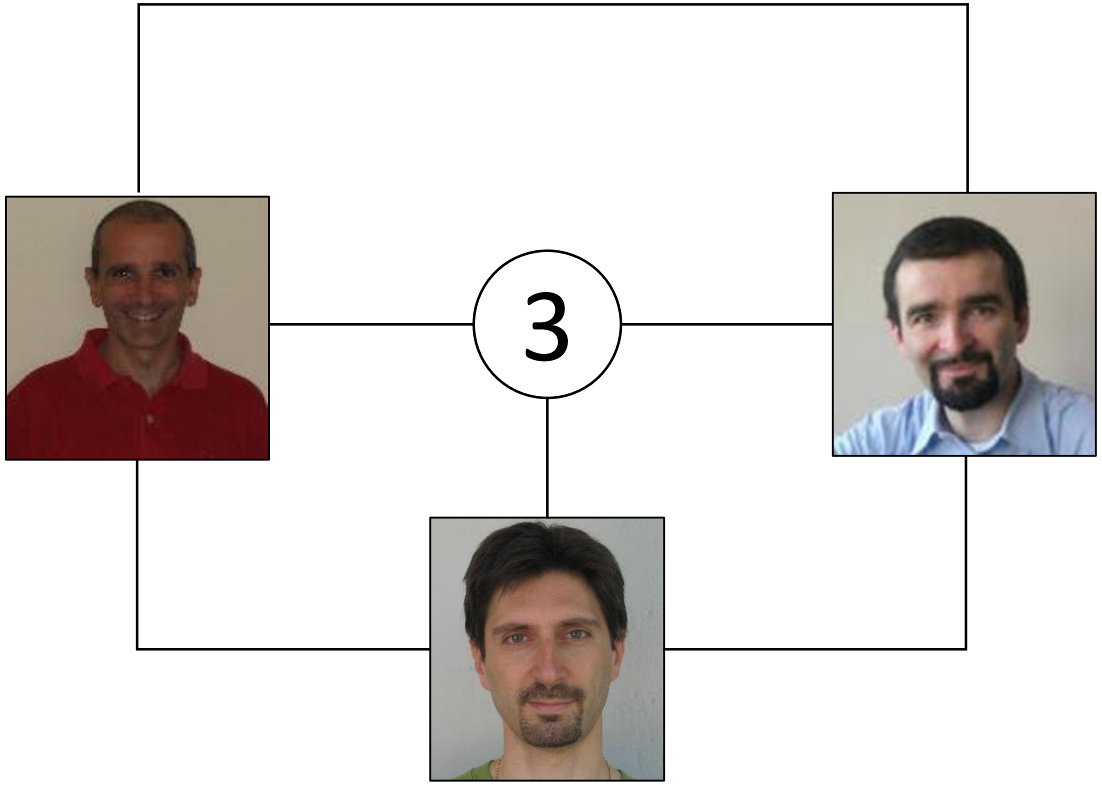
Bend-minimum Orthogonal Drawings in Quadratic Time



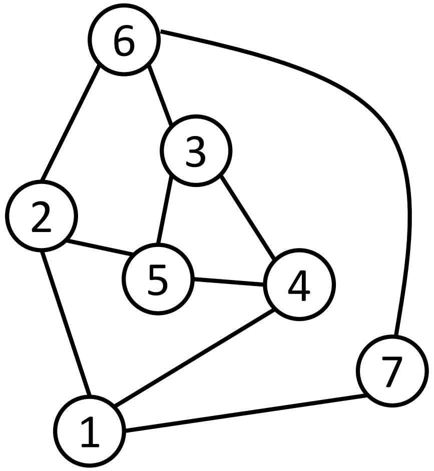
Giuseppe Liotta

Maurizio Patrignani

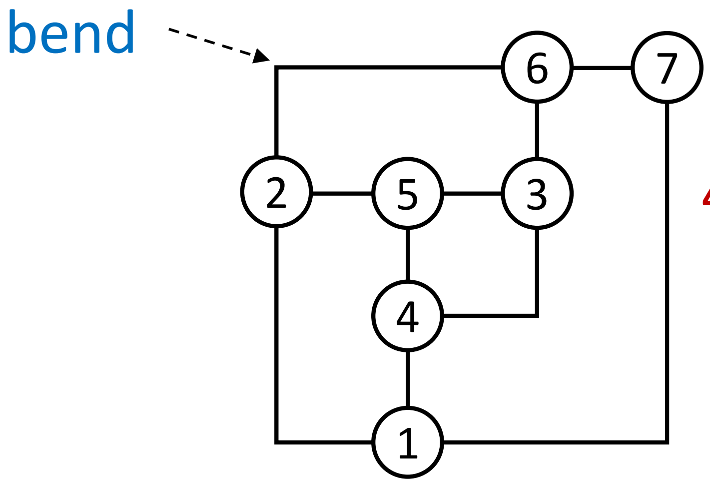
Walter Didimo

The problem

Problem: planar **3-graph** ⟹ planar **bend-minimum** orthogonal drawing

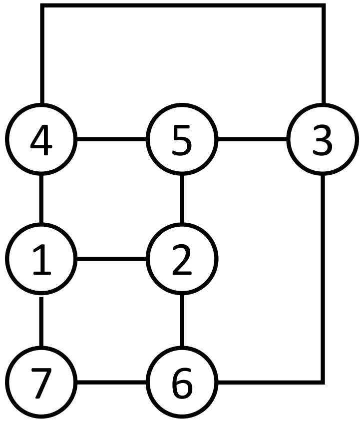


plane 3-graph



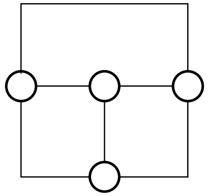
4 bends

bend-min orthogonal drawing
(**fixed embedding**)



3 bends

bend-min orthogonal drawing
(**variable embedding**)



A bit of history

Bend-min orthogonal drawings: **fixed embedding**

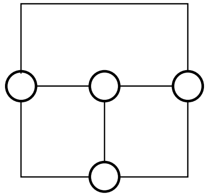
- plane 4-graphs
  - $O(n^2 \log n)$ [Tamassia (1987)]
  - $O(n^{7/4} \sqrt{\log n})$ [Garg and Tamassia (2001)]
  - $O(n^{1.5})$ [Cornelsen and Karrenbauer (2011)]

  based on min-cost flow

- plane 3-graphs

  $O(n)$ [Rahman and Nishizeki (2002)]

  not based on flow techniques



A bit of history

Bend-min orthogonal drawings: **variable embedding**

- planar 4-graphs: NP-hard [Garg and Tamassia (2001)]
- planar 3-graphs

1998

$O(n^5 \log n)$
Di Battista-Liotta-Vargiu

2011

$O(n^{4.5})$
consequence of Cornelsen-Karrenbauer

2017

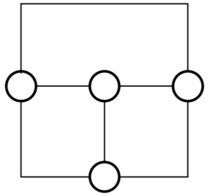
$O(n^{2.43} \log^k n)$
Chang and Yen

2018

$O(n^2)$
our result

?

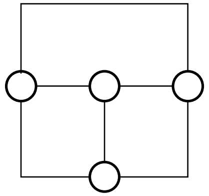
Can we do better?



Our result

Theorem. Let G be an n-vertex (simple) planar 3-graph. There exists an $O(n^2)$-time algorithm that computes a bend-minimum orthogonal drawing of G, with at most two bends per edge.

P. S. the algorithm takes O(n) time if we require that a prescribed edge of G is on the external face



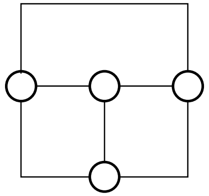
General strategy for biconnected graphs

input: G biconnected planar 3-graph with n vertices

output: bend-min orthogonal drawing Γ of G

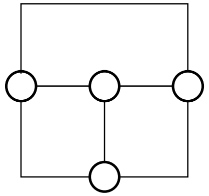
- for each edge *e* of G
  - $\Gamma_e \leftarrow$ min-bend orthogonal drawing of G with *e* on the external face
- return $\Gamma \leftarrow$ min-bends $\{\Gamma_e\}$

Γ_e is computed in linear time



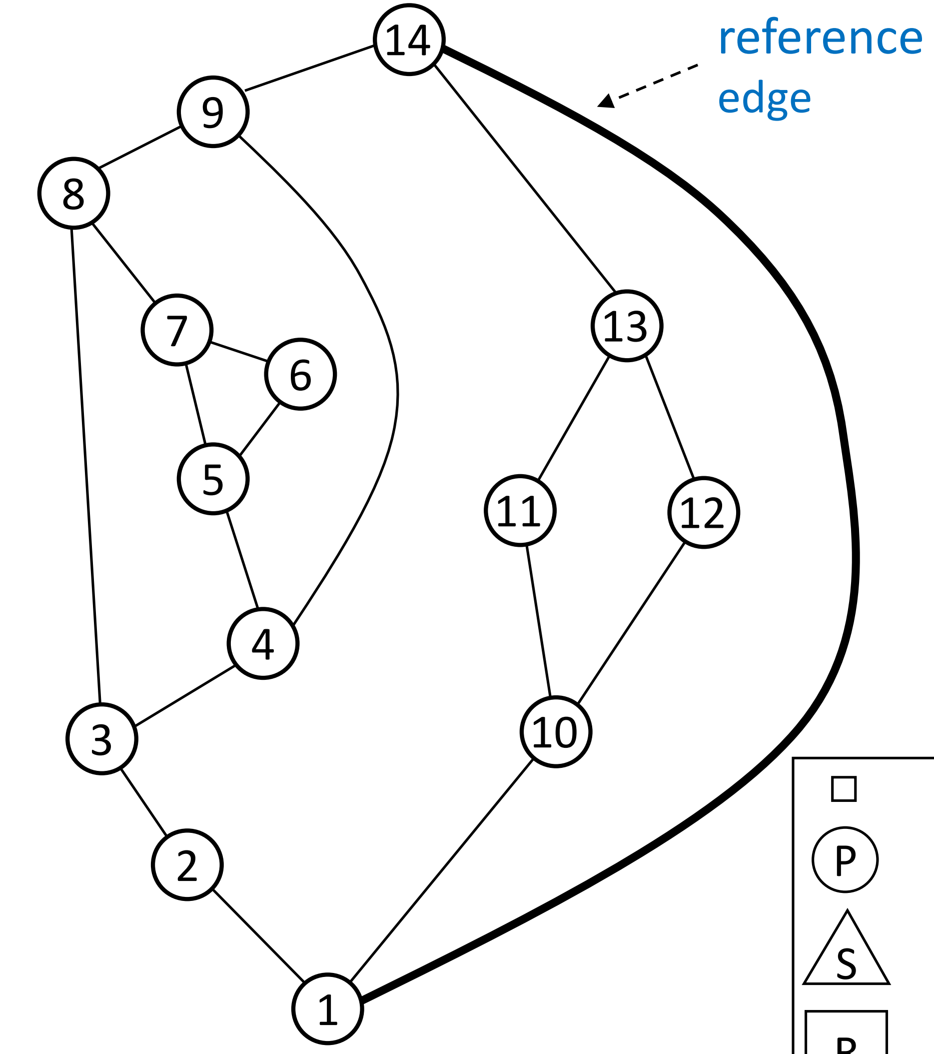
Strategy for the linear-time algorithm

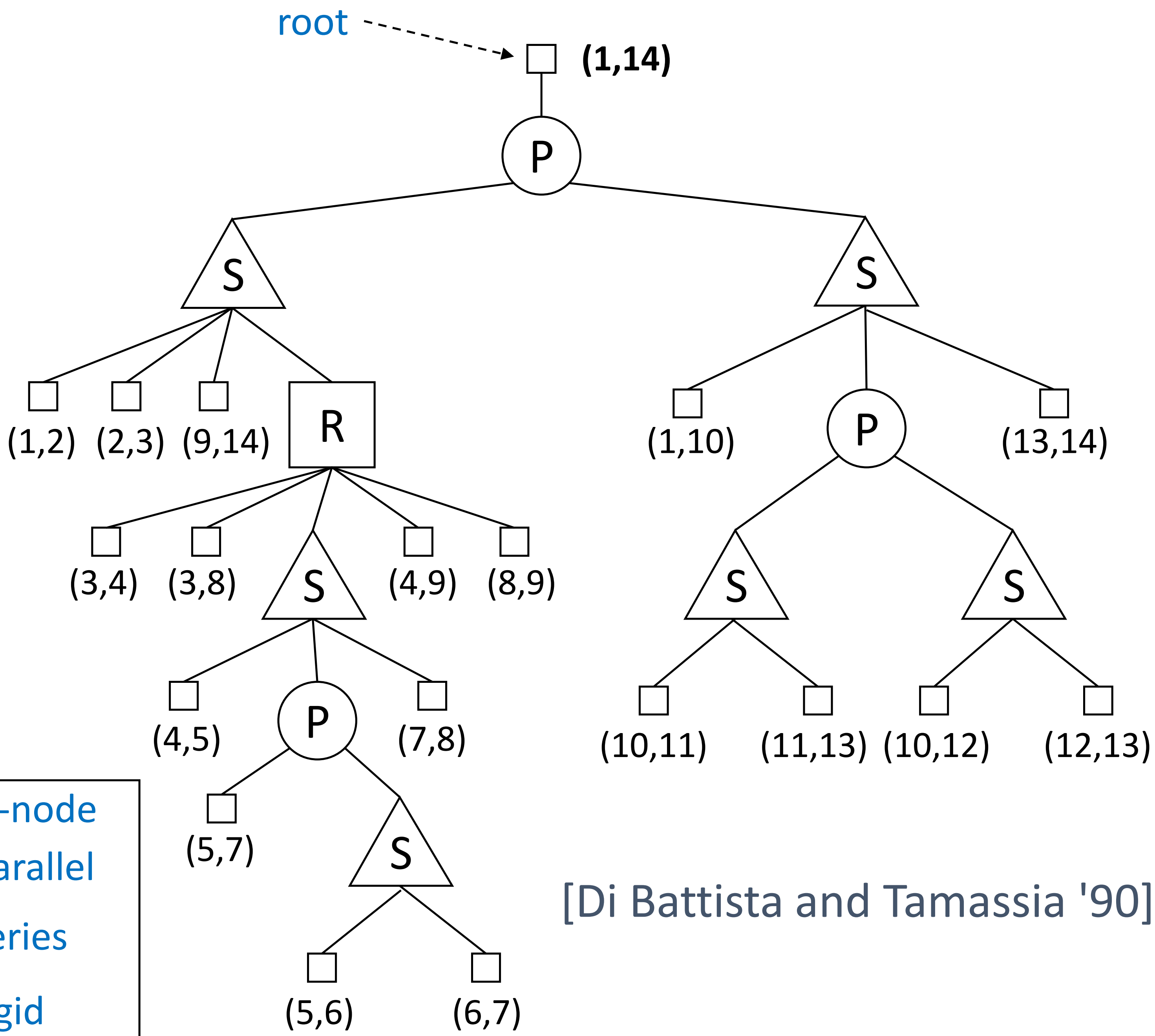
- Incremental construction of Γ_e
  1. bottom-up visit of the SPQR-tree + *orthogonal spirality* (similar to Di Battista, Liotta, Vargiu 1998)
  2. new properties of bend-min orthogonal drawings of planar 3-graphs
  3. non-flow based computation of bend-min orthogonal drawings for the rigid components

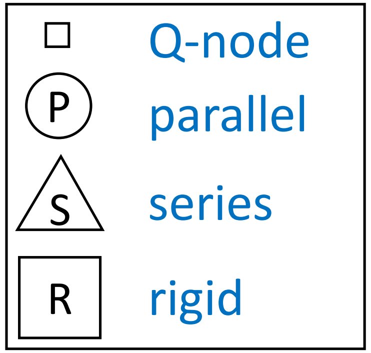


\begin{SPQR-trees}

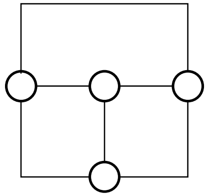
SPQR-trees



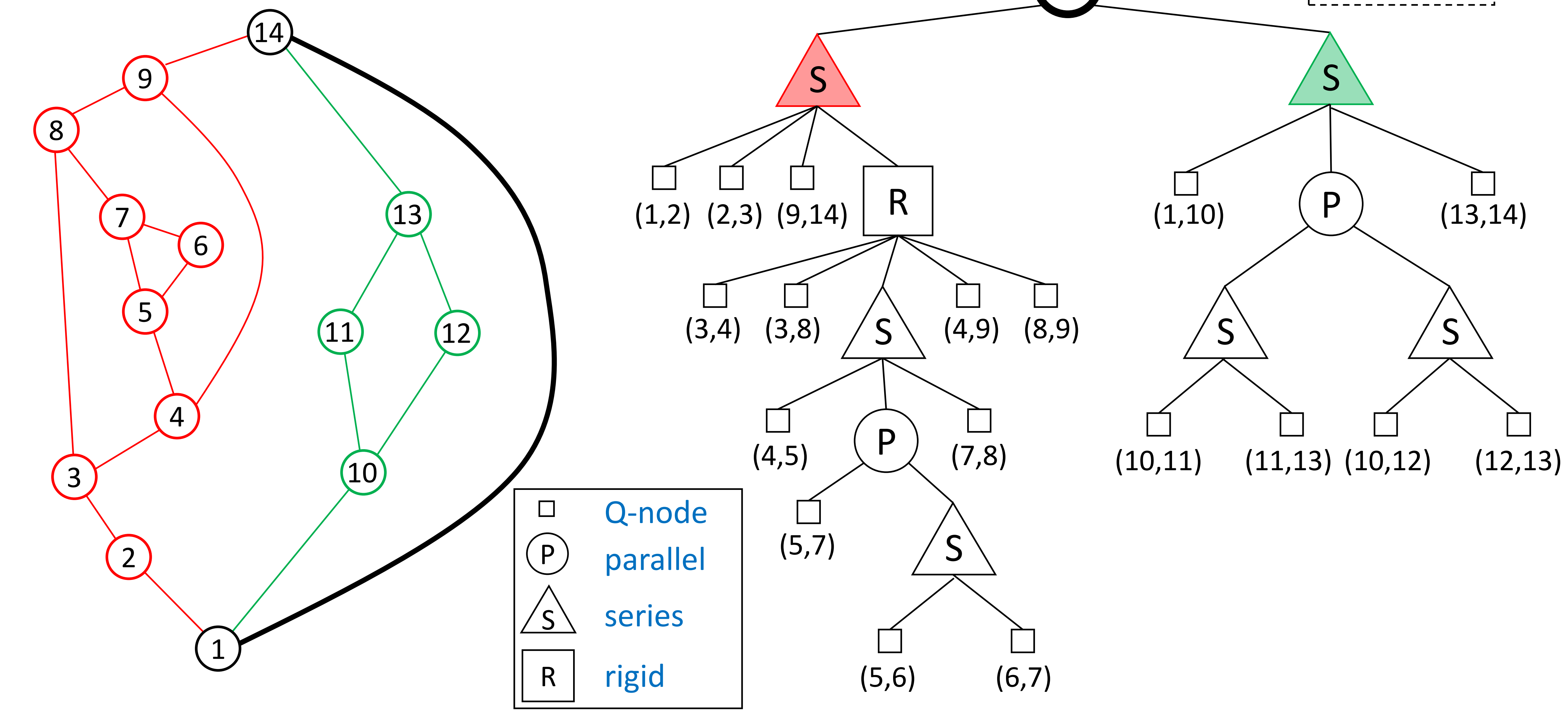




[Di Battista and Tamassia '90]



SPQR-trees



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Q-node
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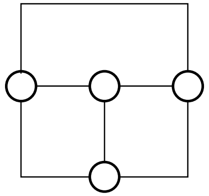
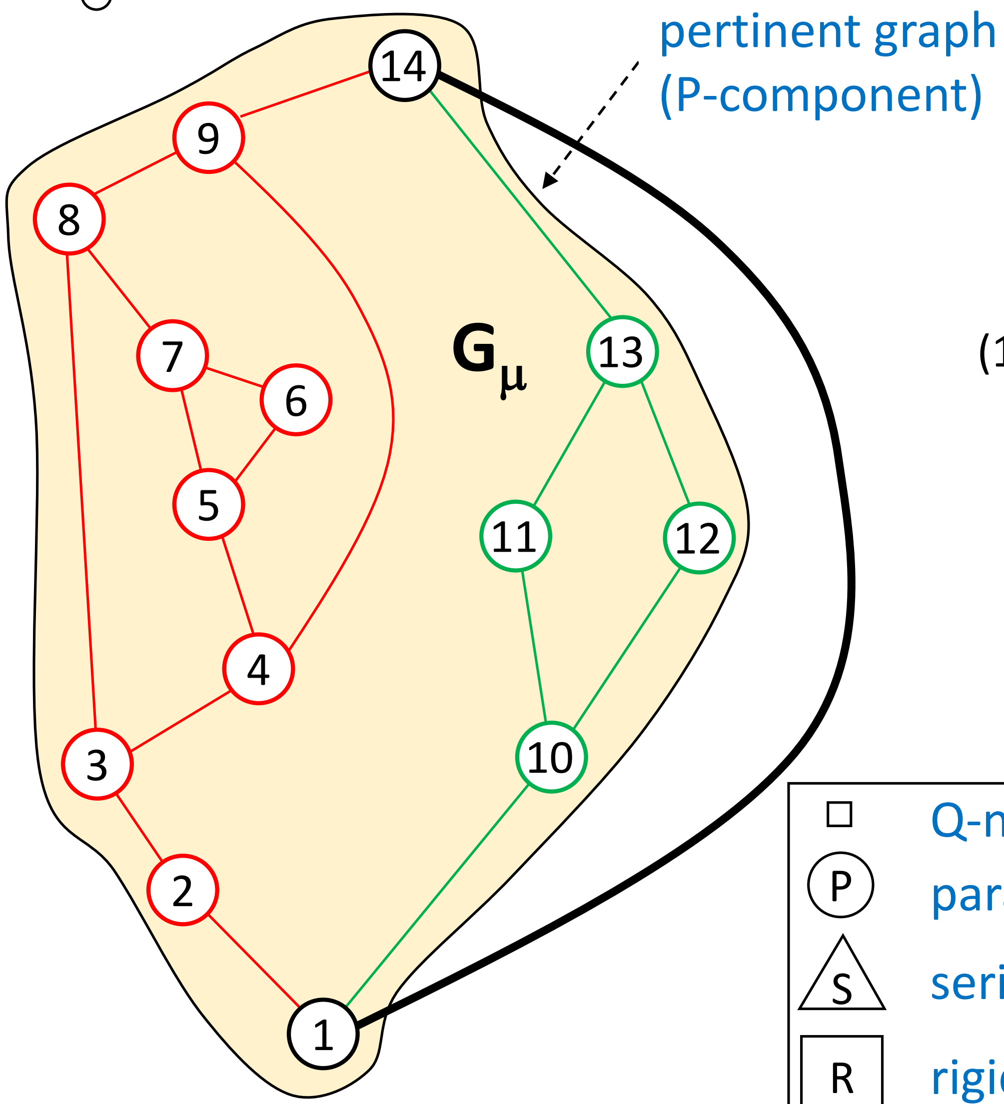
rigid



SPQR-trees


pertinent graph
(P-component)
G_μ
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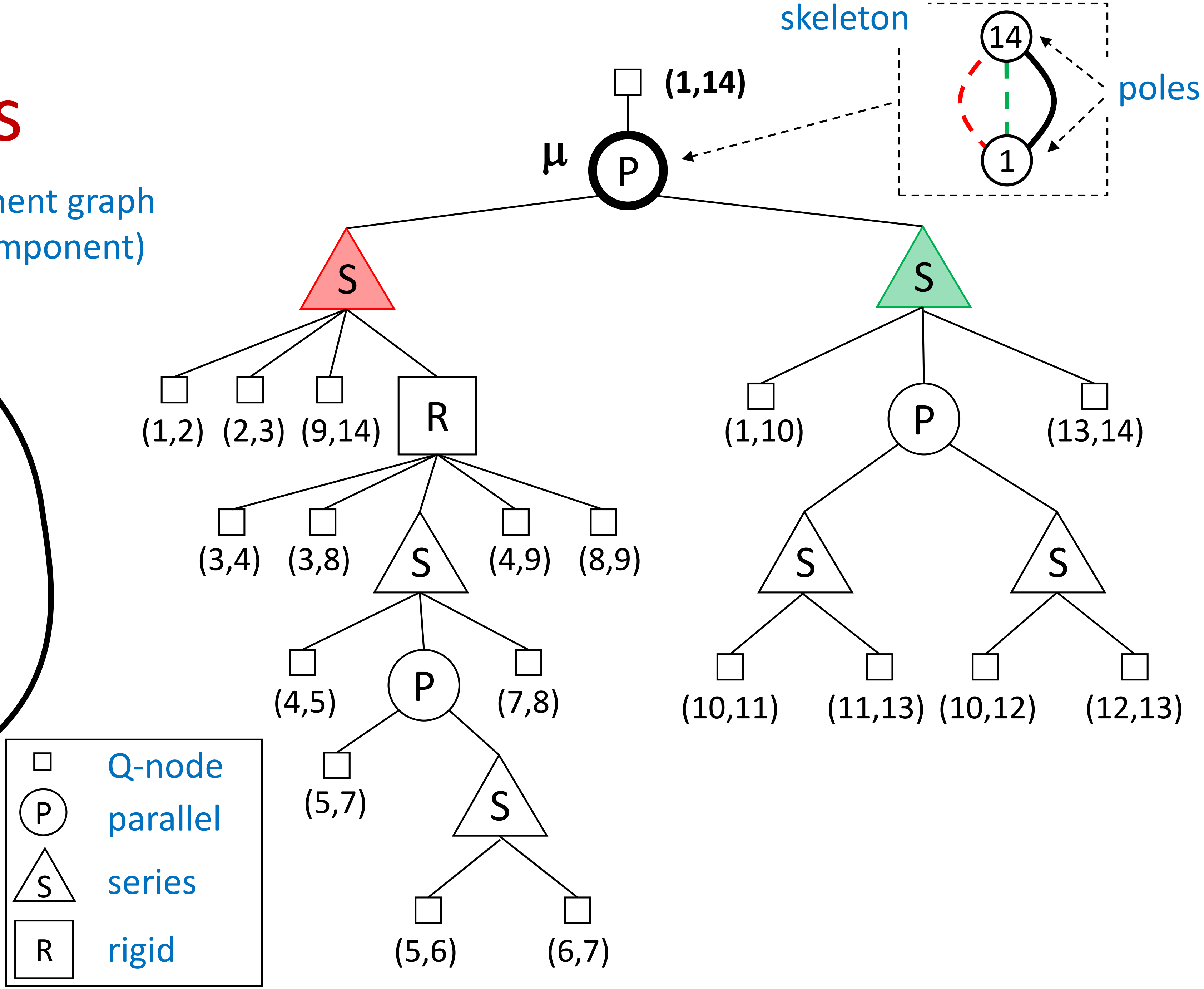
skeleton
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Q-node
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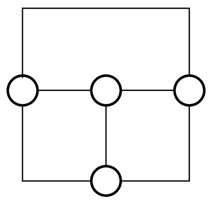
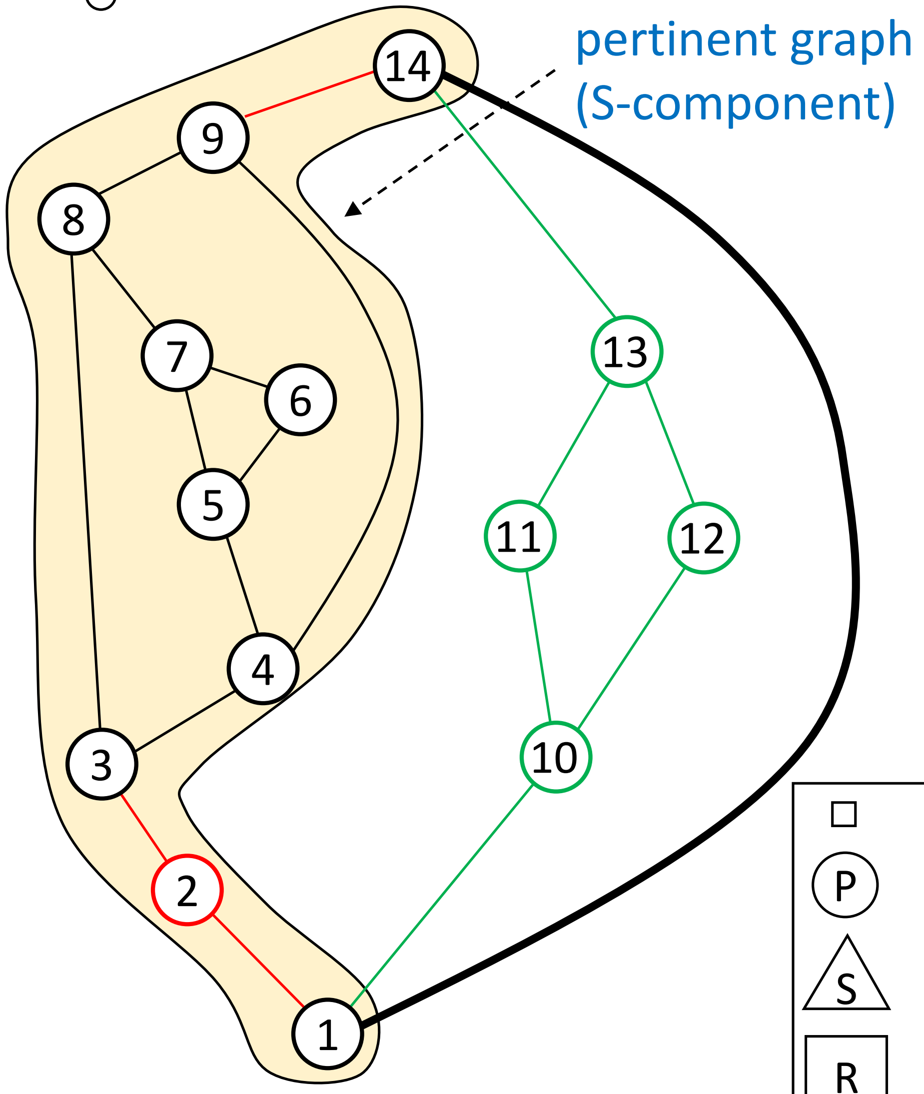


SPQR-trees


pertinent graph
(S-component)
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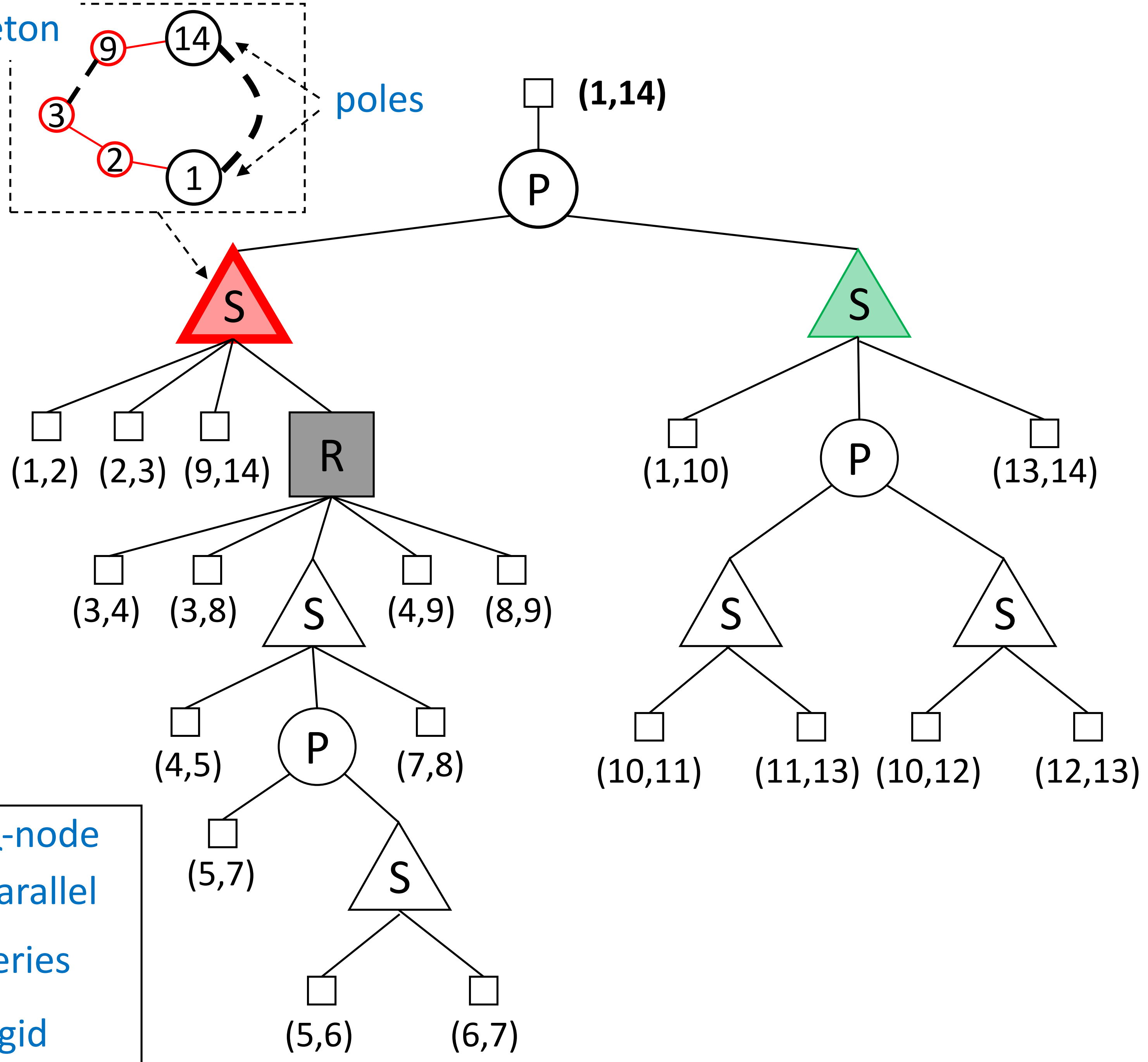
skeleton
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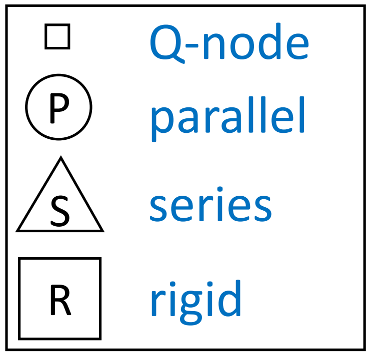
Q-node
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parallel
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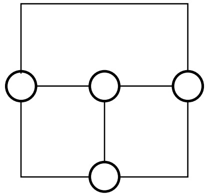
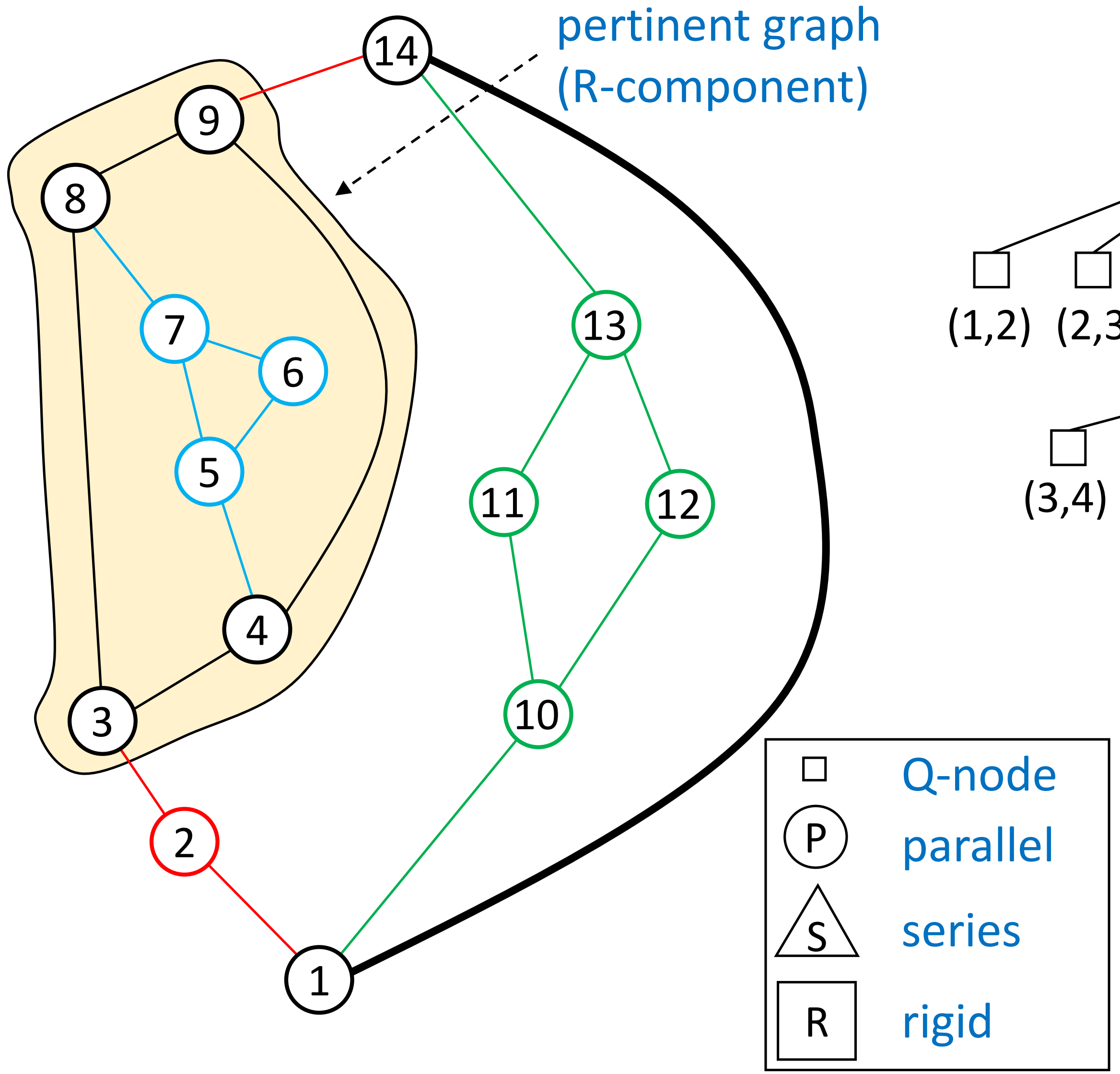
series
R
rigid



SPQR-trees



pertinent graph
(R-component)
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Q-node
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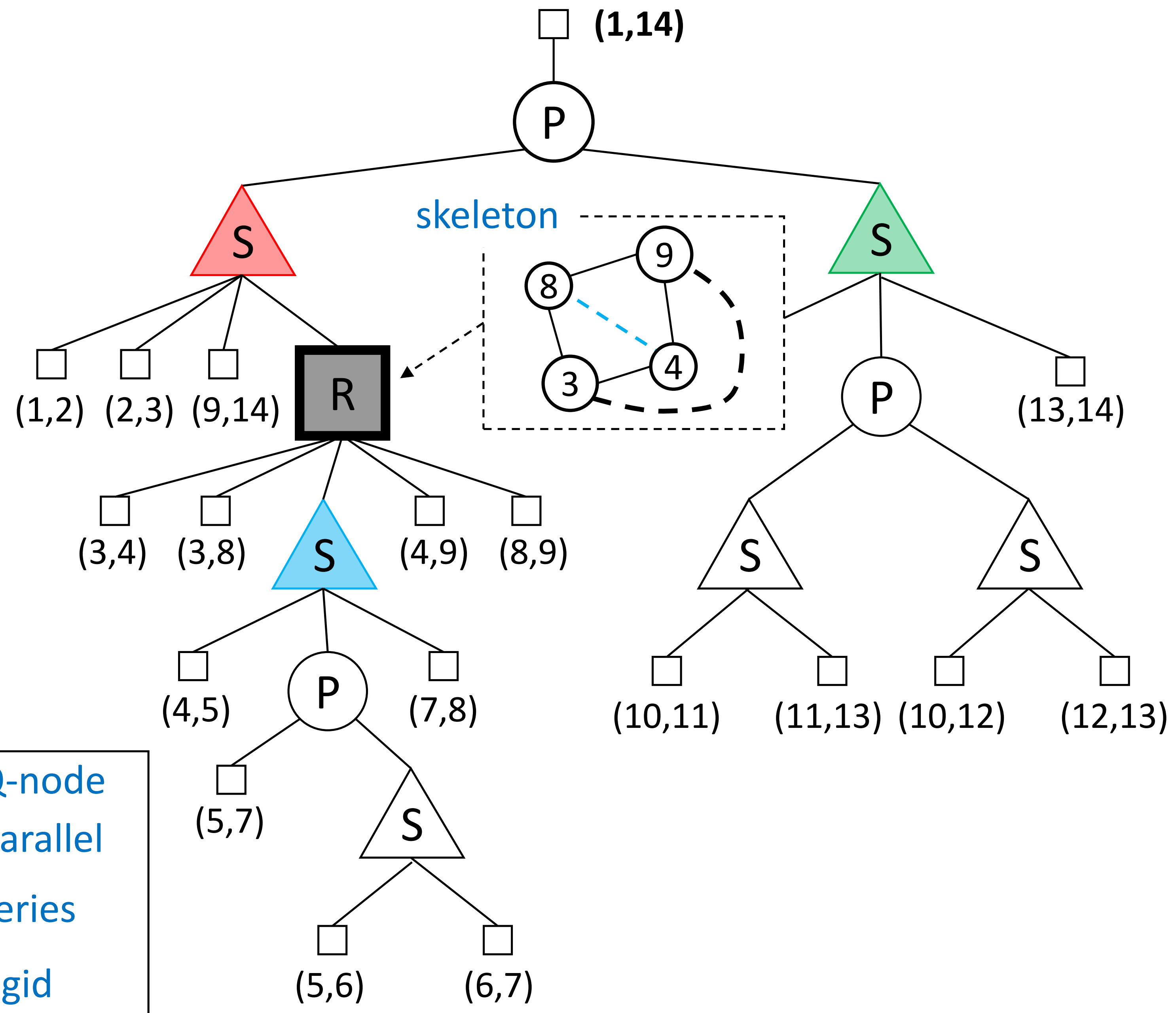

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skeleton
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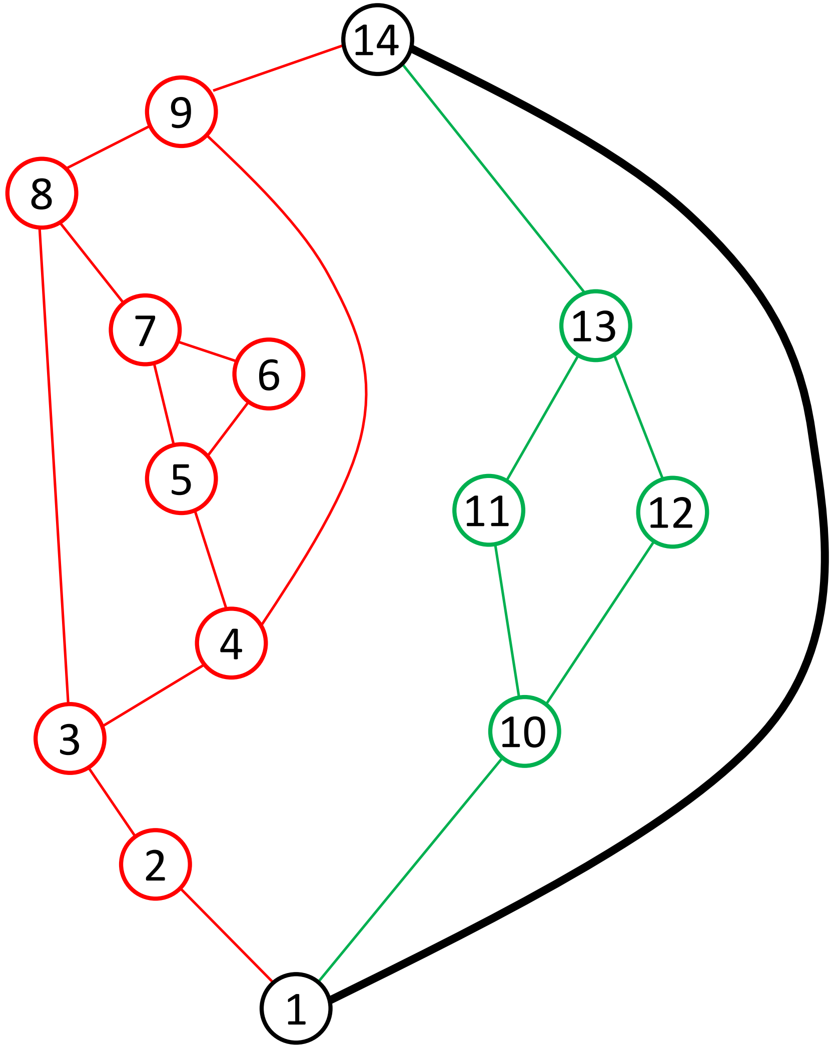
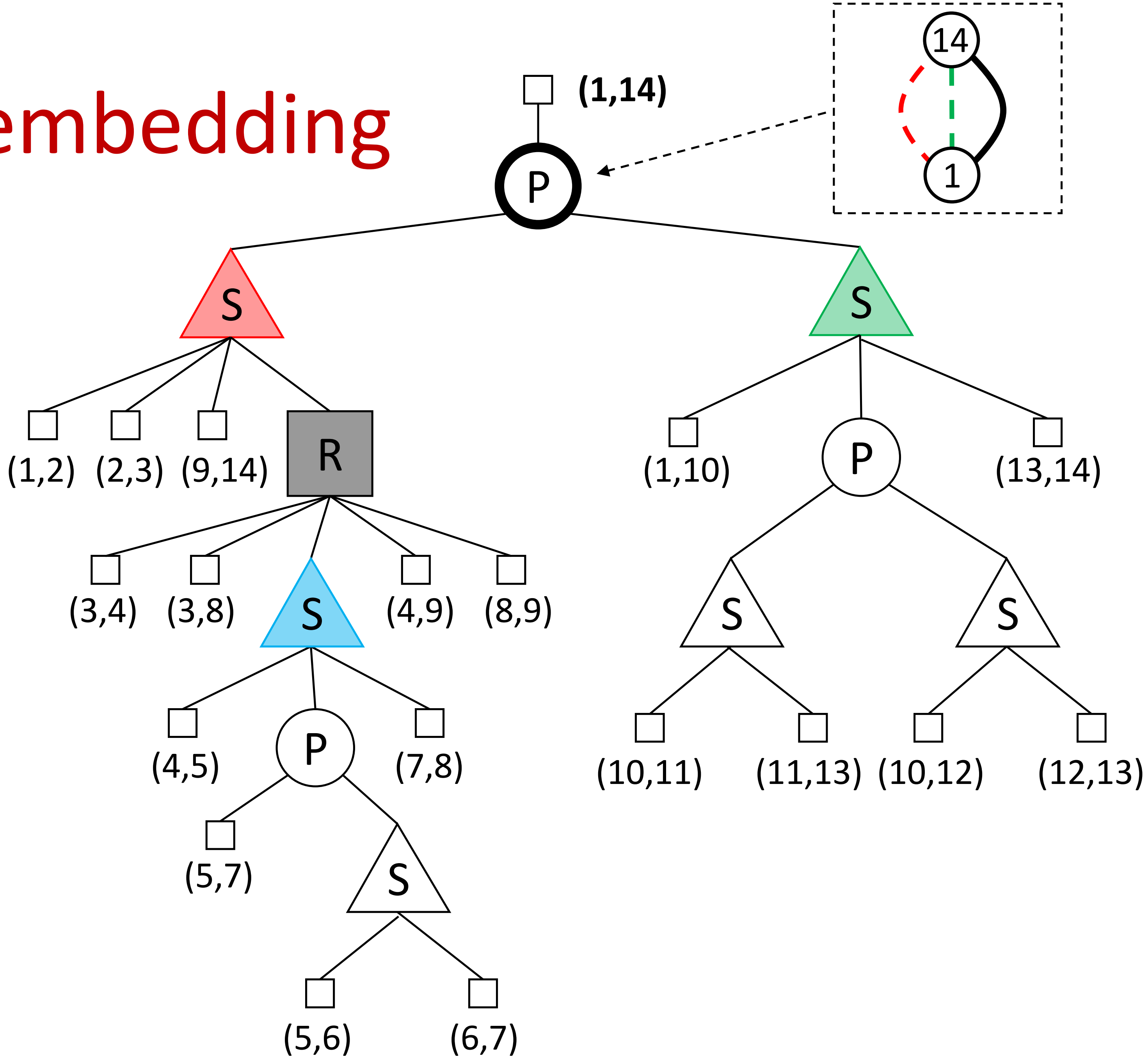
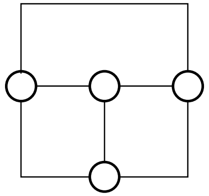
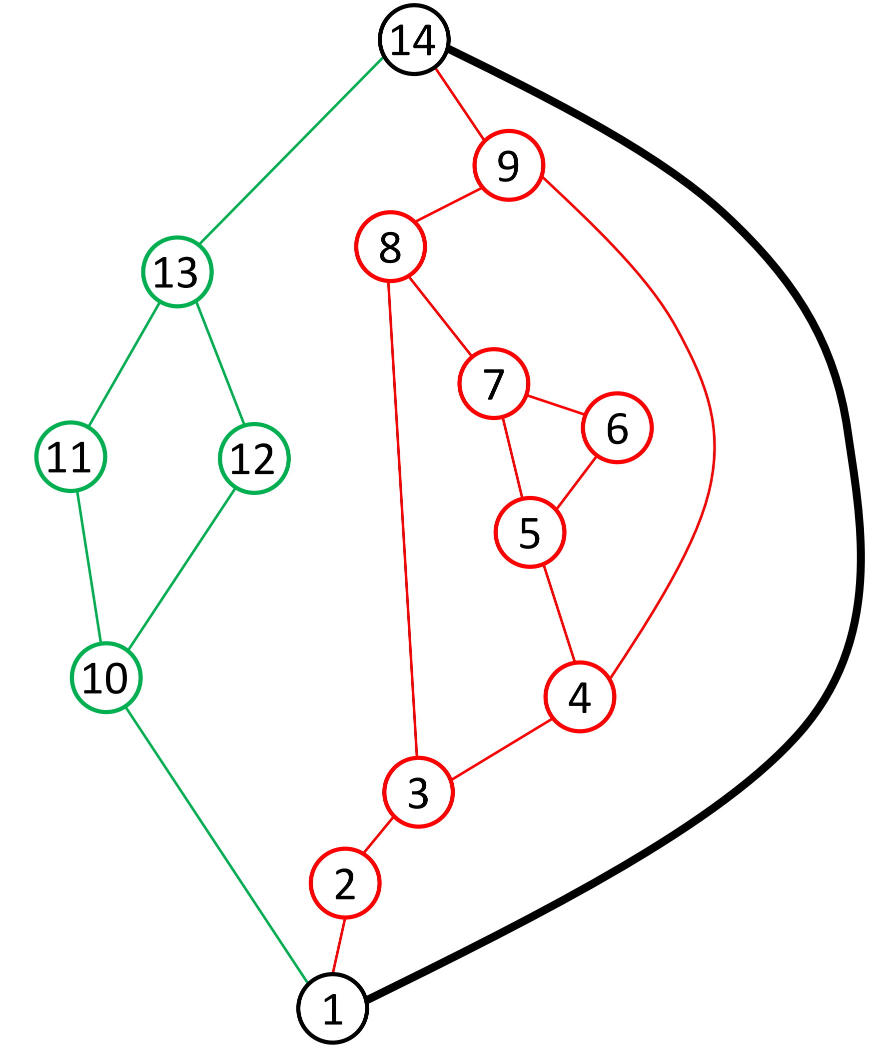
Changing the embedding

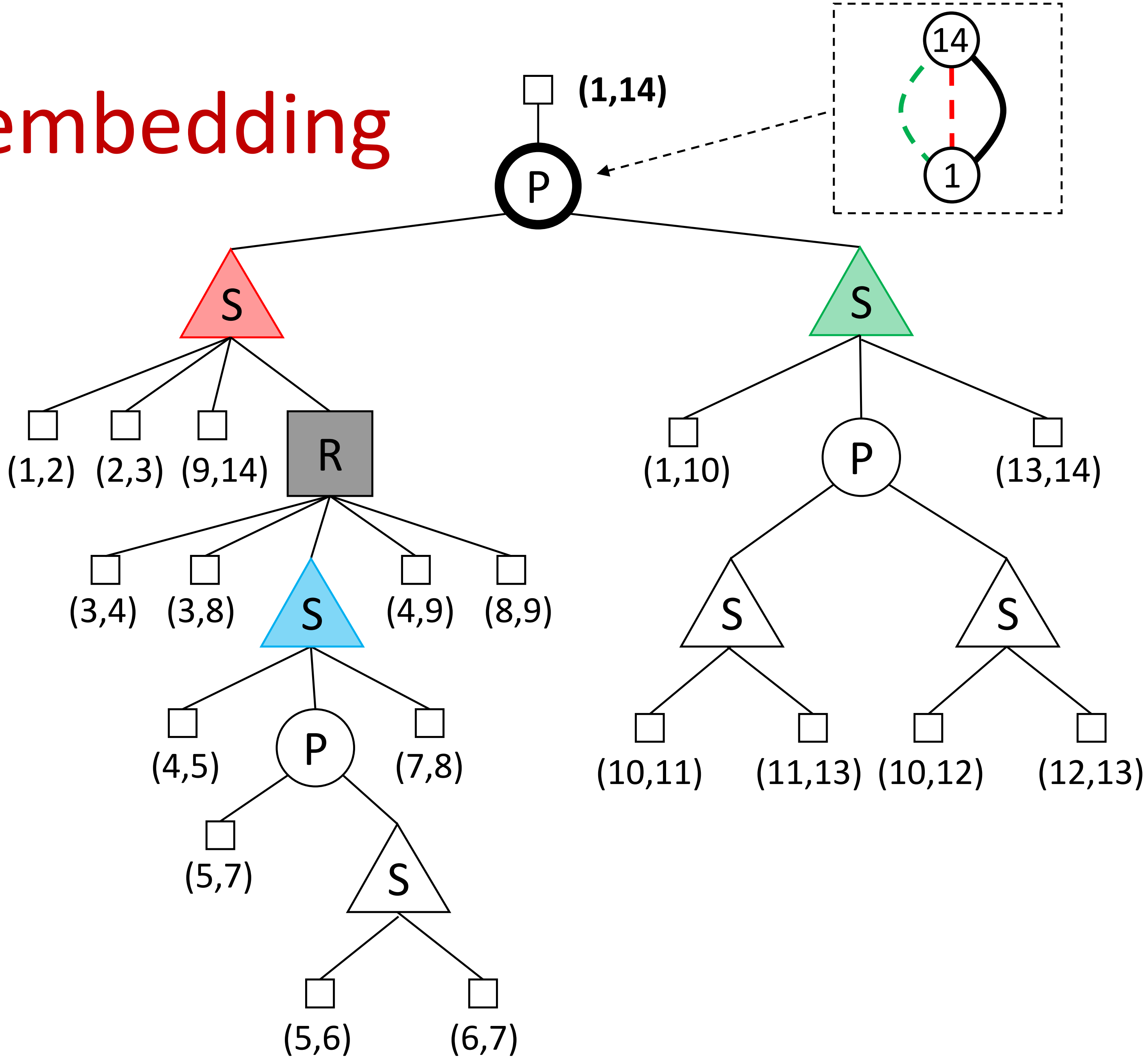




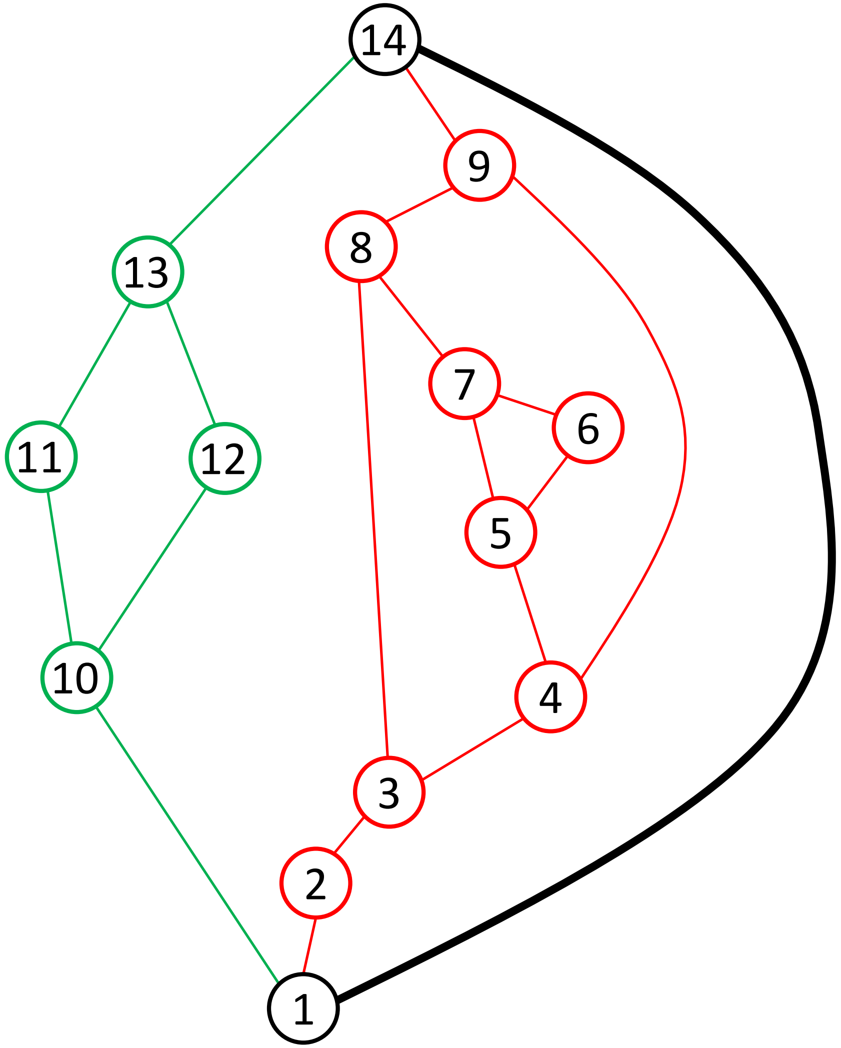


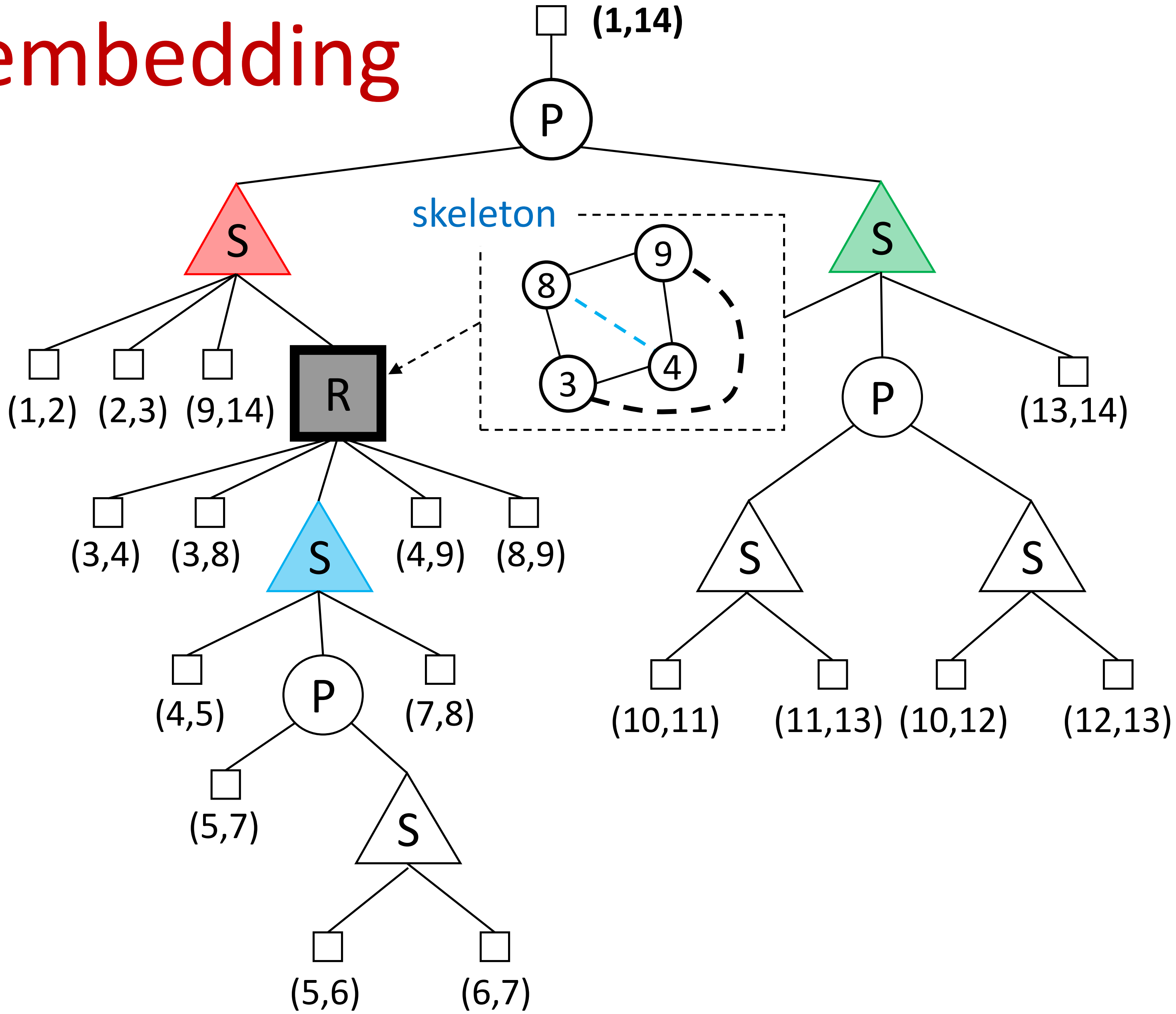
Changing the embedding



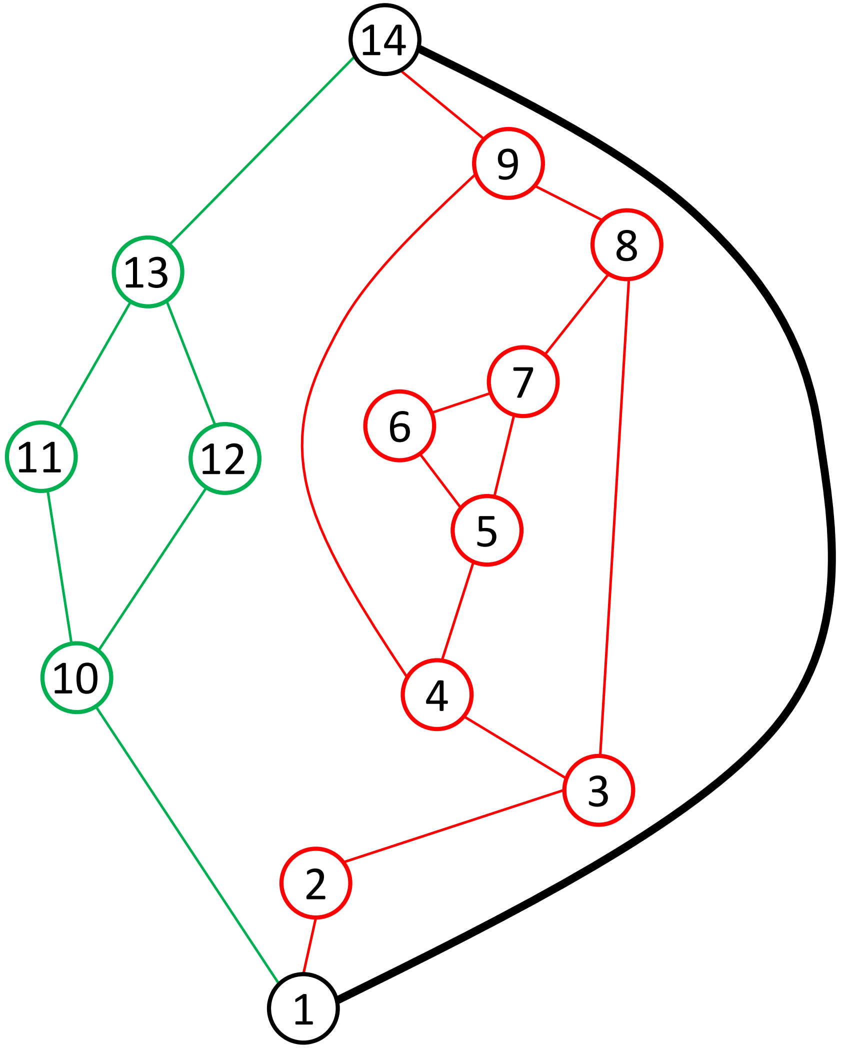


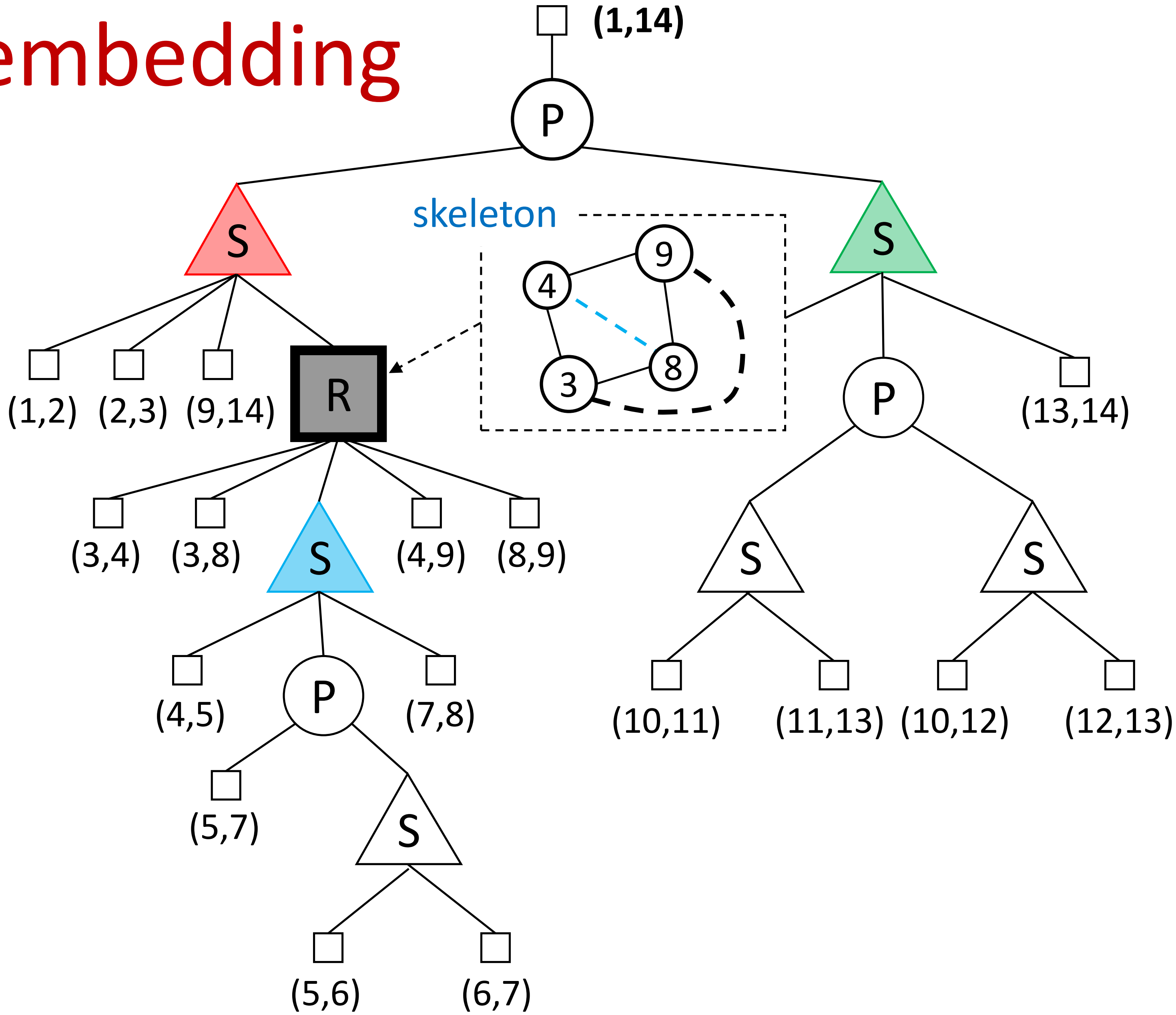
Changing the embedding

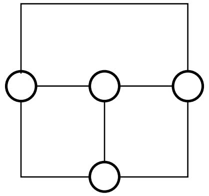




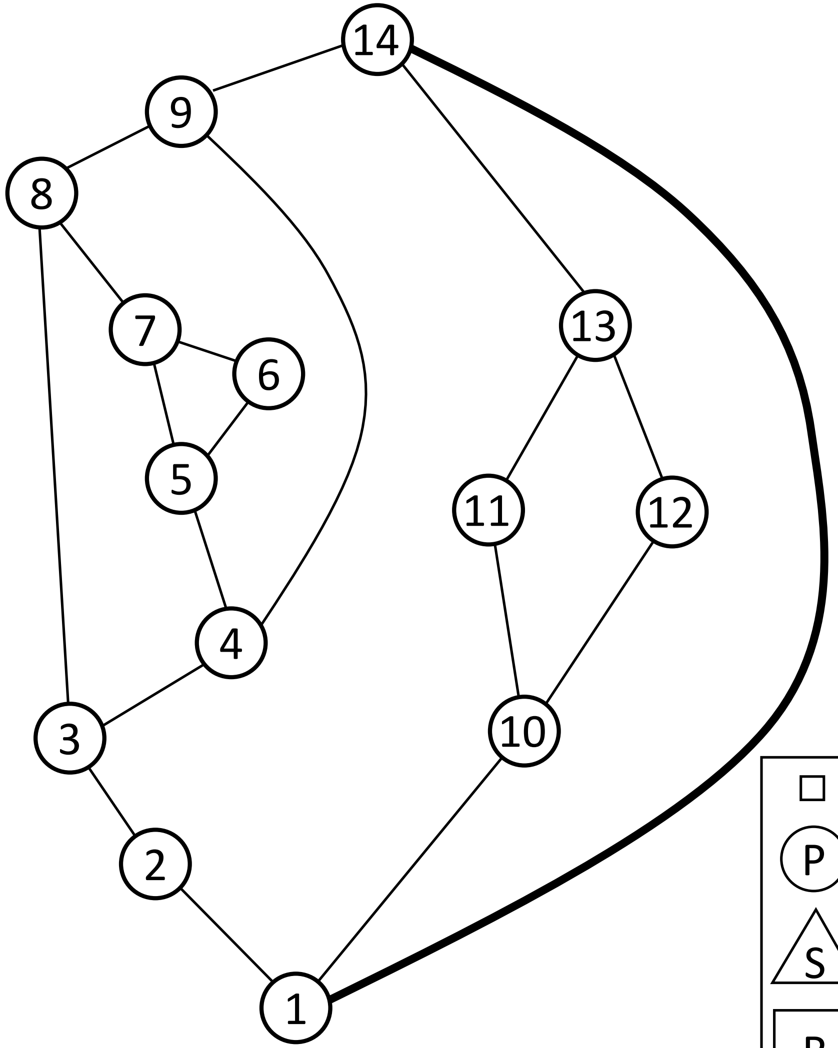
Changing the embedding







SPQR-trees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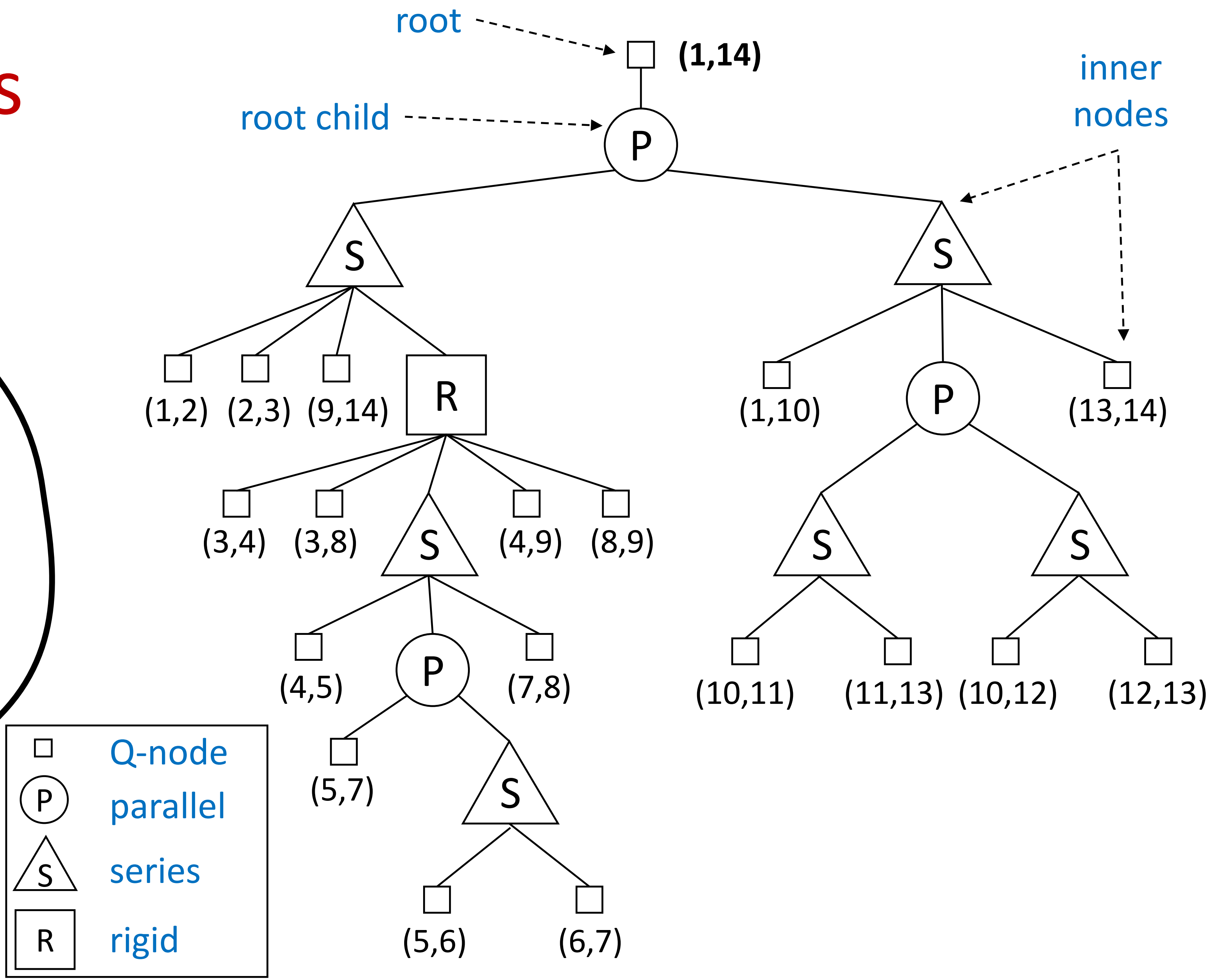
root
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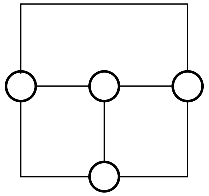
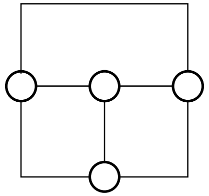


\end{SPQR-trees}



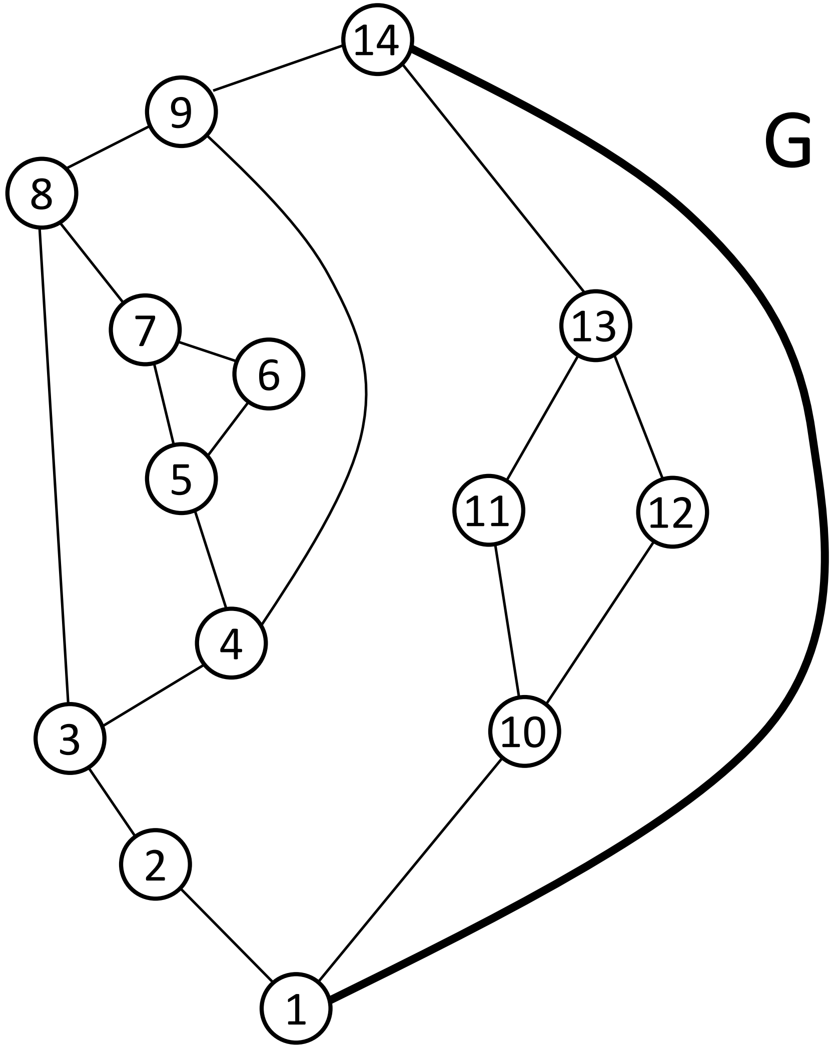
Orthogonal representations

orthogonal representation = equivalence class of orthogonal drawings with the same vertex angles and the same sequence of bends along the edges

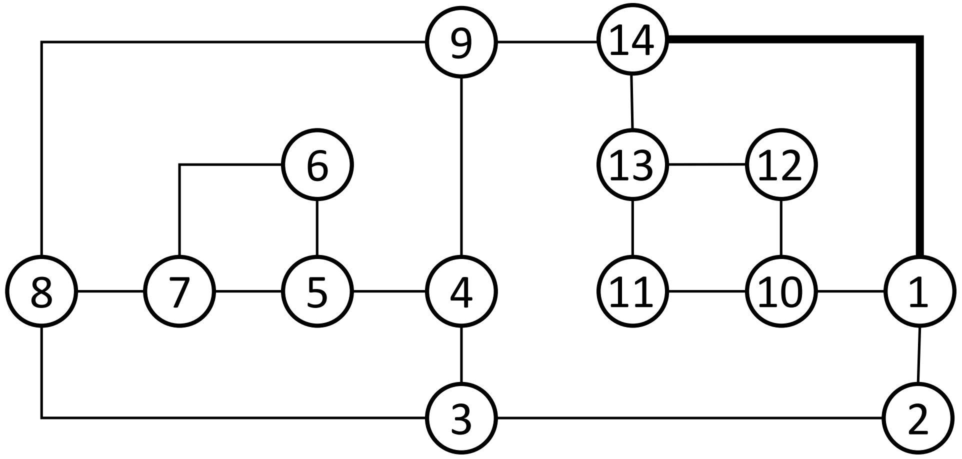
- a drawing of an orthogonal representation can be computed in linear time [Tamassia '97]

orthogonal component = orthogonal representation H_μ of a component G_μ

Orthogonal components: example

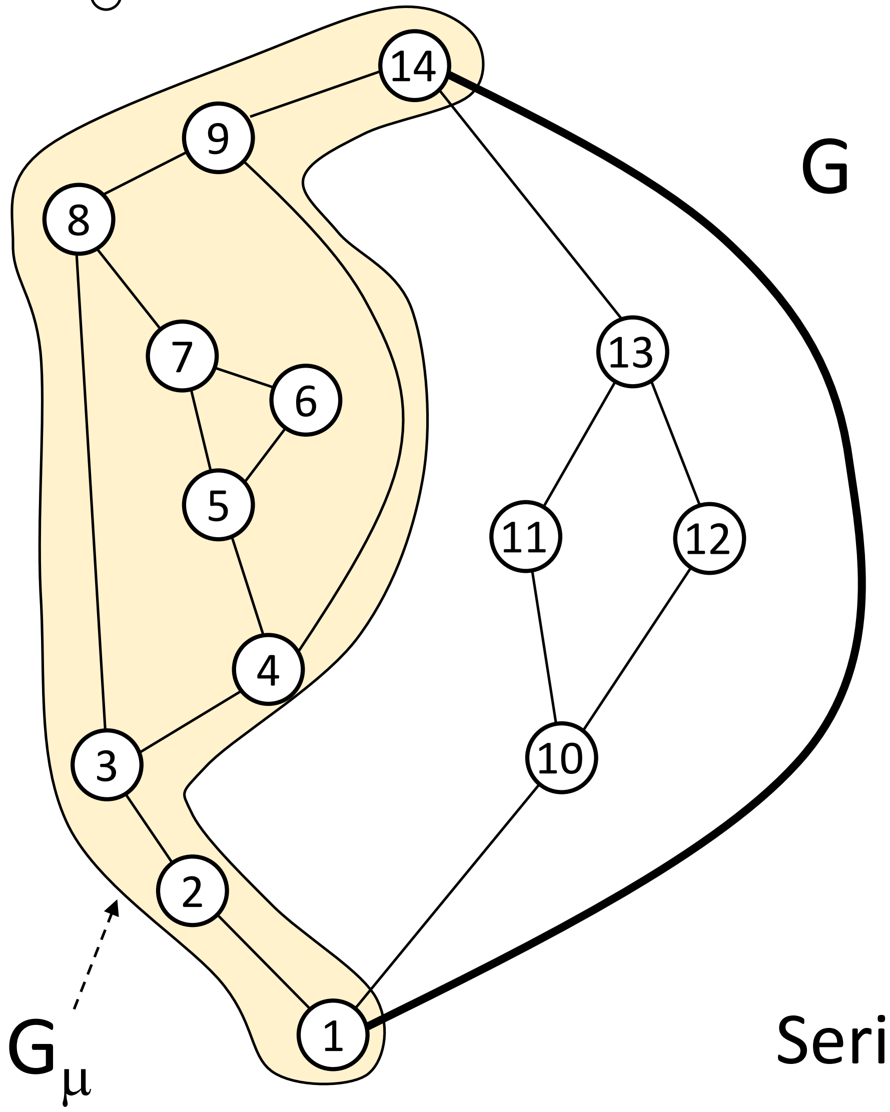


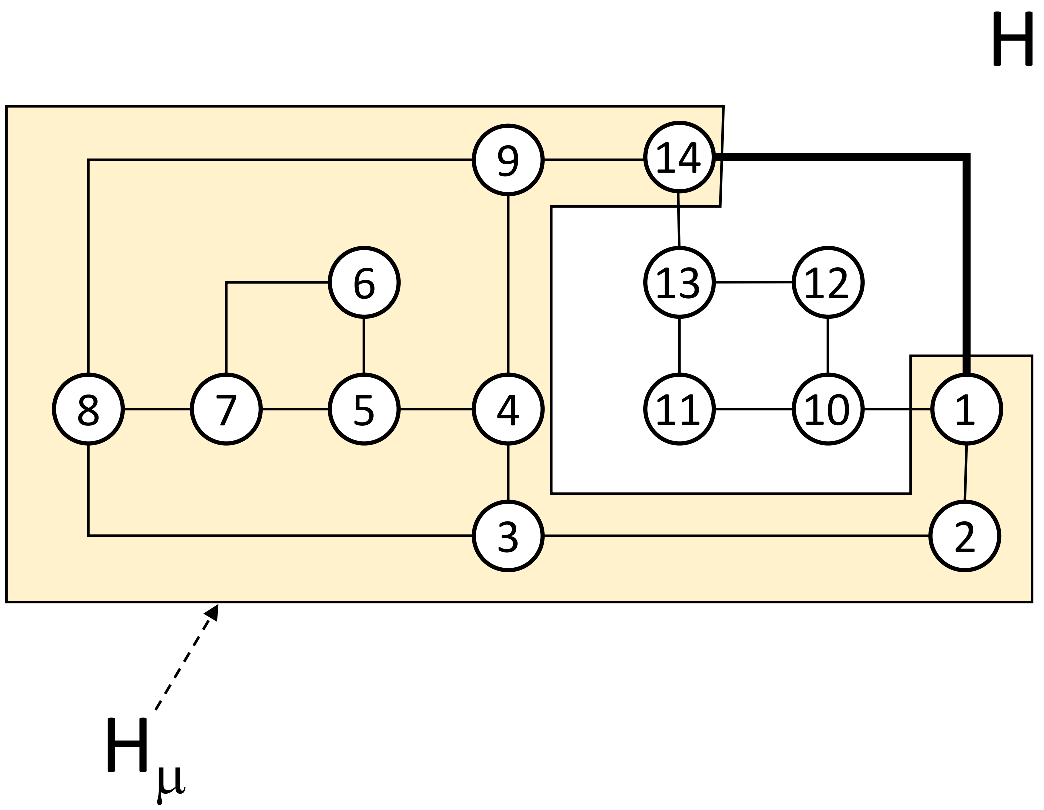
G



H

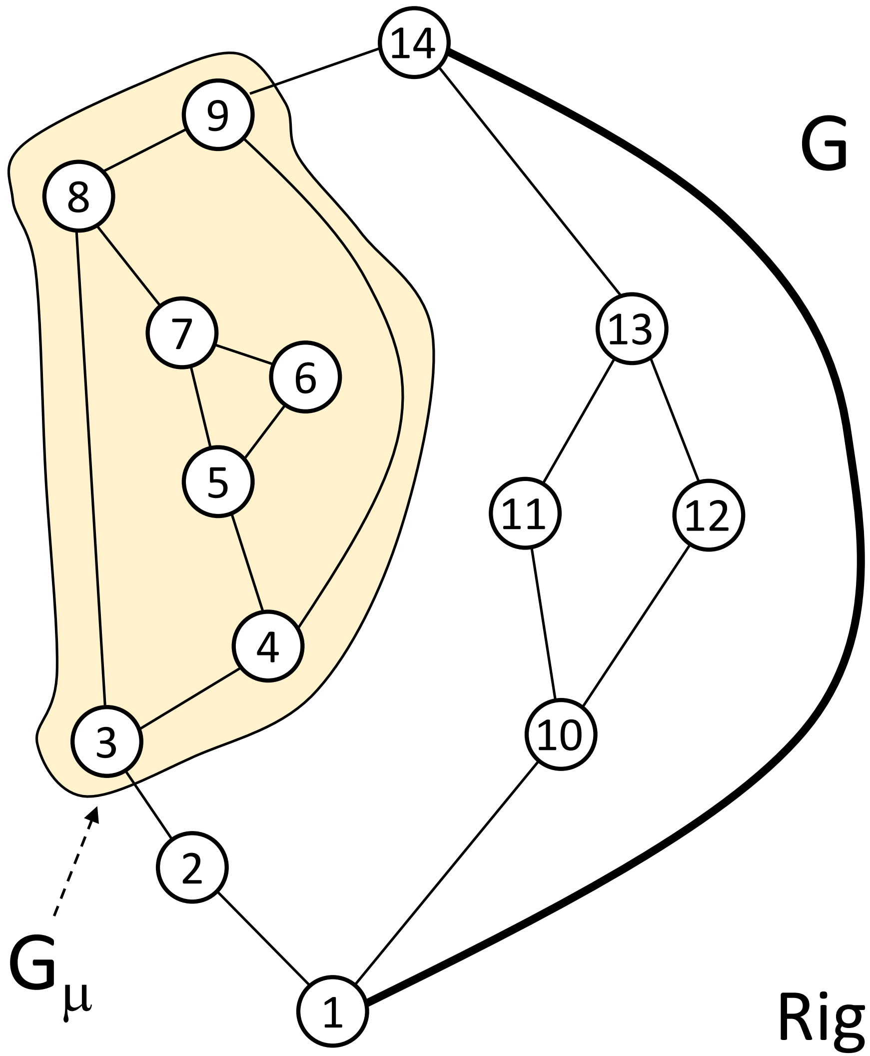
Orthogonal components: examples

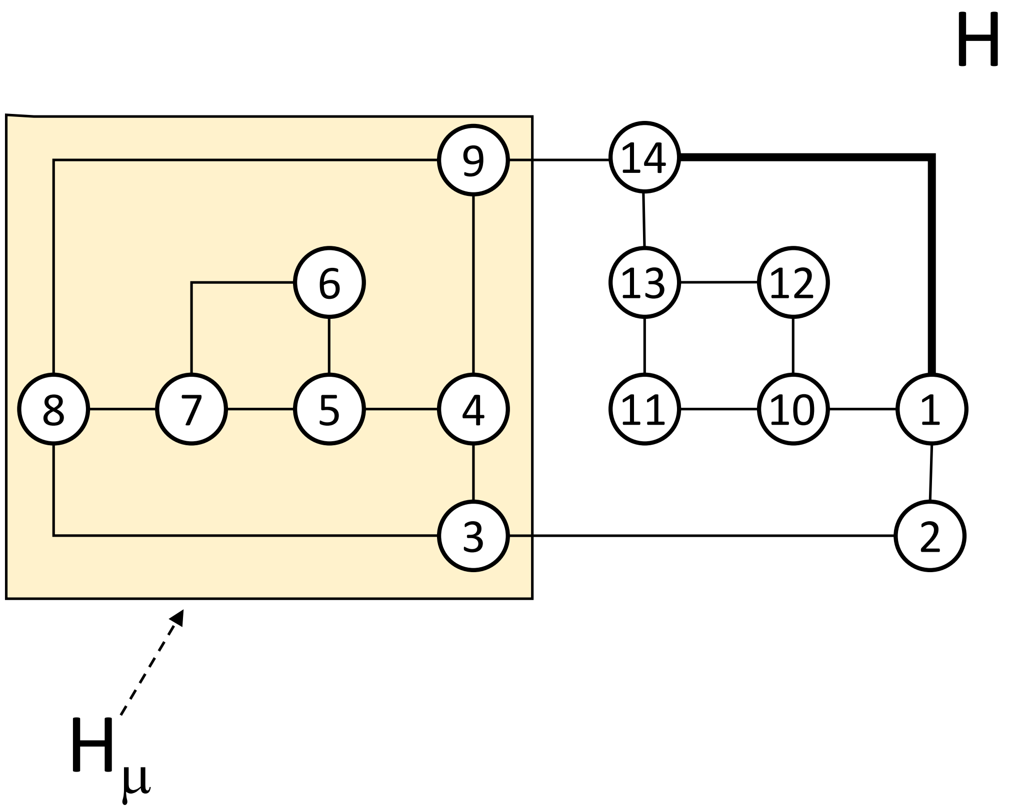




Series (orthogonal) component

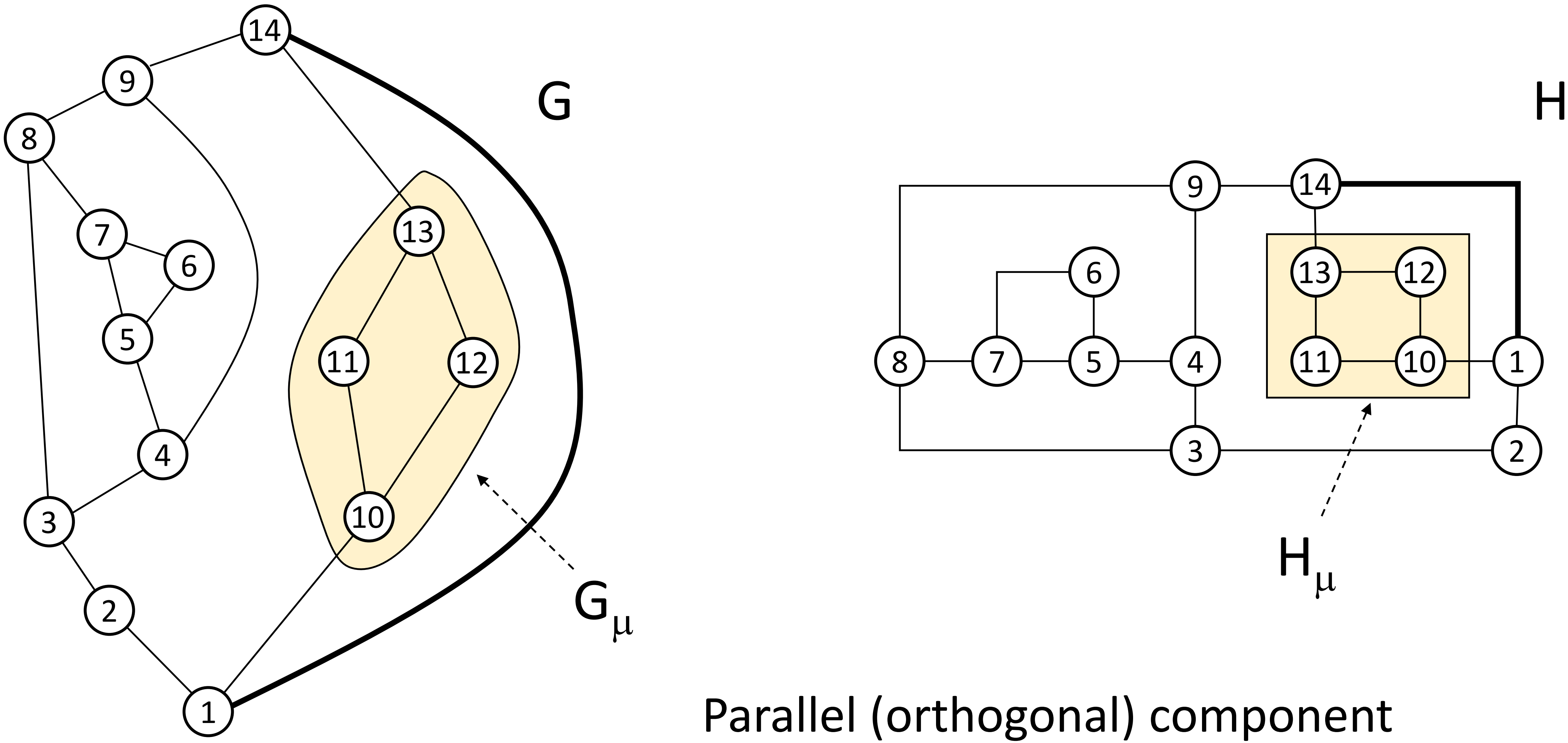
Orthogonal components: examples



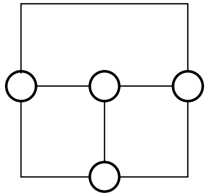


Rigid (orthogonal) component

Orthogonal components: examples

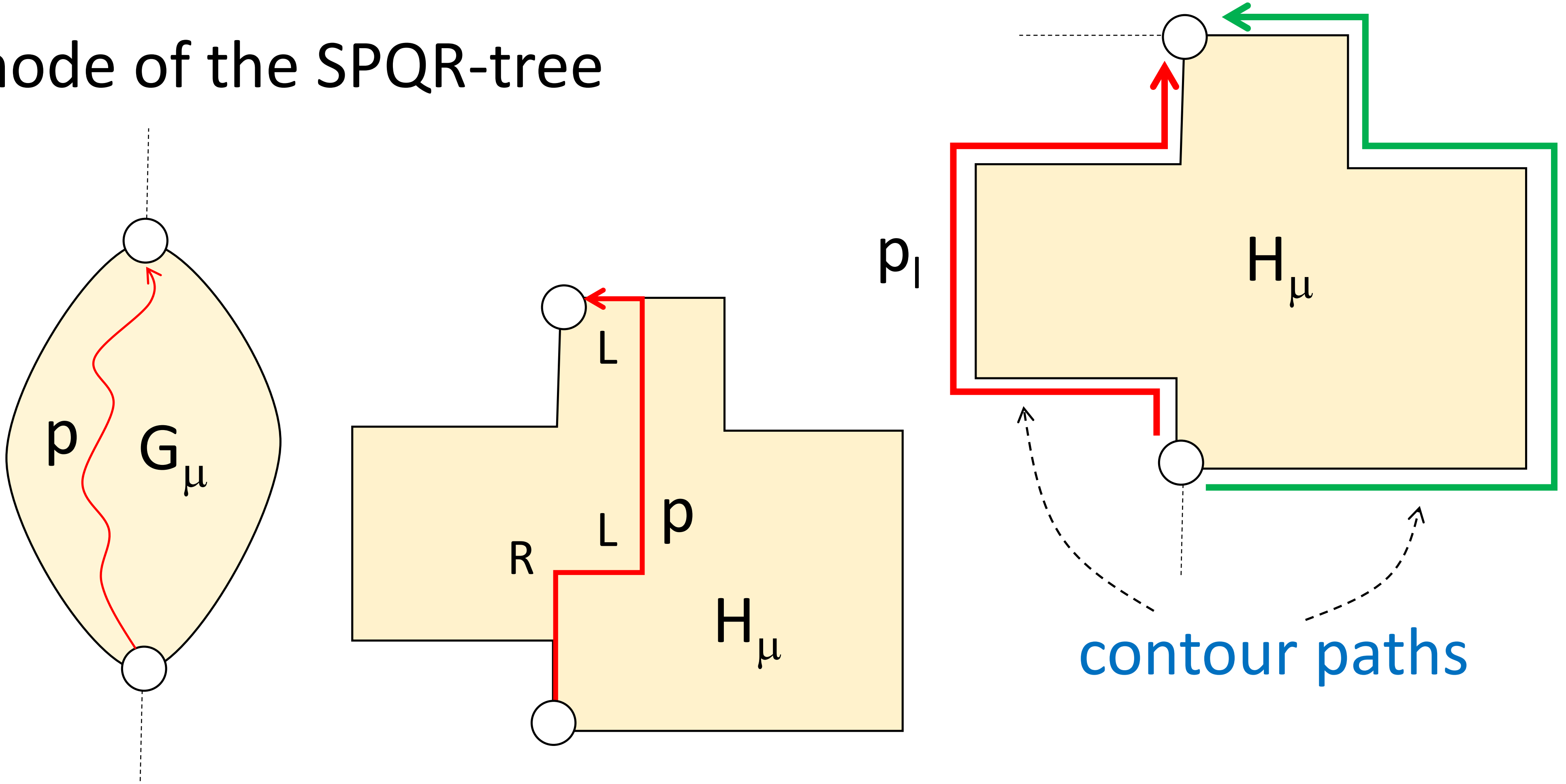


Parallel (orthogonal) component



Turn number and contour paths

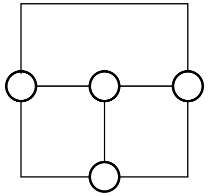
μ = node of the SPQR-tree



$t(p_l) = 0$

$t(p_r) = 2$

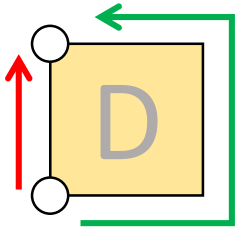
t(p) = turn number = |#left turns – # right turns| (along p)



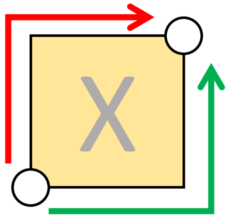
P- and R-components: shapes

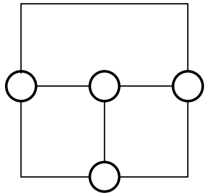
μ = P-node or R-node

H_μ is D-shaped $\Leftrightarrow$ $t(p_l) = 0$ and $t(p_r) = 2$ or vice versa



H_μ is X-shaped $\Leftrightarrow$ $t(p_l) = t(p_r) = 1$

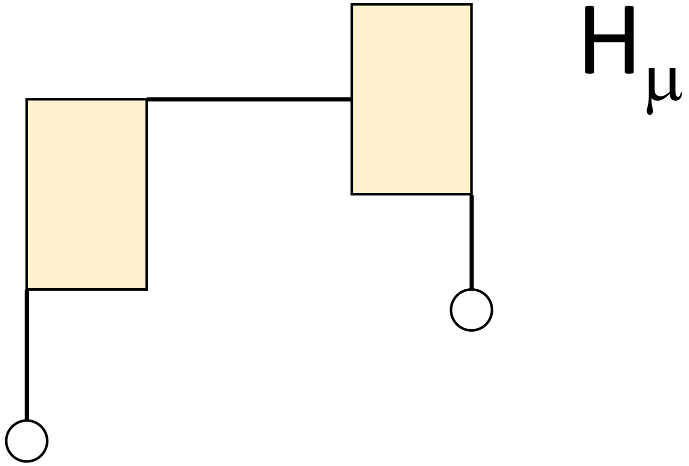


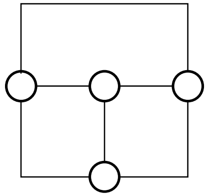


Inner S-components: spirality

μ = inner S-node

Lemma. All paths between the poles of an orthogonal component H_μ have the same turn number

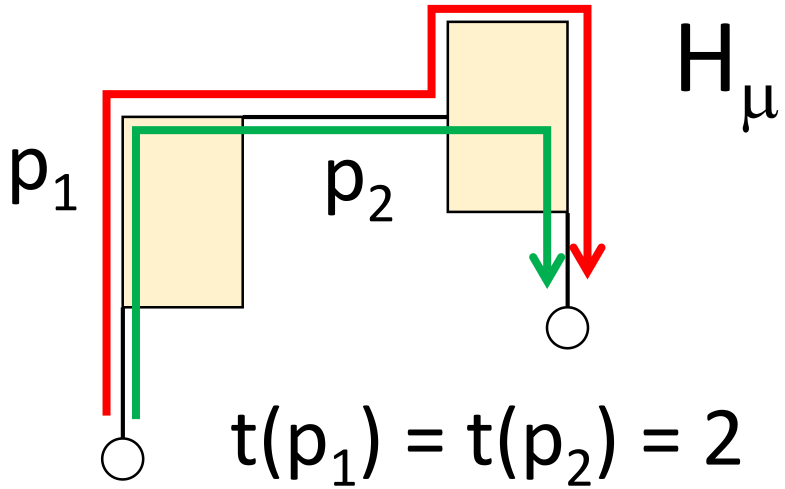




Inner S-components: spirality

μ = inner S-node

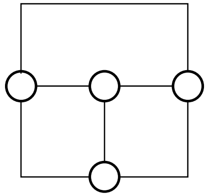
Lemma. All paths between the poles of a H_μ have the same turn number



$t(p) = k$

H_μ is k-spiral

H_μ has spirality k

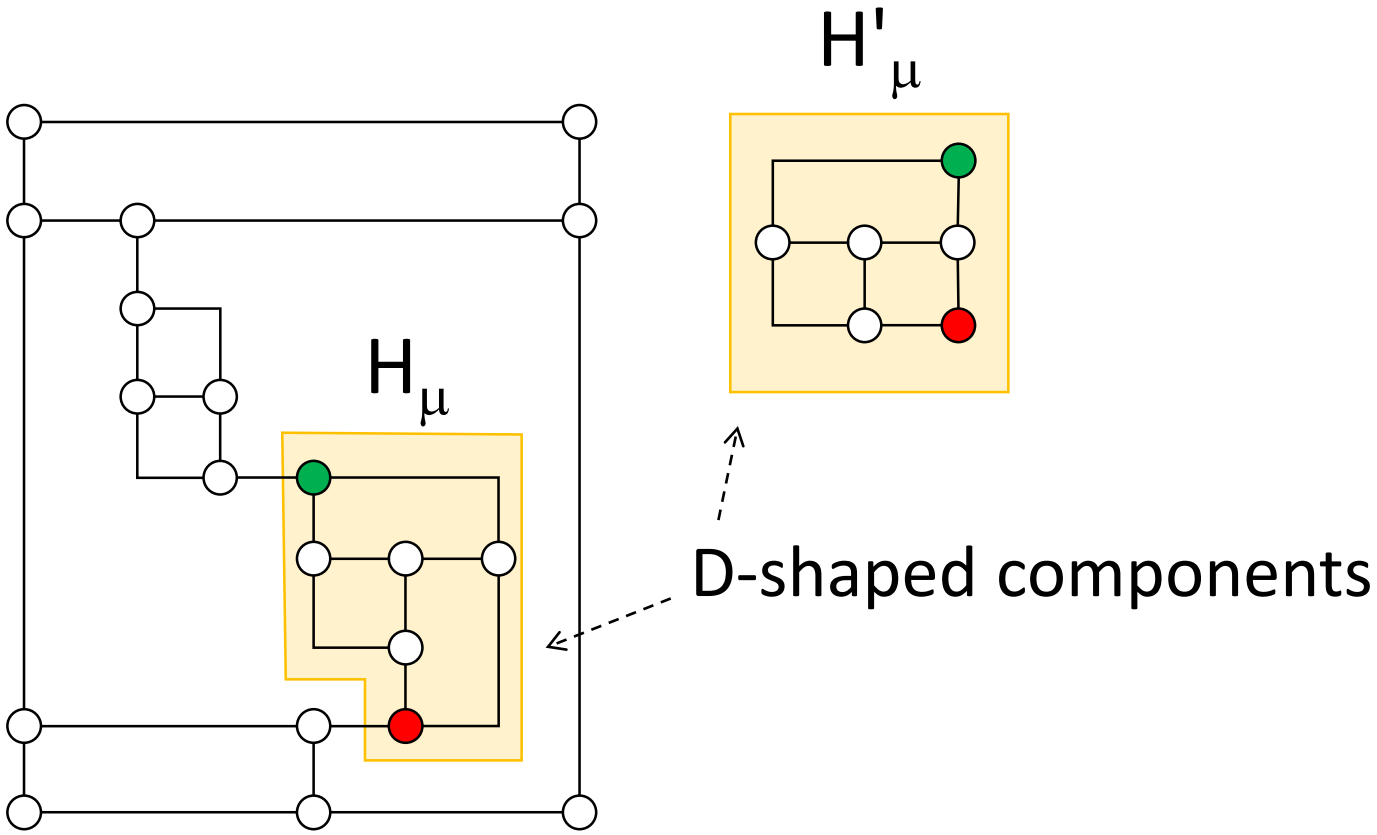


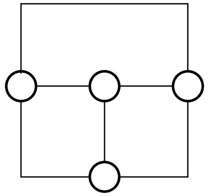
Equivalent orthogonal components

- H_μ and H'_μ = two distinct orthogonal representations of G_μ
- H_μ and H'_μ are equivalent if:
  - μ is a P- or an R-node and H_μ, H'_μ are both D-shaped or both X-shaped
  - μ is an S-node and H_μ, H'_μ have the same spirality

Equivalent orthogonal components

Theorem (substitution). Equivalent orthogonal components are interchangeable





Equivalent orthogonal components

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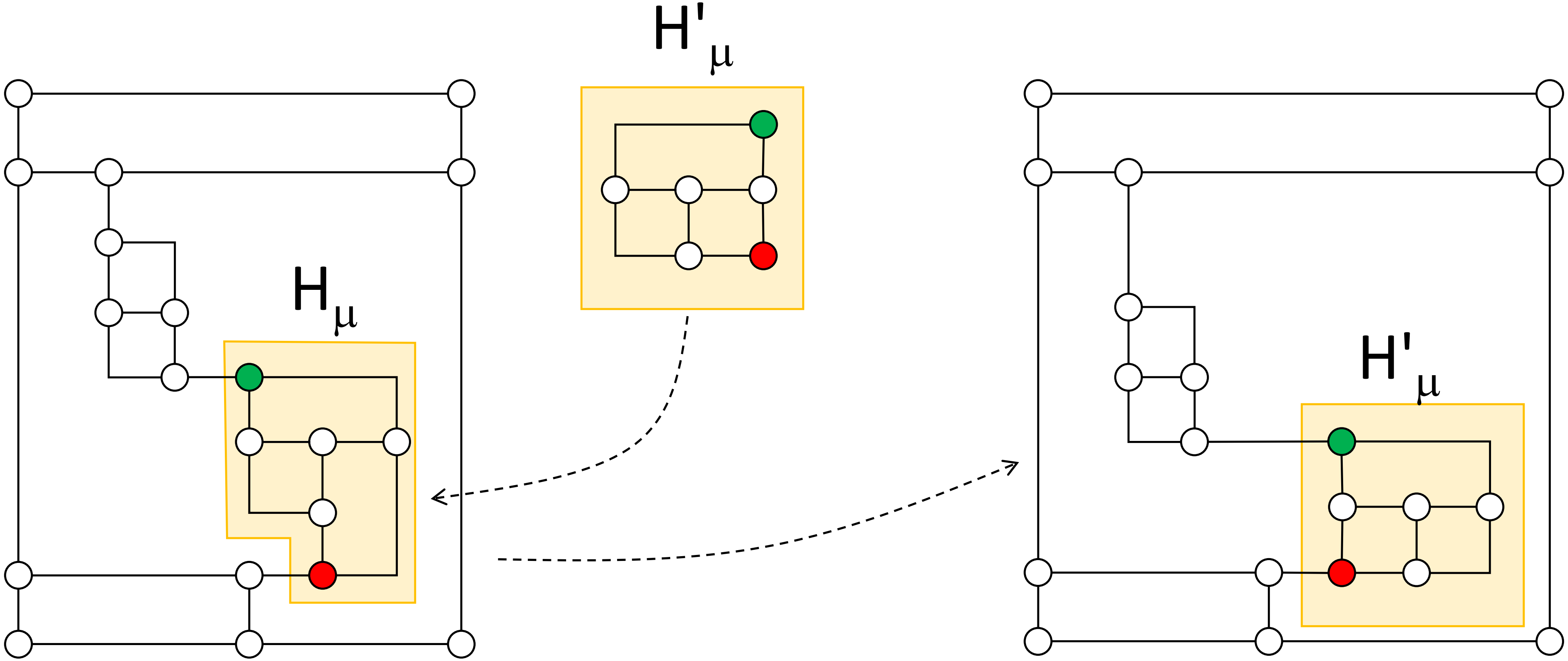

Equivalent orthogonal components

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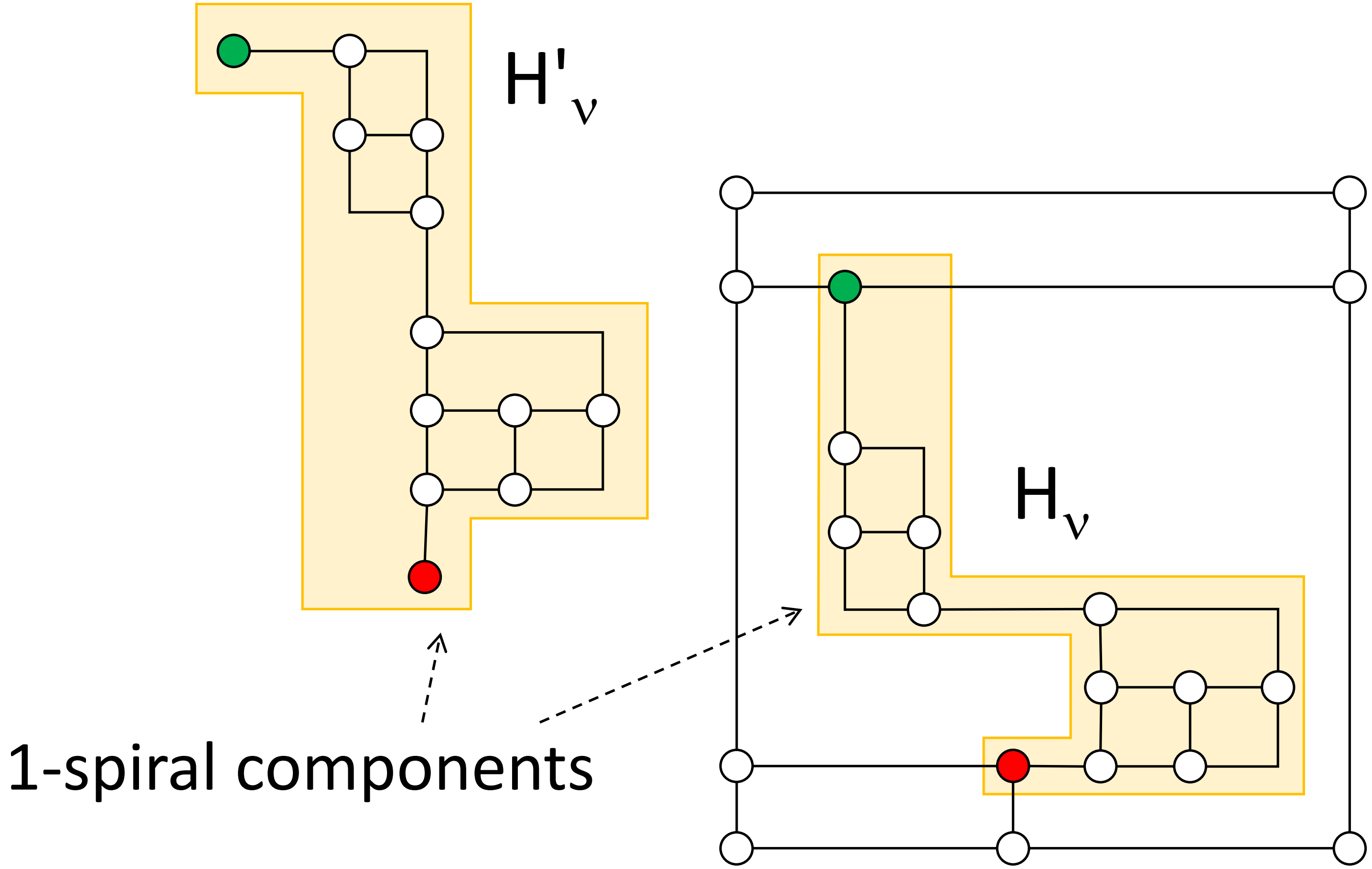
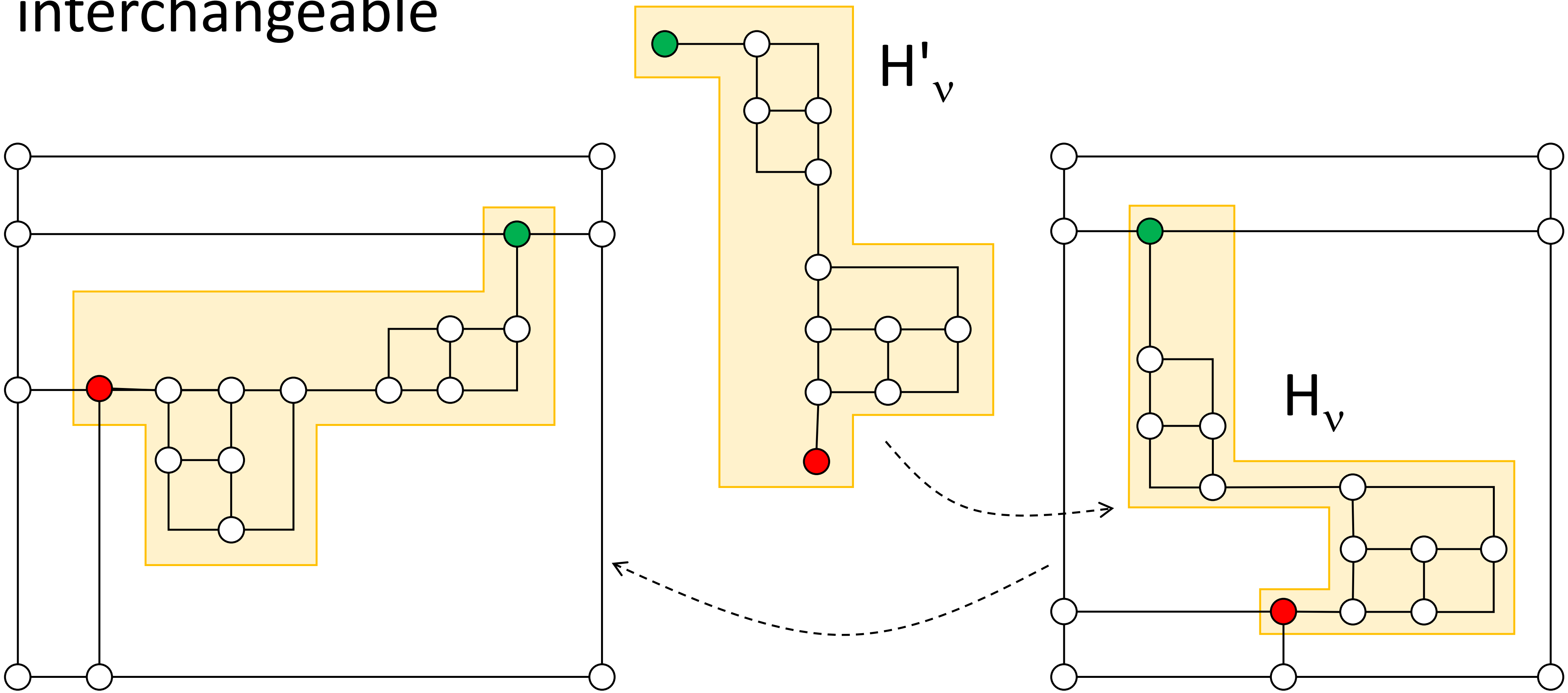


Equivalent orthogonal components

Theorem (substitution). Equivalent orthogonal components are interchangeable



Key lemma

Key-Lemma. Every biconnected planar 3-graph with a given edge *e* admits a bend-min orthogonal representation with *e* on the external face such that:

- **O1.** every edge has at most two bends
- **O2.** every inner P- or R-component is D- or X-shaped
- **O3.** every inner S-component has spirality at most 4

Key lemma

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proof: based on a characterization of no-bend orthogonal representations [Rahman, Nishizeki, Naznin 2003]

Key lemma: consequence

Key-Lemma. Every biconnected planar 3-graph with a given edge *e* admits a bend-min orthogonal representation with *e* on the external face such that:

O1. every edge has at most two bends

O2. every inner P- or R-component is D- or X-shaped

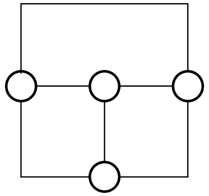
O3. every inner S-component has spirality at most 4

Consequence: we can restrict our algorithm to search for a bend-min representation that satisfies O1, O2, and O3.

Algorithm

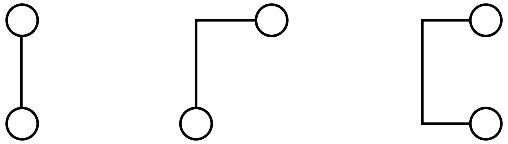
- **input**: biconnected planar 3-graph G with a reference edge *e*
- **output**: bend-min representation H of G with *e* on the external face

1. compute the SPQR-tree T of G with respect to *e*
2. visit the nodes μ of T bottom-up:
   - **μ inner node** $\Rightarrow$ store in μ a set of candidate bend-min representations of G_μ- one for each distinct equivalence class, thanks to the substitution theorem
   - **μ the root child** $\Rightarrow$ construct H by suitably merging *e* with the candidate representations stored at the children of μ; consider {0, 1, 2} bends for *e*, thanks to O1 of the key-lemma

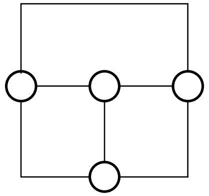


Candidate sets of inner nodes

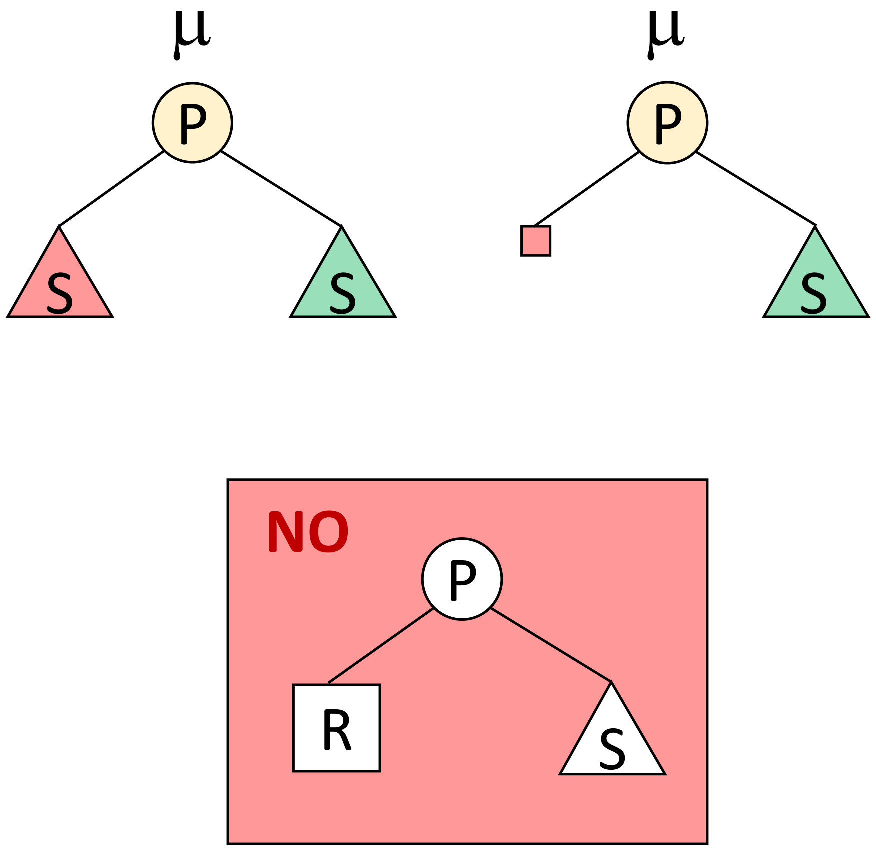
- **Q-node:** a representation for each number of bends in {0, 1, 2}
  - thanks to O1 of the key-lemma

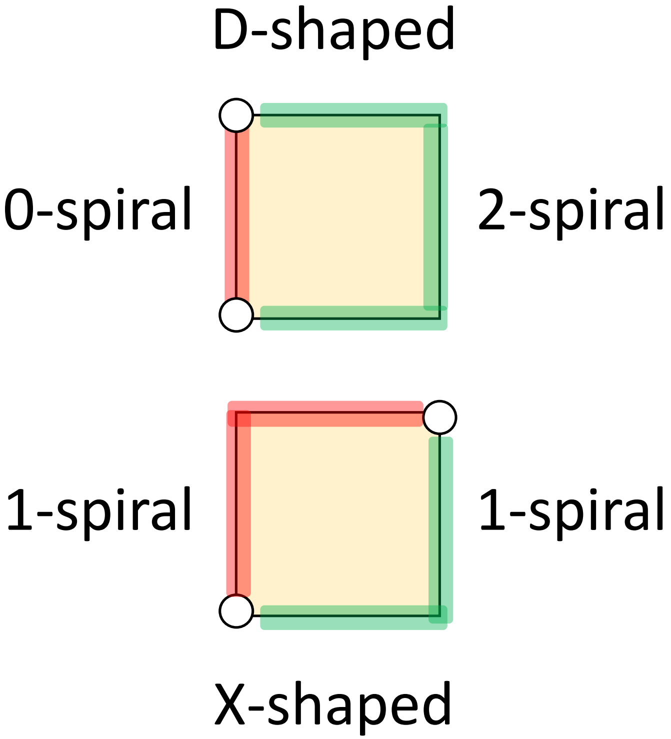


- **P/R-node:** the cheapest D-shaped and the cheapest X-shaped representations
  - thanks to O2 of the key-lemma
- **S-node:** the cheapest representation for each value of spirality in {0, 1, 2, 3, 4}
  - thanks to O3 of the key-lemma



Candidate set of an inner P-node

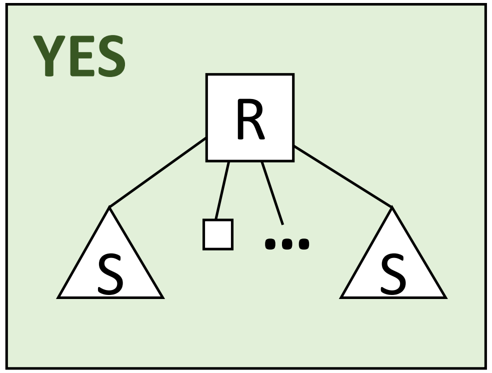


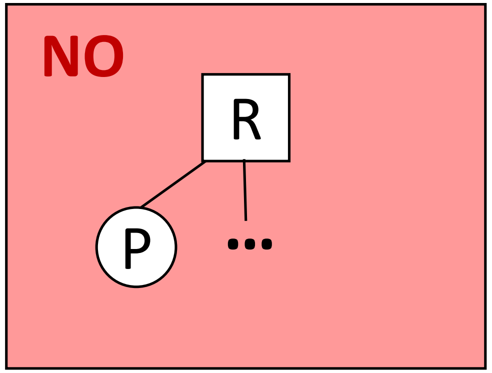


O(1) time

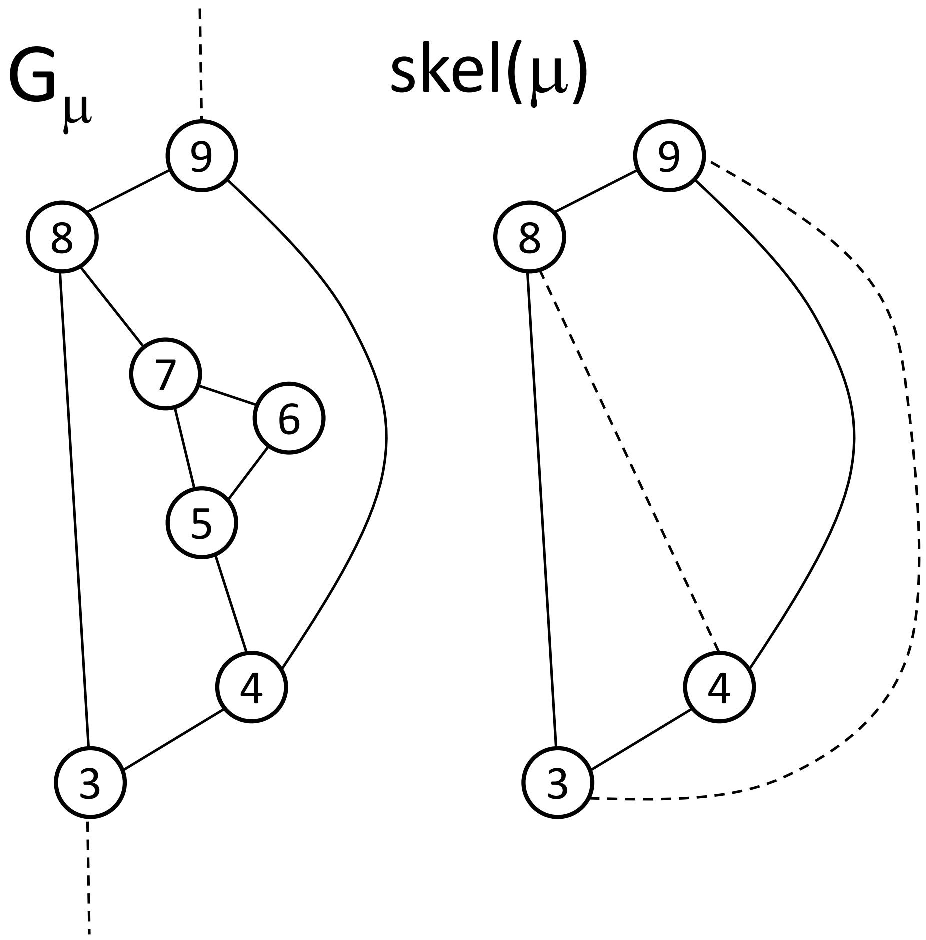
Candidate set of an inner R-node

Each child of an R-node is either a Q- or an S-node





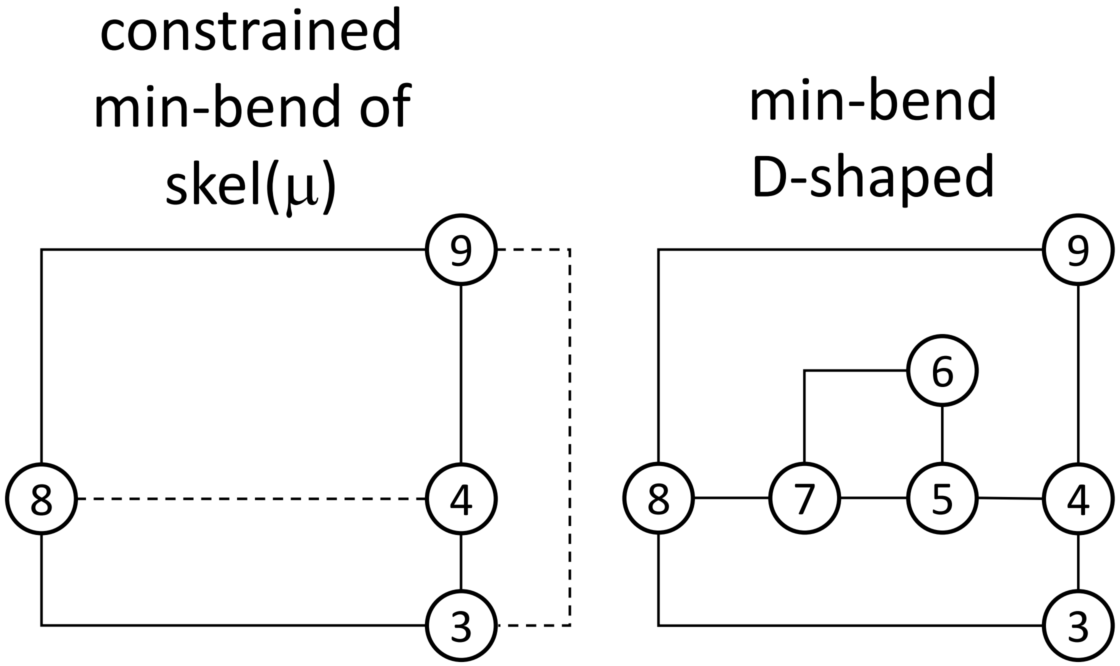
Candidate set of an inner R-node



O(n)-time variant of [RNN'99]

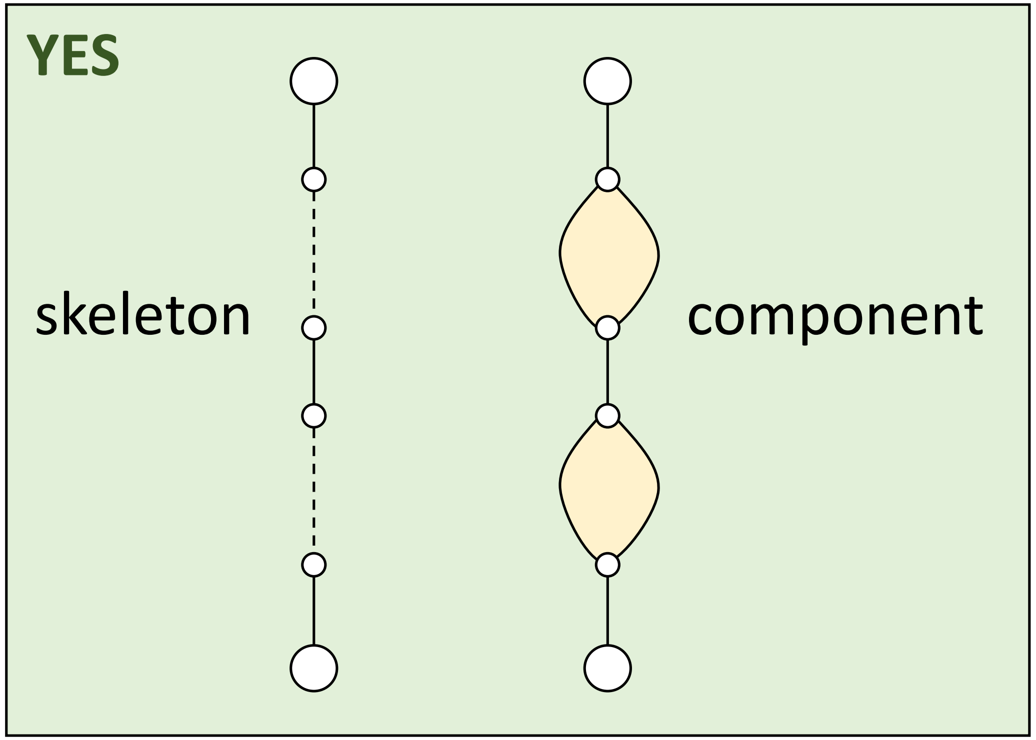
⇒

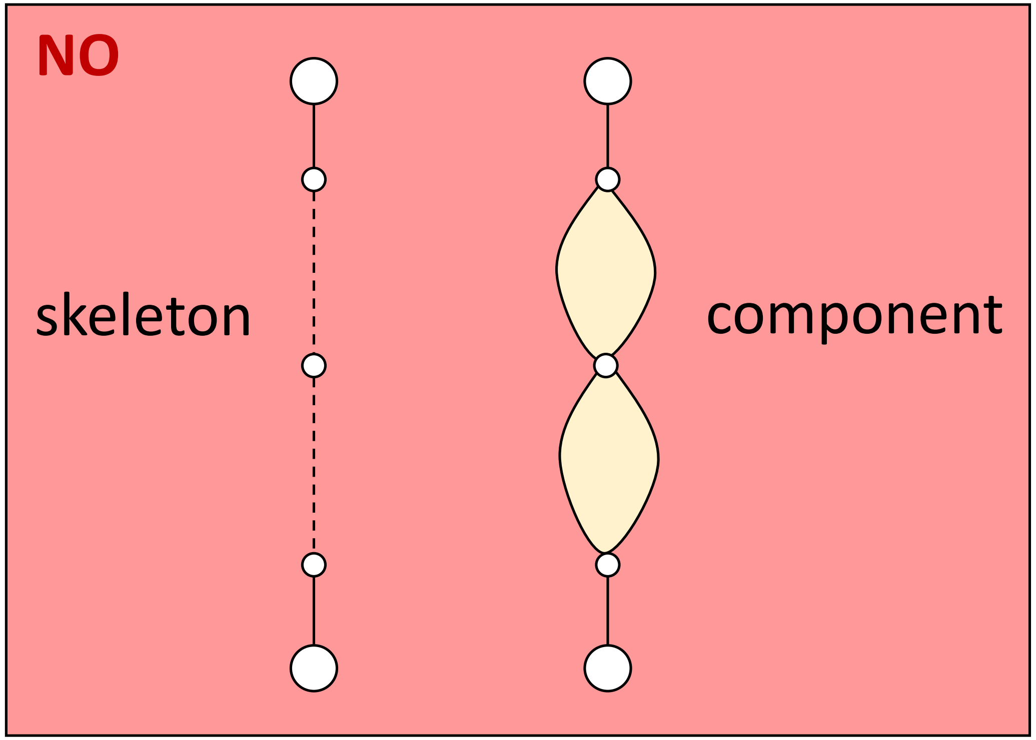
min-bend 3-connected cubic (with constraints)

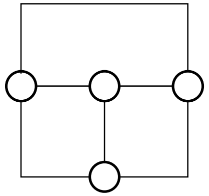


$O(n_\mu)$ time

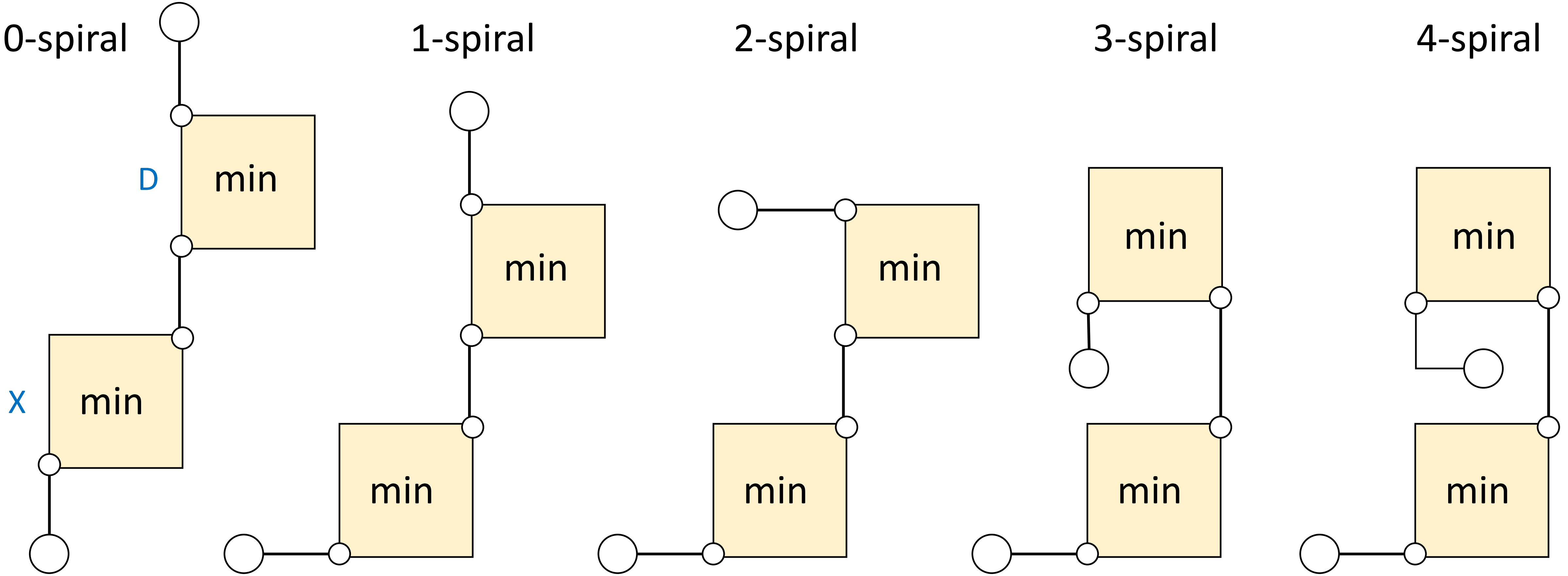
Candidate set of an inner S-node







Candidate set of an inner S-node

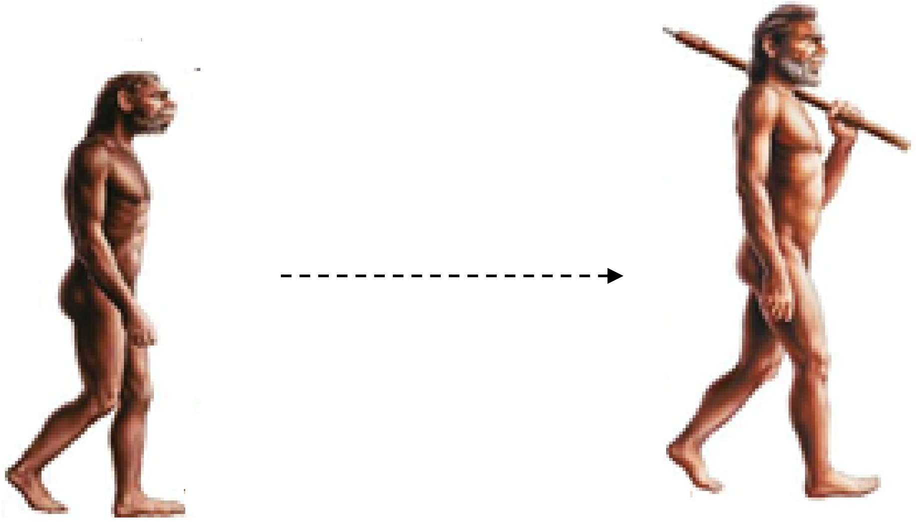


#(extra bends) = max{0, spirality – (#D-shaped + #Q-nodes – 1)}

O(n_μ) time

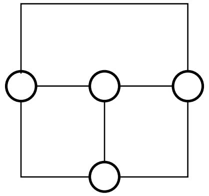
Open problems

- **Problem 1.** Is there a subquadratic-time algorithm to compute a bend-minimum orthogonal drawing of a planar 3-graph?



$O(n^2)$
our result

Can we do better?

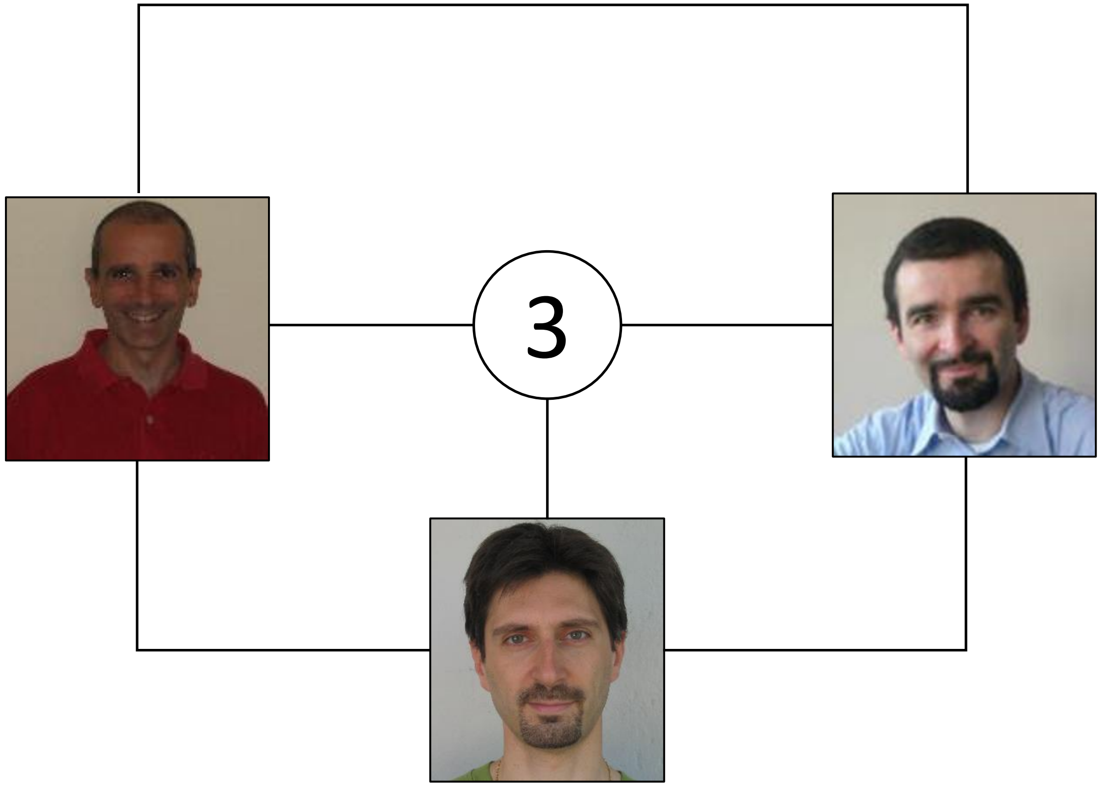


Open problems: even more prehistoric

- **Problem 2.** Is there a linear-time algorithm to compute a bend-minimum orthogonal drawing of a *plane* 4-graph (in the *fixed embedding setting*)?

Thank you!

Giuseppe Liotta



Maurizio Patrignani