On the geometric thickness of 2-degenerate graphs

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Abstract

A graph is 2-degenerate if every subgraph contains a vertex of degree at most 2. We show that every 2-degenerate graph can be drawn with straight lines such that the drawing decomposes into 4 plane forests. Therefore, the geometric arboricity, and hence the geometric thickness, of 2-degenerate graphs is at most 4. On the other hand, we show that there are 2-degenerate graphs that do not admit any straight-line drawing with a decomposition of the edge set into 2 plane graphs. That is, there are 2-degenerate graphs with geometric thickness, and hence geometric arboricity, at least 3. This answers two questions posed by Eppstein [Separating thickness from geometric thickness. In *Towards a Theory of Geometric Graphs*, vol. 342 of *Contemp. Math.*, AMS, 2004].

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1 Introduction

A graph is planar if it can be drawn without crossings on a plane. Planar graphs exhibit many nice properties, which can be exploited to solve problems for this class more efficiently compared to general graphs. However, in many situations, graphs cannot be assumed to be planar even if they are sparse. It is therefore desirable to define graph classes which extend planar graphs. Several approaches for extending planar graphs have been established over the last years [3, 12]. Often these classes are defined via drawings, for which the types of crossings and/or the number of crossings are restricted. A natural way to describe how close a graph is to being a planar graph is provided by the graph parameter *thickness*. The thickness of a graph G is the smallest number $\theta(G)$ such that the edges of G can be partitioned into $\theta(G)$ planar subgraphs of G. Related graph parameters are *geometric thickness* and *book thickness*. Geometric thickness was introduced by Kainen under the name *real linear thickness* [15]. The geometric thickness $\bar{\theta}(G)$ of a graph G is the smallest number of colors that is needed to find an edge-colored geometric drawing (i.e., one with edges drawn as straight-line segments) of G with no monochromatic crossings. For the book thickness $\mathsf{bt}(G)$, we only consider geometric drawings with vertices in convex position.

An immediate consequence from the definitions of thickness, geometric thickness and book thickness is that for every graph G we have $\theta(G) \leq \bar{\theta}(G) \leq \mathsf{bt}(G)$. Eppstein shows that the three thickness parameters can be arbitrarily "separated". Specifically, for any number k there exists a graph with geometric thickness 2 and book thickness at least k [9] as well as a graph with thickness 3 and geometric thickness at least k [10]. The latter result is particularly notable since any graph of thickness k admits a k-edge-colored drawing of G with no monochromatic crossings if edges are not required to be straight lines. This follows from a result by Pach and Wenger [20], stating that any planar graph can be drawn without crossings on arbitrary vertex positions with polylines.

Related to the geometric thickness is the *geometric arboricity* $\bar{\mathsf{a}}(G)$ of a graph G, introduced by Dujmović and Wood [5]. It denotes the smallest number of colors among all edge-colored geometric drawings of G without monochromatic crossings where every color

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class is acyclic. As every such plane forest is a plane graph, we have $\bar{\theta}(G) \leq \bar{\mathsf{a}}(G)$. Moreover, every plane graph can be decomposed into three forests [22], and therefore $3\bar{\theta}(G) \geq \bar{\mathsf{a}}(G)$.

Bounds on the geometric thickness are known for several graph classes. Due to Dillencourt et al. [4] we have $\frac{n}{5.646} + 0.342 \leq \bar{\theta}(K_n) \leq \frac{n}{4}$ for the complete graph K_n. Graphs with bounded degree can have arbitrarily high geometric thickness. In particular, as shown by Barárt et al. [1], there are d-regular graphs with n vertices and geometric thickness at least $c\sqrt{d}n^{1/2-4/d-\varepsilon}$ for every $\varepsilon > 0$ and some constant c. However, due to Duncan et al. [7], if the maximum degree of a graph is 4, its geometric thickness is at most 2. For graphs with treewidth t, Dujmović and Wood [5] showed that the maximum geometric thickness is $\lceil t/2 \rceil$. Hutchinson et al. [13] showed that graphs with n vertices and geometric thickness 2 can have at most $6n - 18$ edges. As shown by Durocher et al. [8], there are n-vertex graphs for any $n \geq 9$ with geometric thickness 2 and $2n - 19$ edges. In the same paper, it is proven that it is NP-hard to determine if the geometric thickness of a given graph is at most 2. Computing thickness [16] and book thickness [2] are also known to be NP-hard problems. For bounds on the thickness for several graph classes, we refer to the survey of Mutzel et al. [17]. An overview on bounds for book thickness is given on the webpage of Pupyrev [21].

A graph G is *d-degenerate* if every subgraph contains a vertex of degree at most d. So we can repeatedly find a vertex of degree at most d and remove it, until no vertices remain. The reversal of this vertex order (known as a *degeneracy order*) yields a construction sequence for G that adds vertex by vertex and each new vertex is connected to at most d previously added vertices (called its *predecessors*). Adding a vertex with exactly two predecessors is also known as a Henneberg 1 step [11]. In particular, any 2-degenerate graph is a subgraph of a so-called Laman graph, however not every Laman graph is 2-degenerate. Laman graphs are the generically minimal rigid graphs and they are exactly those graphs constructable from a single edge by some sequence of Henneberg 1 and Henneberg 2 steps (the latter step consists of subdividing an arbitrary existing edge and adding a new edge between the subdivision vertex and an arbitrary, yet non-adjacent vertex). All d-degenerate graphs are (d, ℓ)-sparse, for any $\binom{d+1}{2} \geq \ell \geq 0$, that is, every subgraph on n vertices has at most $dn - \ell$ edges.

Our Results. In this paper, we study the geometric thickness of 2-degenerate graphs. Due to the Nash-Williams theorem [18, 19], every 2-degenerate graph can be decomposed into 2 forests and hence has arboricity at most 2 and therefore thickness at most 2. On the other hand, as observed by Eppstein [9], 2-degenerate graphs can have unbounded book thickness. Eppstein's examples of graphs with thickness 3 and arbitrarily high geometric thickness are 3-degenerate graphs [10]. Eppstein asks whether the geometric thickness of 2-degenerate graphs is bounded by a constant from above and whether there are 2-degenerate graphs with geometric thickness greater than 2. The currently best upper bound of $O(\log n)$ follows from a result by Duncan for graphs with arboricity 2 [6]. We improve this bound and answer both of Eppstein's questions with the following two theorems.

▶ **Theorem 1.** *For each 2-degenerate graph G we have $\bar{\theta}(G) \leq \bar{\mathsf{a}}(G) \leq 4$.*

▶ **Theorem 2.** *There is a 2-degenerate graph G with $\bar{\mathsf{a}}(G) \geq \bar{\theta}(G) \geq 3$.*

We give proof ideas for these theorems in Sections 2 and 3, respectively.

2 The upper bound

In this section, we outline the proof of Theorem 1. We describe, for any 2-degenerate graph, a construction for a straight-line drawing such that the edges can be colored using four colors,

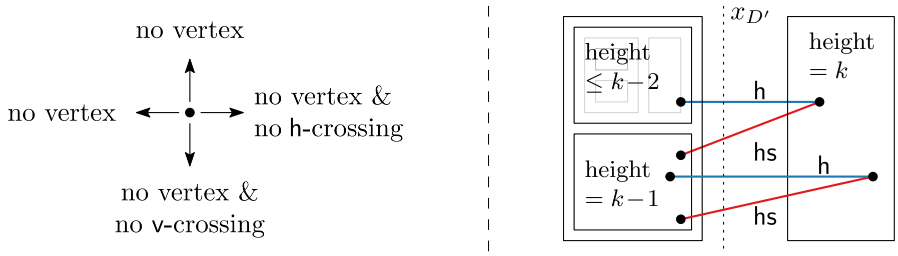

Figure 1 Left: For each vertex v in a feasible drawing, there are no other vertices on the vertical and the horizontal line through v. Moreover, v is h-open to the right and v-open to the bottom. Right: All vertices in the highest level (of height k) are placed to the right of all vertices of smaller height. Each vertex in that level is incident to one edge of color h and one edge of color hs.

avoiding monochromatic crossings and monochromatic cycles. This shows that 2-degenerate graphs have geometric arboricity, and hence geometric thickness, at most four.

For a graph G we denote its edge set with $E(G)$ and its vertex set with $V(G)$. Consider a 2-degenerate graph G with a given, fixed degeneracy order. We define the *height* of a vertex v in G as the length t of a longest path $u_0 \cdots u_t$ with $u_t = v$ such that for each i, with $1 \leq i \leq t$, the vertex u_{i-1} is a predecessor of u_i. The height of G is the largest height among its vertices. The set of vertices of the same height is called a *level* of G.

Our construction process embeds G level by level with increasing height. The levels are placed alternately either strictly below or strictly to the right of the already embedded part of the graph. If a level is placed below, then we use specific colors v and vs (short for "vertical" and "vertical slanted", respectively) for all edges between this level and levels of smaller height. Similarly, we use specific colors h and hs (short for "horizontal" and "horizontal slanted", respectively) if a level is placed to the right. See Figure 1 (right).

To make our construction work, we need several additional constraints to be satisfied in each step which we will describe next. For a point p in the plane, we use the notation $\mathsf{x}(p)$ and $\mathsf{y}(p)$ to refer to the x- and y-coordinates of p, respectively. Consider a drawing D of a 2-degenerate graph G together with a coloring of the edges with colors $\{\mathsf{h}, \mathsf{hs}, \mathsf{v}, \mathsf{vs}\}$. For the remaining proof, we assume that each vertex of G has either 0 or exactly 2 predecessors. If not, we add a dummy vertex without predecessors to the graph and make it the second predecessor of all those vertices which originally only had 1 predecessor. Let k denote the height of G. We say that D is *feasible* if it satisfies the following constraints:

(C1) For each vertex in G the edges to its predecessors are colored differently. If $k > 0$, then each vertex of height k in G is incident to edges of colors h and hs only.

(C2) There exists some $x_D \in \mathbb{R}$ such that for each vertex $v \in V(G)$ we have $\mathsf{x}(v) > x_D$ if and only if v is of height k in G.

(C3) There is no monochromatic crossing.

(C4) No two vertices of G lie on the same horizontal or vertical line.

(C5) Each $v \in V(G)$ is h-*open to the right*, that is, the horizontal ray emanating at v directed to the right avoids all h-edges.

(C6) Each $v \in V(G)$ is v-*open to the bottom*, that is, the vertical ray emanating at v directed downwards avoids all v-edges.

These constraints are schematized in Figure 1. We now show how to construct a feasible drawing for G. We prove this using induction on the height of the graph. The base case $k = 0$ is trivial, as there are no edges in the graph. Assume that $k \geq 1$ and the theorem is true for all 2-degenerate graphs with height $k - 1$. Let H denote the subgraph of G induced by vertices with height less than k. By induction, there is a feasible drawing D of H.

As a first step, we reflect the drawing D at the straight line $y = -x$. Additionally, we swap the colors hs and vs as well as the colors h and v. Let D' denote the resulting drawing. From now on, all appearing coordinates of vertices refer to coordinates in D'. By construction, D' satisfies (C3–C6). Applying (C1) to D shows that in D' each vertex of height $k-1$ is incident to one edge of color v and one edge of color vs. Applying (C2) to D shows that there exists $y_{D'} \in \mathbb{R}$ such that for each vertex $v \in V(H)$ we have $\mathsf{y}(v) < y_{D'}$ in D' if and only if v is of height $k-1$.

As the second (and last) step, we place the points of height k of G such that the resulting drawing is feasible. We only give a rough description of this placement here and refer to the full version of this paper [14, Section 2] for a precise formulation. Let L_k denote the set of these vertices and let $x_{D'}$ denote the largest x-coordinate among all vertices in D'. We choose a sufficiently small, positive slope m such that for any distinct $u, v \in V(H)$ with $\mathsf{y}(u) < \mathsf{y}(v)$, the horizontal line through v and the straight line through u with slope m intersect at a point p with $\mathsf{x}(p) > x_{D'}$. For each vertex $w \in L_k$ let u and v be the two predecessors of w in H with $\mathsf{y}(u) < \mathsf{y}(v)$ and let p^w denote the intersection point of the straight line of slope m passing through u (called a slanted line) and the horizontal line passing through v. We place w at point p^w and connect w to v using an edge of color h and we connect w to u using an edge of color hs. Then (C1), (C2) and (C6) are clearly satisfied. However, this placement comes with some issues: Several vertices in L_k might have the same predecessors and, hence, are placed on the same point, new edges of the same color with a common endpoint in H overlap (along a horizontal or slanted line), and (C3–C5) might not be satisfied, yet. To address these issues, we use a small perturbation, moving each point $w \in L_k$ slightly to the bottom-right (along a straight line of slope $-1/m$ through p^w) such that all vertices $w \in L_k$ are placed at different distances to their respective point p^w. In the full version of this paper [14, Section 2] we describe such a perturbation which yields a feasible drawing of G. This eventually shows that the geometric arboricity, and hence the geometric thickness, of G is at most four.

3 The lower bound

In this section, we shall describe a 2-degenerate graph with geometric thickness at least 3. For a positive integer n let $G(n)$ denote the graph constructed as follows. Start with a vertex set Λ_0 of size n and for each pair of vertices from Λ_0 add one new vertex adjacent to both vertices from the pair. Let Λ_1 denote the set of vertices added in the last step. For each pair of vertices from Λ_1 add 89 new vertices, each adjacent to both vertices from the pair. Let Λ_2 denote the set of vertices added in the last step. For each pair of vertices from Λ_2 add one new vertex adjacent to both vertices from the pair. Let Λ_3 denote the set of vertices added in the last step. This concludes the construction. Observe that for each $i = 1, 2, 3$, each vertex in Λ_i has exactly two neighbors in Λ_{i-1}. Hence, $G(n)$ is 2-degenerate. We claim that for sufficiently large n the graph $G(n)$ has geometric thickness at least 3. Due to limited space we briefly sketch of our arguments here. A complete proof is provided in the full version of this paper [14, Section 3].

Consider a geometric drawing of $G(n)$, for large n, and assume that there is a partition of its edge set into two plane subgraphs $\mathbb{A}$ and $\mathbb{B}$. In the first step, we find a large, and particularly nice grid structure (called a tidy grid) formed by edges between Λ_0 and Λ_1 where many disjoint $\mathbb{A}$-edges cross many disjoint $\mathbb{B}$-edges. We additionally ensure that there is a large subset $\Lambda'_1 \subseteq \Lambda_1$ spread out over many cells of this grid. Next, we consider

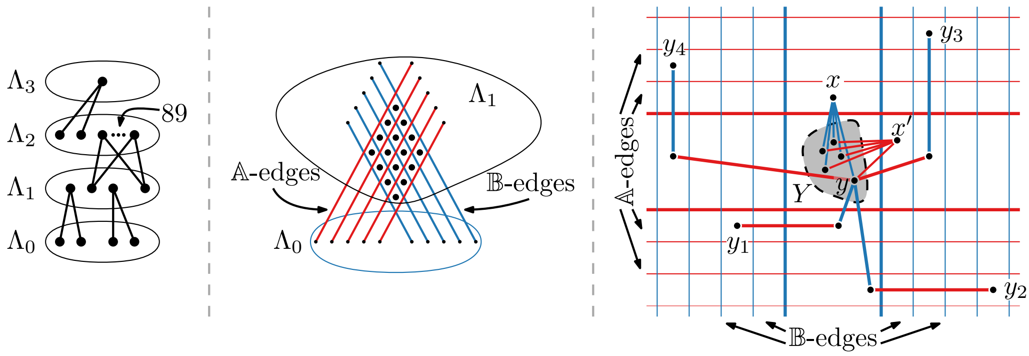

Figure 2 Left: Sketch of the graph $G(n)$. Middle: A tidy grid. Right: The situation leading to a contradiction in the proof of Theorem 2 with x, $x' \in \Lambda_1$, $Y \subseteq \Lambda_2$, and y_1, y_2, y_3, $y_4 \in \Lambda_3$.

the connections of vertices from Λ'_1 via the edges towards Λ_2. We show that the drawing restrictions imposed by the surrounding grid edges force many of the edges between Λ'_1 and Λ_2 to stay within the grid. In particular, this gives a large subset $\Lambda'_2 \subseteq \Lambda_2$ spread out over many cells of the grid. Similarly to the previous argument, we then find many of the edges between Λ'_2 and Λ_3 staying within the grid. We eventually arrive at a situation depicted in Figure 2 (right): A cell with a set Y of five vertices from Λ_2 with the same predecessors in Λ_1, such that for each $y \in Y$ there are four vertices $y_1, \ldots, y_4 \in \Lambda_2$ (one from the bottom-left, one from the bottom-right, one from the top-right, and one from the top-left part of the grid) and for each i the common neighbor of y and y_i from Λ_3 lies in the grid. It turns out, that each $y \in Y$ either has an $\mathbb{A}$-edge to the left and an $\mathbb{A}$-edge to the right or it has a $\mathbb{B}$-edge to the top and a $\mathbb{B}$-edge to the bottom (using directions from Figure 2). As this is impossible to realize for all five vertices in Y simultaneously, the geometric thickness of $G(n)$ is at least 3.

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