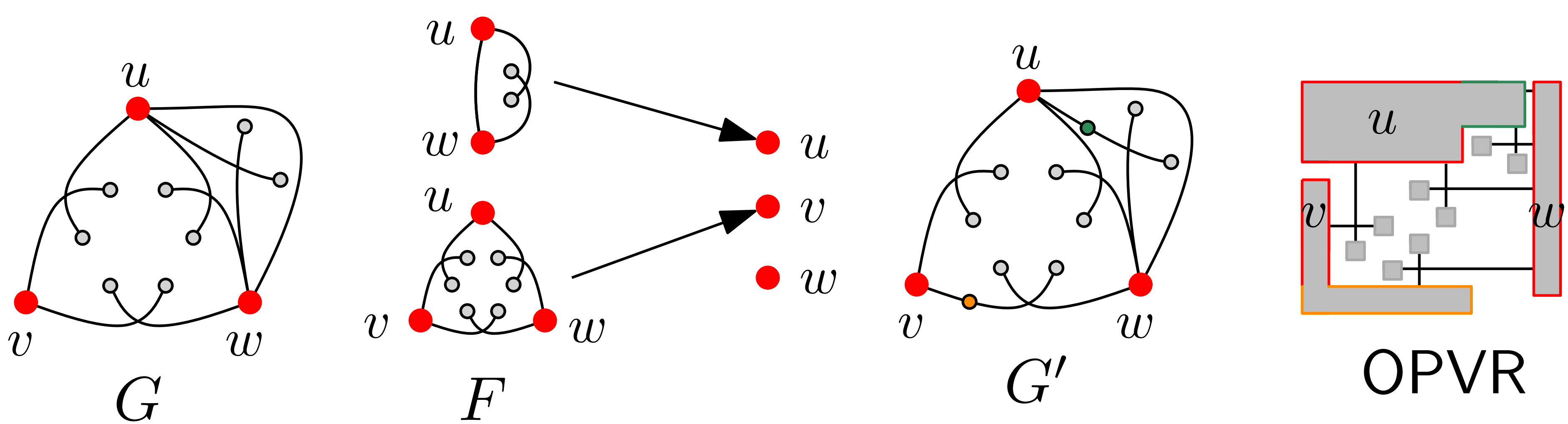

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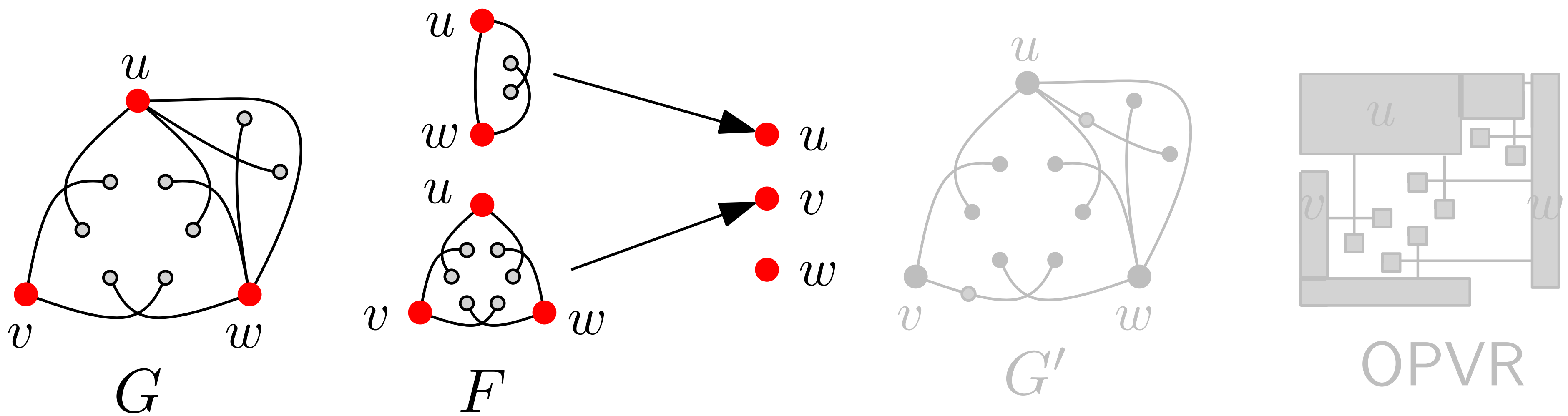
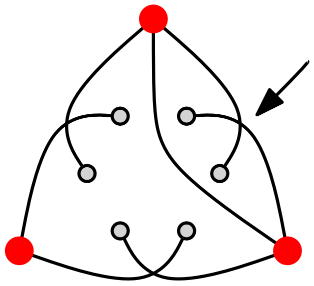
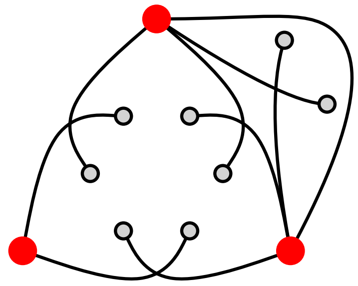


# Some definitions

Two configurations that do not share any crossing are called independent, or dependent otherwise



Two dependent configurations



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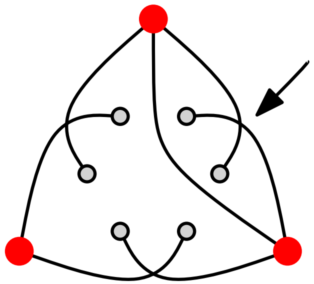

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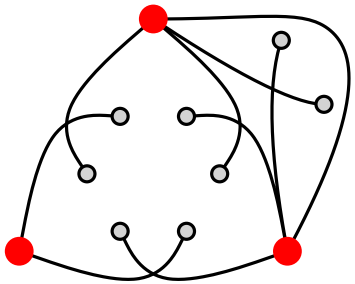


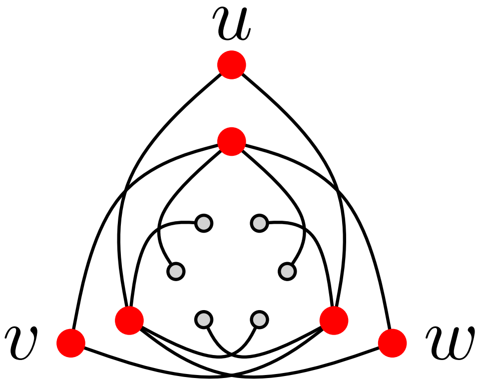
Two independent configurations

A set $F$ of configurations of a $3$-connected $1$-plane graph $G$ is non-redundant if it contains:

- All B-configurations of $G$;
- All T-configurations of $G$ that are independent of B-configurations.
- The W-configuration, if any.

# A special case

A T-configuration is separating if it contains a pole of another configuration in its interior region

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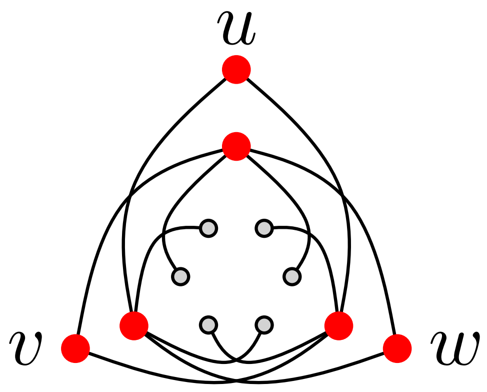


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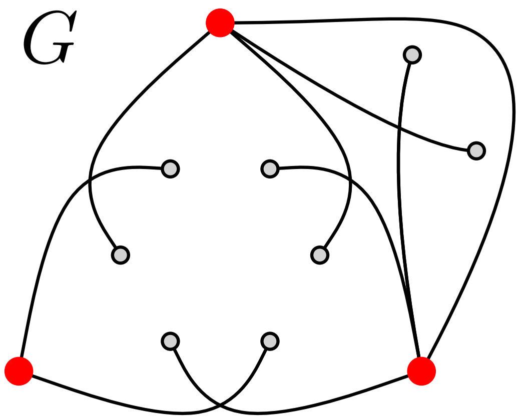
$G$: $3$-connected $1$-plane graph with set of poles $P$
$F$: non-redundant set of configurations of $G$
$\beta$: number of B-configurations in $G$
$\tau$: number of T-configurations in $G$

**Lemma 1** *If $G$ has no separating T-configurations and no W-configurations, then $|F| \leq 4|P| - 8$. Also, $|F| = 4|P| - 8$ if and only if $\beta = 3|P| - 6$ and $\tau = |P| - 2$*

# A special case: sketch of proof

We construct an auxiliary graph $G_A$ from $G$:



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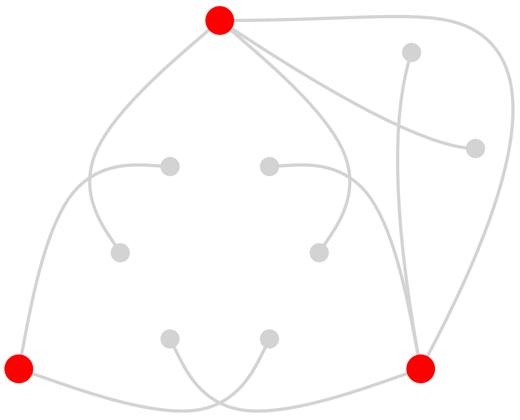
We construct an auxiliary graph $G_A$ from $G$:

Remove everything except the poles



# A special case: sketch of proof

We construct an auxiliary graph $G_A$ from $G$:

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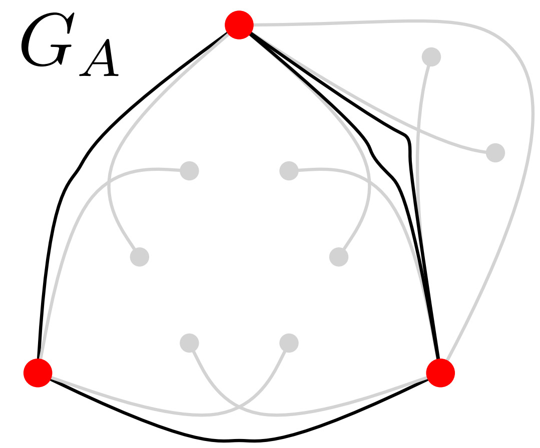
For each pair of crossing edges in a configuration of $F$, draw a crossing-free edge (close to the removed edges)



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We construct an auxiliary graph $G_A$ from $G$:

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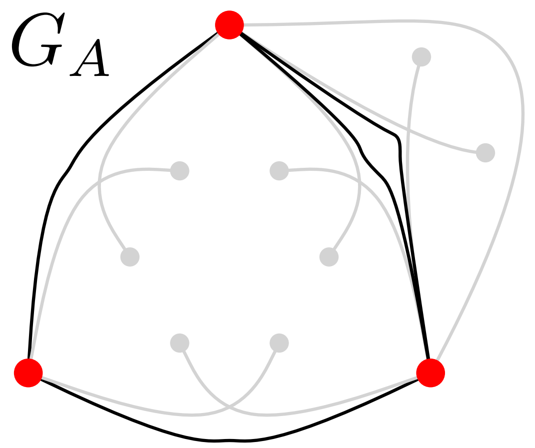


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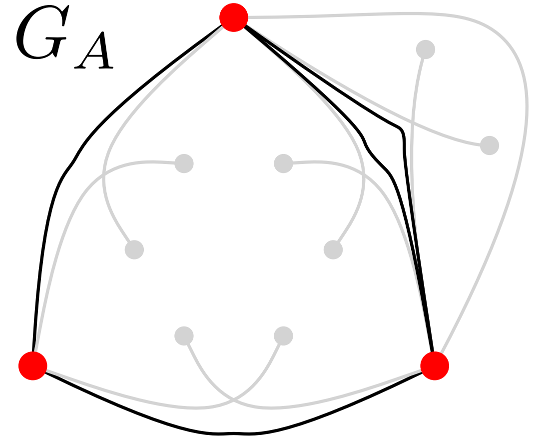

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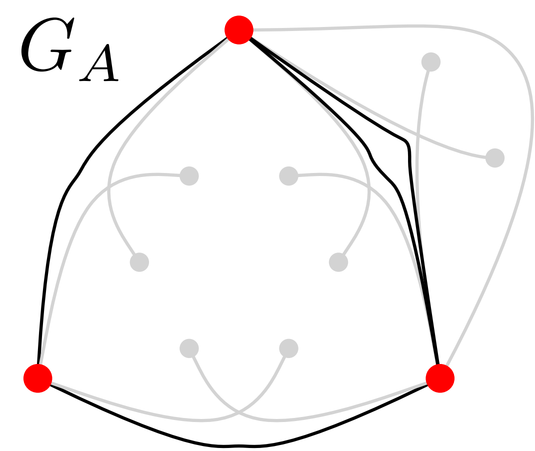


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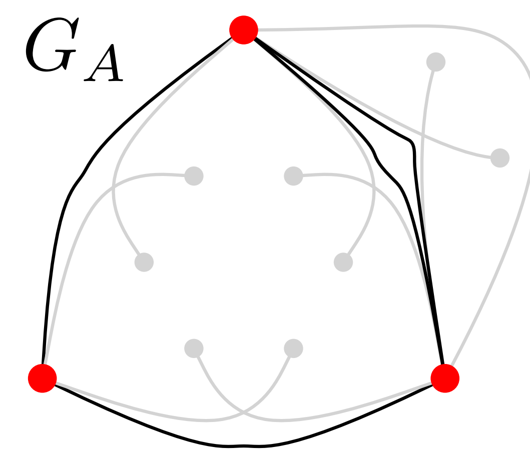


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- No two B-configurations can share the same pair of poles by $3$-connectivity $\rightarrow \beta \leq 3|P| - 6$

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$\rightarrow$ $f(\beta, \tau) \leq 4|P| - 8$ in all points of $D$
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By Hall's theorem, there is a $5$-matching from $F$ into $P$, as desired.

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# THANK YOU!