On the Arrangement of Hyperplanes Determined by n Points

Michal Opler[1], Pavel Valtr[2], and Tung Anh Vu[3]

1 **Czech Technical University in Prague**
   `michal.opler@fit.cvut.cz`
2 **Charles University**
   `valtr@kam.mff.cuni.cz`
3 **Charles University**
   `tung@kam.mff.cuni.cz`

—— **Abstract** ——

For any $d \leq 6$ and for any n, we determine the maximum number of cells in the arrangement of hyperplanes determined by n points in $\mathbb{R}^d$. It is shown that this number can be expressed as a polynomial in n of degree d^2 for any fixed d, and exact formulas for the first $d-1$ coefficients of this polynomial are given.

1 Introduction

Arrangements of lines in the plane and their higher-dimensional generalization, arrangements of hyperplanes in $\mathbb{R}^d$, are a basic geometric structure. If a finite set of hyperplanes is *in general position*, which means that the intersection of every k hyperplanes is $(d-k)$-dimensional, $k = 2, 3 \ldots, d+1$, the arrangement is called *simple*. If the hyperplanes of a hyperplane arrangement $\mathcal{A}$ are removed from $\mathbb{R}^d$, the remaining part of $\mathbb{R}^d$ consists of connected components called *cells of* $\mathcal{A}$. The following proposition implies that the number of cells of a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is a function of n and d only, and is thus independent of the arrangement.

▶ Proposition 1. *The number of cells in a simple arrangement of n hyperplanes in $\mathbb{R}^d$ is*

$$\Phi_d(n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}.$$

For $d = 2$, Proposition 1 says that n lines in general position in the plane partition the plane into $\Phi_2(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} = \frac{n^2}{2} + \frac{n}{2} + 1$ cells. This fact is well known also outside of the discrete and computational geometry community due to the fact that it has several elementary proofs which nicely demonstrate the principle of mathematical induction. Proposition 1 for general d also has simple proofs using mathematical induction.

In this paper we consider arrangements of all hyperplanes in $\mathbb{R}^d$ determined by d-element subsets of a given set of n points in general position in $\mathbb{R}^d$. In particular, we are interested in the (maximum) number of cells in such arrangements.

Let P be a set of $n \geq d$ points in general position in $\mathbb{R}^d$. The affine hull of each d-tuple of points of P is a hyperplane. We denote the arrangement of these $\binom{n}{d}$ hyperplanes by $\mathcal{A}(P)$, or by $\mathcal{A}(p_1, \ldots, p_n)$ if $P = \{p_1, \ldots, p_n\}$.

We find it a very natural question to ask how many cells $\mathcal{A}(P)$ can have. Surprisingly, as far as we know, this question has been considered only in dimensions 2 and 3 so far. We study this question for general d. If P is in a "sufficiently general" position, the number of cells, denoted $f_d(n)$, depends only on n and d. By a computer assisted proof, we determine

$f_d(n)$ for $d \leq 5$ and any $n \geq d$. We can also determine $f_6(n)$ using a result of Koizumi et al. [3] who studied the so-called characteristic polynomial of hyperplane arrangements in vector spaces. It turns out that Koizumi et al. compute the characteristic polynomial of arrangements up to dimension six which are equivalent to $\mathcal{A}(P)$. This is discussed in more detail in the initial part of Section 3. We also show that for any fixed d, $f_d(n)$ can be expressed as a polynomial in n of degree d^2, and in Theorem 3.2 we give exact formulas for the first two coefficients of this polynomial, which shows the growth rate of $f_d(n)$ for any fixed d. In the second part of Theorem 3.2 we actually show a stronger result that for any d, the first $d-1$ coefficients of $f_d(n)$ and $\Phi_d(\binom{n}{d})$ are equal.

When trying to find some results about the numbers $f_d(n)$, we found a discussion on mathoverflow [2] where the question was asked by Min Wu in February 2020. Few days later Richard Stanley outlined on the same place how to obtain $f_2(n)$ and $f_3(n)$. The computation of $f_3(n)$ required some case distinction and relatively complicated formulas appeared in the computation. Stanley wrote that there could be an error in the computation. It turns out that his formula for $f_3(n)$ was not quite correct but the method works. We managed to correct the formula and extend the method to higher dimensions.

We now prepare for the definition of the type of "sufficiently general" position which is suitable for us. Let P be a set of $n \geq d$ points in general position in $\mathbb{R}^d$. We say that a hyperplane arrangement $\mathcal{B}$ is *central* if it has a non-empty intersection, i.e., if $\bigcap_{H \in \mathcal{B}} H \neq \emptyset$. We associate with every central subarrangement $\mathcal{B}$ of $\mathcal{A}(P)$ a poset $\mathcal{P}_\mathcal{B}$ of sets ordered by inclusion defined as

$$\mathcal{P}_\mathcal{B} := \{F \subseteq P \mid \exists H \in \mathcal{B} : \cap \mathcal{B} \subseteq \mathrm{aff}(F) \subseteq H\}.$$

In other words, $\mathcal{P}_\mathcal{B}$ contains all tuples of points from P spanning an affine space that (i) contains the common intersection of $\mathcal{B}$, and simultaneously (ii) is contained in some hyperplane $H \in \mathcal{B}$. Our definition of a "sufficiently general" position ensures that if a subarrangement $\mathcal{B}$ of $\mathcal{A}(P)$ is central then the intersection $\bigcap \mathcal{B}$ is given by the structure of the minimal elements of $\mathcal{P}_\mathcal{B}$. The *support* $\mathcal{S}_\mathcal{B}$ *of* $\mathcal{B}$ is the set system consisting of the minimal elements of $\mathcal{P}_\mathcal{B}$. If the sets in $\mathcal{S}_\mathcal{B}$ are denoted by $S_1, \ldots, S_k$, we have $\bigcap \mathcal{B} = \bigcap_{i=1}^{k} \mathrm{aff}(S_i)$. It follows from a basic result in linear algebra that, under the assumption that $\mathcal{B}$ is a central arrangement, $\mathrm{codim}(\bigcap \mathcal{B}) \leq \sum_{i=1}^{k} \mathrm{codim}(\mathrm{aff}(S_i))$. Intuitively, the previous inequality is strict in case of certain degeneracy.

We say that P is in a *very general position*, if for any central subarrangement $\mathcal{B} \subseteq \mathcal{A}(P)$ with support $\{S_1, \ldots, S_k\}$ we have

$$\mathrm{codim}\left(\bigcap \mathcal{B}\right) = \sum_{i=1}^{k} \mathrm{codim}(\mathrm{aff}(S_i)), \tag{1}$$

which is equivalent to

$$\dim\left(\bigcap \mathcal{B}\right) = \sum_{i=1}^{k} |S_i| - (k-1) \cdot (d+1) - 1. \tag{2}$$

An example of six points in general position in the plane which are not in very general position is depicted in Figure 1, where the arrangement $\mathcal{B}$ of the three lines $p_i q_i$, $i = 1, \ldots, 3$, intersecting in a common point has support $\mathcal{S}_\mathcal{B} = \{\{p_1, q_1\}, \{p_2, q_2\}, \{p_3, q_3\}\}$, and it holds that $2 = \mathrm{codim}(\bigcap \mathcal{B}) \neq \sum_{i=1}^{3} \mathrm{codim}(\mathrm{aff}(S_i)) = 1 + 1 + 1 = 3$.

In the full version of this paper, it is shown that any set in general position can be perturbed to a set in a very general position. This proposition is used in the proof of the
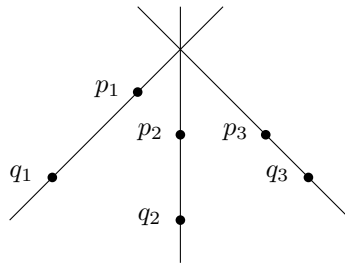
Figure 1 The lines given by pairs of points $\{p_i, q_i\}, i = 1, 2, 3$, have a common intersection.

main results, and it also easily implies that any hyperplane arrangement determined by a set of n points in $\mathbb{R}^d$ has at most $f_d(n)$ cells. Indeed, if we slightly perturb any point set to a set in general position and then to a set in very general position, the number of cells cannot decrease and it reaches exactly the value of $f_d(n)$.

We were able to find the related sequences in the On-Line Encyclopedia of Integer Sequences [7]. Sequence `A055503` [6] corresponds to the number of cells in an arrangement of lines determined by n points in very general position in the plane, i.e. $f_2(n)$. Sequence `A002817` [4] corresponds to the number of such cells which are bounded. Sequence `A037255` [5] corresponds to the number of cells in an arrangement of lines determined by a generic set of n points in the real projective plane. (The sequence for the real projective plane can be obtained from the previous two sequences as the arithmetic mean of the two sequences, since if we embed a real plane containing a line arrangement to a real projective plane then each pair of opposite unbounded cells merges into a single cell.)

Open problems

A natural open problem is to determine or estimate the maximum number of k-faces in a hyperplane arrangement determined by a set of n points in $\mathbb{R}^d$. It is widely open if there is a closed formula for $f_d(n)$ similar to the one given for $\Phi_d(n)$ in Proposition 1.

2 Warm-up: two-dimensional space

In this section, we count the number of cells in a line arrangement determined by a set P of n points in very general position in the two-dimensional space. For each cell we assume without loss of generality that the bottommost point is unique if it exists. Let P' be the set of intersections between all pairs of lines given by $\binom{P}{2}$, and let $Q = P' \setminus P$. Each cell C satisfies exactly one of the following three conditions:

1. the bottommost point of C belongs to P,
2. the bottommost point of C belongs to Q,
3. C does not have a bottommost point.

In the first case, every point $p \in P$ can be a bottommost point of a cell. There are $n-1$ lines passing through p, so there are $n-2$ cells with p as its bottommost point, see Figure 2 for an example. Thus, the number of cells satisfying condition 1. is

$$n \cdot (n-2). \tag{3}$$
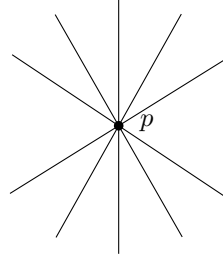
Figure 2 There are five lines passing through p and only four regions with p as its bottommost point.

In the second case, the cell C is given by a point $q \in Q$. The number of such points, and in turn the number of cells satisfying condition 2. is

$$\frac{1}{2} \cdot \binom{n}{2} \cdot \binom{n-2}{2}. \tag{4}$$

To count the number of cells that are unbounded from below, consider a horizontal line ℓ which lies below all points of P'. Then each unbounded cell is intersected by ℓ exactly once. Let L be the set of lines given by all pairs of points of P, and let Q be the set of points in which lines of L and ℓ intersect. Projecting each unbounded cell onto ℓ lets us count the number of unbounded cells as the number of line segments on ℓ given by Q and two additional cells are not bounded from either the left or the right. As such, the number of unbounded cells is

$$\binom{n}{2} + 1. \tag{5}$$

By (3), (4), and (5) we have

$$f_2(n) = n(n-2) + \binom{n}{2}\left(\frac{1}{2}\binom{n-2}{2} + 1\right) + 1. \tag{6}$$

3   Computing the characteristic polynomial

In this section, we focus on computing the characteristic polynomial of the arrangement using a mechanical method that can be implemented by a program. The number of cells as well as the number of bounded cells can then easily be retrieved from the polynomial.

First, let us formally introduce the characteristic polynomial. Recall that a hyperplane arrangement $\mathcal{A}$ is central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. The *rank* of $\mathcal{A}$, denoted by $\mathrm{rank}(\mathcal{A})$, is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. For any central arrangement $\mathcal{A}$, we have $\mathrm{rank}(\mathcal{A}) = \mathrm{codim}(\bigcap \mathcal{A})$. The *characteristic polynomial* of a hyperplane arrangement $\mathcal{A}$, denoted by $\chi_{\mathcal{A}}(t)$, is defined as

$$\chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{|\mathcal{B}|} t^{d - \mathrm{rank}(\mathcal{B})} = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (-1)^{|\mathcal{B}|} t^{\dim(\bigcap \mathcal{B})}.$$

Note that the characteristic polynomial is typically defined using a so-called intersectional lattice associated with $\mathcal{A}$ and its Möbius function. The equivalence of the definition above is due to Whitney's theorem [8, Lemma 2.3.8]. We chose to omit the standard definition as our approach really boils down to computing the characteristic polynomial as a sum over all central subarrangements of $\mathcal{A}$. Note that this differs from the approach of Koizumi et al. [3]

who compute the Möbius function of the intersection lattice and only subsequently recover the polynomial via its more usual definition.

The connection between the characteristic polynomial and the number of cells of a hyperplane arrangement is a celebrated result by Zaslavsky [9]. The number of cells of an hyperplane arrangement $\mathcal{A}$ in a real d-dimensional space is equal to $(-1)^d \chi_{\mathcal{A}}(-1)$, while the number of bounded cells is obtained as $(-1)^{\mathrm{rank}(\mathcal{A})} \chi_{\mathcal{A}}(1)$.

3.1 Algorithm for general d-dimensional space

Inspired by the approach used in Section 2, we devise an algorithm that computes the characteristic polynomial of n points in very general position in d-dimensional space expressed as a polynomial in both t and n.

▶ **Theorem 3.1.** *There is an algorithm that receives an integer $d \in \mathbb{N}$ as input and outputs a polynomial $Q(t, n)$ such that for arbitrary integer $n \geq d$ and an arbitrary set of n points $P_n \subset \mathbb{R}^d$ in very general position, we have $Q(t, n) = \chi_{\mathcal{A}(P_n)}(t)$.*

Now, we briefly try to sketch the basic idea behind the algorithm. Recall that we associate with any subarrangement $\mathcal{B}$ a poset $\mathcal{P}_{\mathcal{B}}$ of sets ordered by inclusion defined in Section 1. Furthermore, recall that we defined the support $\mathcal{S}_{\mathcal{B}}$ as the collection of all minimal sets of $\mathcal{P}_{\mathcal{B}}$. Our goal is to show how central arrangements with different supports contribute to the characteristic polynomial.

We notice that there are only finitely many ways how the support of a central subarrangement can look like. We call these classes of isomorphic supports *types*. It is easy to prove that there are only finitely many non-isomorphic types as each can contain at most d sets. For example in three-dimensional space, a central arrangement $\mathcal{B}$ intersecting in a common line can have the following three possible types of support: (i) a single pair of points $\{p, q\}$ in the case when all the hyperplanes in $\mathcal{B}$ contain the line spanned by p and q, (ii) two disjoint triples of points $\{p_1, q_1, r_1\}, \{p_2, q_2, r_2\}$ in the case when $\mathcal{B}$ contains precisely the two hyperplanes spanned by these triples, and (iii) two triples sharing one common point $\{p, q_1, r_1\}, \{p, q_2, r_2\}$ which again corresponds to an arrangement $\mathcal{B}$ containing precisely the two hyperplanes.

The algorithm computes $Q(t, n)$ by enumerating all possible support types and summing the contributions of all central subarrangements with a given support type. However, we remark that this is far from a full description of the algorithm since there is a great deal of non-trivial care needed to handle overcounting.

3.2 Three-, four- and five-dimensional space

We were able to successfully compute the characteristic polynomials for $d = 4$ and $d = 5$ by implementing the algorithm of Subsection 3.1. We include below only the polynomials $f_3(n), f_4(n)$ and the first three terms of the polynomial $f_5(n)$ counting the number of cells determined by the hyperplane arrangement determined by n points in very general position. The full characteristic polynomials can be found in [1].

$$f_3(n) = \frac{1}{1296}\,n^9 - \frac{1}{144}\,n^8 - \frac{1}{27}\,n^7 + \frac{61}{72}\,n^6 - \frac{2237}{432}\,n^5 + \frac{2231}{144}\,n^4 - \frac{14945}{648}\,n^3 + \frac{41}{3}\,n^2$$
$$- \frac{13}{18}\,n + 1$$

$$f_4(n) = \frac{1}{7962624}\,n^{16} - \frac{1}{331776}\,n^{15} + \frac{65}{1990656}\,n^{14} - \frac{157}{497664}\,n^{13} + \frac{1315}{442368}\,n^{12} - \frac{923}{124416}\,n^{11}$$
$$- \frac{486709}{1990656}\,n^{10} + \frac{198593}{55296}\,n^9 - \frac{201042623}{7962624}\,n^8 + \frac{108860747}{995328}\,n^7 - \frac{103295189}{331776}\,n^6$$
$$+ \frac{73347065}{124416}\,n^5 - \frac{120791941}{165888}\,n^4 + \frac{3824591}{6912}\,n^3 - \frac{259219}{1152}\,n^2 + \frac{531}{16}\,n + 1$$

$$f_5(n) = \frac{1}{2985984000000}\,n^{25} - \frac{1}{59719680000}\,n^{24} + \frac{47}{119439360000}\,n^{23} + O(n^{22})$$

3.3 Asymptotic behavior

Although we were unable to compute the exact number of cells $f_d(n)$ for $d > 6$, we can use our techniques to obtain at least their asymptotic growth. It is not hard to deduce that $f_d(n)$ must be a polynomial in n of degree d^2. We can however precisely determine its first $d - 1$ coefficients, which, somewhat surprisingly, are exactly the same as if the $\binom{n}{d}$ hyperplanes of $\mathcal{A}(P)$ were in a general position.

▶ **Theorem 3.2.** *For every $d \geq 3$*

$$f_d(n) = \frac{1}{(d!)^{d+1}} \cdot n^{d^2} + \frac{d^2 - d^3}{2 \cdot (d!)^{d+1}} \cdot n^{d^2-1} + O(n^{d^2-2}).$$

In fact, the first $d - 1$ coefficients of $\Phi_d\left(\binom{n}{d}\right)$ and $f_d(n)$ are equal.

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