

A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces

Camille Lanuel^{*1}, Francis Lazarus^{†2}, and Rudi Pendavingh³

¹ Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

² G-SCOP, CNRS, UGA, Grenoble, France

³ Department of Mathematics and Computer Science, Eindhoven University of Technology, 5600 MB Eindhoven, Netherlands

Abstract

The Colin de Verdière graph parameter $\mu(G)$ was introduced in 1990 by Y. Colin de Verdière. It is defined via spectral properties of a certain type of matrices, called Schrödinger operators, associated to a graph G . We provide a combinatorial and self-contained proof that for all graphs G embedded on a surface S , the Colin de Verdière parameter $\mu(G)$ is upper bounded by $7 - 2\chi(S)$, where $\chi(S)$ is the Euler characteristic of S .

Related Version <https://arxiv.org/abs/2303.00556>

1 Introduction

In this paper, we establish an upper bound for the Colin de Verdière’s graph parameter μ for graphs that can be embedded on a fixed surface. This parameter was introduced by Colin de Verdière [4] in analogy with the multiplicity of the second eigenvalue of Schrödinger operators on a Riemannian surface. The exact definition of $\mu(G)$ resorts to a transversality condition between the space of so-called discrete Schrödinger operators on a graph $G = (V, E)$ and a certain stratification of the space of symmetric matrices of dimension $V \times V$. This *Strong Arnold Hypothesis* (SAH), as coined by Colin de Verdière [3], expresses a stability property and ensures that μ is minor-monotone. It can thus be applied to the graph minor theory of Robertson and Seymour. We refer to the survey by van der Holst, Lovász and Schrijver [13] for more properties on μ .

Here, we are interested in an upper bound for the parameter μ of the minor-closed family of graphs that can be embedded on a surface S . It is relatively easy to show that $\mu(K_n) = n - 1$ for K_n , the complete graph with n vertices [13]. On the other hand, the largest n such that K_n embeds on S is known as the Heawood number

$$\gamma(S) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi(S)}}{2} \right\rfloor,$$

where $\chi(S)$ is the Euler characteristic of S . Colin de Verdière [2] conjectured that the maximum of μ for all graphs that can be embedded in S is attained at $K_{\gamma(S)}$. In other words, μ is upper bounded by $\gamma(S) - 1$. In practice, the known upper bounds have been proved in the realm of Riemannian surfaces where μ is defined for each Riemannian metric

^{*} This work was conducted when this author was at the G-SCOP laboratory in Grenoble. This author is (partially) supported by the grant ANR-17-CE40-0033 of the French National Research Agency ANR (project SoS)

[†] This author is partially supported by LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01) funded by the French program Investissement d’avenir.

as the maximum multiplicity of the second eigenvalue of Schrödinger differential operators (based on the Laplace-Beltrami operator associated to the given metric). In this framework, Besson [1] obtained the bound $7 - 2\chi(S)$. When the Euler characteristic is negative, this bound was further decreased by 2 by Nadirashvili [8], who thus showed the upper bound $5 - 2\chi(S)$. Sévenec [10] eventually divided by two the dependency in the characteristic to obtain the upper bound $5 - \chi(S)$. Note that all those bounds are linear in the genus of S and remain far from the square root bound in the conjecture of Colin de Verdière. It can be proved that any upper bound for μ in the Riemannian world holds for graphs, cf. [6, Th. 6.3] and [2, Th. 7.1]. However, the proof relies on the construction of Schrödinger differential operators from combinatorial ones and is not particularly illuminating from the combinatorial viewpoint. The goal of this paper is to propose a purely combinatorial and self-contained proof of the bound of Besson in the combinatorial framework of graphs. In the following, S is a compact surface, orientable or not.

► **Theorem 1.1.** *Let G be a graph that can be embedded on a surface S , then*

$$\mu(G) \leq 7 - 2\chi(S).$$

Our proof completes and slightly simplifies a proof of Pendavingh that appeared in his PhD thesis [9]. Before describing the general strategy of the proof, we provide some relevant definitions and basic facts.

2 Background

2.1 Schrödinger operators and μ

Let $G = (V, E)$ be a connected simplicial¹ graph with at least two vertices. A *Schrödinger operator* on G , sometimes called a generalized Laplacian, is a symmetric $V \times V$ matrix such that for $i \neq j \in V$, its ij coefficient is negative if $ij \in E$ and zero otherwise. (There is no condition on the diagonal coefficients.) We say that a Schrödinger operator L satisfies the *Strong Arnold Hypothesis (SAH)* if there is no nonzero symmetric matrix $X = (X_{ij})$ such that $LX = 0$ and $X_{ij} = 0$ whenever $i = j$ or $ij \notin E$, in other words, if the only symmetric matrix satisfying these conditions is zero. This is an algebraic translation of a transversality condition between the space of the discrete Schrödinger operators on G and a certain stratification of the space of symmetric matrices of dimension $V \times V$. It is not necessary to understand the SAH for the purpose of this paper.

It follows from Perron–Frobenius theorem that the first (smallest) eigenvalue of a Schrödinger operator L has multiplicity one [5]. Now, if λ_2 is the second eigenvalue of L , then the first eigenvalue of $L - \lambda_2 \text{Id}$, where Id is the identity matrix, is negative and its second eigenvalue is zero. This translation by $-\lambda_2 \text{Id}$ does not change the sequence of multiplicities of the eigenvalues of L nor the stability of L with respect to the SAH. Consequently, we can safely restrict to Schrödinger operators whose second eigenvalue is zero. We can now define the *Colin de Verdière graph parameter* $\mu(G)$ as the maximal corank (dimension of the kernel $\ker(L)$) of any Schrödinger operator L satisfying the Strong Arnold Hypothesis. In other words, $\mu(G)$ is the largest integer p such that there exists a Schrödinger operator L with $\dim \ker L = p$ and L satisfies the SAH.

Recall that a *minor* of G is a graph obtained from G by deleting edges, vertices or contracting edges. As a fundamental property, μ is minor-monotone.

¹ A graph is *simplicial*, or *simple*, if it has no loops or multiple edges.

► **Theorem 2.1** ([4]). *If H is a minor of G , then $\mu(H) \leq \mu(G)$.*

It also characterizes planar graphs.

► **Theorem 2.2** ([4]). *A graph G is planar if and only if $\mu(G) \leq 3$.*

Colin de Verdière made the following conjecture in [4].

► **Conjecture 2.3.** *The chromatic number of a graph G satisfies $\text{chr}(G) \leq \mu(G) + 1$.*

This conjecture contains the four colour theorem thanks to Theorem 2.2. It is implied by Hadwiger's conjecture (a graph which is not colourable with k colours has the complete graph K_{k+1} as a minor), using the fact that $\mu(K_n) = n - 1$ [13] and the minor-monotone property of μ .

We view a vector of $\mathbb{R}^V$ as a discrete map $V \rightarrow \mathbb{R}$, so that a Schrödinger operator acts linearly on the set of discrete maps. For $f : V \rightarrow \mathbb{R}$, we denote by V_f^+, V_f^0, V_f^- the subsets of vertices where f takes respectively positive, null and negative values. The *support* of f is the subset $V_f^+ \cup V_f^-$ of vertices with nonzero values. As a simple property of Schrödinger operators we have

► **Lemma 2.4** ([13]). *Let L be a Schrödinger operator of G and let $f \in \ker L$. Then, a vertex $v \in V_f^0$ is adjacent to a vertex of V_f^+ if and only if v is adjacent to a vertex of V_f^- .*

A discrete version of the nodal theorem of Courant reads as follows.

► **Theorem 2.5** ([5, 12]). *Let L be a Schrödinger operator of G and let $f \in \ker L$ be a nonzero map with minimal support². Then, the subgraphs of G induced respectively by V_f^+ and V_f^- are nonempty and connected.*

2.2 Surfaces and Euler characteristic

By a *surface* of finite type we mean a topological space homeomorphic to a compact two dimensional manifold minus a finite number of points. A surface may have nonempty boundary and each of the finitely many boundary components is homeomorphic to a circle. We shall only consider surfaces of finite type and omit to specify this condition. A *closed* surface means a compact surface without boundary. By a *triangulation* of a surface, we mean a simplicial complex together with a homeomorphism between its underlying space and the surface.

The *Euler characteristic* $\chi(X)$ of a finite simplicial complex (and more generally a finite CW complex) is the alternating sum of the numbers of cells of each dimension. In particular, the Euler characteristic of a graph $\Gamma = (V, E)$ is $\chi(\Gamma) = |V| - |E|$. For a surface S we define $\chi(S)$ as the Euler characteristic of a polygonisation of S , that is, a description of S as a CW complex. The Euler characteristic is homotopy invariant: two spaces with the same homotopy type have the same Euler characteristic.

The following property will be needed in our proof. It is a corollary of the Inclusion-exclusion formula [11, p.205]. The proof is not completely trivial and can be found in [7, Section 2].

² A discrete map $f \in \ker L$ has minimal support if it is nonzero and for every nonzero $g \in \ker L$, if $V_g^+ \cup V_g^- \subseteq V_f^+ \cup V_f^-$, then $V_g^+ \cup V_g^- = V_f^+ \cup V_f^-$.

► **Proposition 2.6.** *Let X be a triangulated compact surface and let Y be a subcomplex of X . Then*

$$\chi(X) = \chi(Y) + \chi(X \setminus Y).$$

Here, X and Y should be considered as topological spaces, forgetting about their triangulations. In general, $X \setminus Y$ is not a subcomplex of X .

3 Overview of the proof

In this section, we give the main ideas of the proof of Theorem 1.1. The detailed proof can be found in [7, Section 5].

Let G be a graph embedded on a surface S , which we can assume closed without loss of generality. We may also assume that S is not homeomorphic to a sphere as otherwise Theorem 1.1 follows directly from Theorem 2.2. In a first step, we remove an open disk $D \subset S \setminus G$ whose boundary avoids G , and build a graph³ H embedded in $S \setminus D$ such that

- C1.** H triangulates $S \setminus D$ and subdivides ∂D into a cycle of $\mu(G) - 1$ edges,
- C2.** G is a minor of H , and
- C3.** the length of the shortest closed walk in H that is non-contractible in $S \setminus D$, i.e. the *edgewidth* of H in $S \setminus D$, is $\mu(G) - 1$.

We denote by W the set of vertices of H .

We next choose a Schrödinger operator L for H whose corank achieves $\mu(H)$. Condition C2 and the monotonicity of μ imply $\mu(H) \geq \mu(G)$, so that $\ker L$ has dimension at least $\mu(G)$. By C1, ∂D has $\mu(G) - 1$ vertices, so the vectors of the basis of $\ker L$ restricted to ∂D are linearly dependent. Thus there exists a nonzero vector $f \in \ker L$ such that f cancels on the vertices of ∂D . We pick such an f with minimal support so that by Theorem 2.5 the subsets of vertices W_f^+ and W_f^- induce connected subgraphs of H . We connect the vertices of ∂D by inserting $\mu(G) - 4$ edges in D to obtain a graph H' with the same vertices as H and that triangulates S . We can now extend f linearly on each face of H' to get a piecewise linear map $\bar{f} : S \rightarrow \mathbb{R}$. Let S_f^+, S_f^0, S_f^- denote the subspaces of S where $\bar{f}$ is respectively positive, null, and negative. By Theorem 2.5, S_f^+ and S_f^- are connected open subsurfaces of S , while S_f^0 is a closed subcomplex of some subdivision of the triangulation induced by H' . We can thus apply Proposition 2.6 to write

$$\chi(S) = \chi(S_f^0) + \chi(S_f^+ \cup S_f^-) = \chi(S_f^0) + \chi(S_f^+) + \chi(S_f^-).$$

The subsets S_f^+ and S_f^- cannot be homeomorphic to spheres because they are both proper subsets of S . We deduce from the classification of surfaces, that $\chi(S_f^+) \leq 1$ and $\chi(S_f^-) \leq 1$. It ensues that $\chi(S_f^0) \geq \chi(S) - 2$. The goal is now to provide an upper bound for $\chi(S_f^0)$ in terms of $\mu(G)$ in order to obtain the desired upper bound for $\mu(G)$.

Start by observing S_f^0 : it is formed of plain triangles adjacent by an edge or a vertex, or connected together, via their vertices, by a piece of dimension 1. To provide the upper bound for $\chi(S_f^0)$, we build a graph Γ whose Euler characteristic is larger than S_f^0 by contracting its two dimensional parts, that are the parts formed by the plain triangles. To ensure that this operation results in the desired property on the Euler characteristic of Γ , precautions are taken before the contraction to remove the singularities of the two dimensional parts while keeping the same homotopy type. By definition of $\bar{f}$, $D \subset S_f^0$, so we can define K as the two

³ See [7, Section 4] for a detailed construction of such a graph.

dimensional part containing D . Because of condition C3 on H , we can argue that K has a non-contractible boundary in $S \setminus D$. It follows that this boundary has length at least the edgewidth of H , hence at least $\mu(G) - 1$ by condition C3. Thanks to Lemma 2.4 we may infer that K contracts to a vertex of degree at least $\mu(G) - 1$ in Γ . We also argue thanks to Lemma 2.4 that Γ has no vertex of degree one. By the handshaking lemma applied to Γ , we deduce that $\chi(S_f^0) \leq \chi(\Gamma) \leq (3 - \mu(G))/2$. We finally conclude that $\chi(S) - 2 \leq (3 - \mu(G))/2$, hence $\mu(G) \leq 7 - 2\chi(S)$.

References

- 1 Gérard Besson. Sur la multiplicité de la première valeur propre des surfaces riemanniennes. *Annales de l'institut Fourier*, 30(1):109–128, 1980. URL: <http://www.numdam.org/item/10.5802/aif.777.pdf>.
- 2 Yves Colin de Verdière. Construction de laplaciens dont une partie finie du spectre est donnée. *Annales scientifiques de l'école normale supérieure*, 20(4):599–615, 1987.
- 3 Yves Colin de Verdière. Sur une hypothèse de transversalité d'Arnol'd. *Comment. Math. Helv.*, 63(2):184–193, 1988.
- 4 Yves Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *Journal of Combinatorial Theory, Series B*, 50(1):11–21, 1990.
- 5 Yves Colin de Verdière. Théorèmes de Courant et de Cheng combinatoires. *Séminaire de théorie spectrale et géométrie*, 13:9–13, 1994.
- 6 Yves Colin de Verdière. *Spectres de graphes*. Number 4 in Cours Spécialisés. Société Mathématique de France, 1998.
- 7 Camille Lanuel, Francis Lazarus, and Rudi Pendavingh. A linear bound for the Colin de Verdière parameter μ for graphs embedded on surfaces, 2023. Preprint. URL: <https://arxiv.org/abs/2303.00556>, doi:10.48550/ARXIV.2303.00556.
- 8 Nikolai Semenovitch Nadirashvili. Multiple eigenvalues of the Laplace operator. *Mathematics of the USSR-Sbornik*, 61(1):225, 1988.
- 9 Rudolf Anton Pendavingh. *Spectral and Geometrical Graph Characterizations*. PhD thesis, Amsterdam University, 1998.
- 10 Bruno Sévenec. Multiplicity of the second Schrödinger eigenvalue on closed surfaces. *Mathematische Annalen*, 324(1):195–211, 2002.
- 11 Edwin H. Spanier. *Algebraic topology*. Springer Verlag, corrected reprint of the 1966 original edition, 1989.
- 12 Hein Van der Holst. A short proof of the planarity characterization of Colin de Verdière. *Journal of Combinatorial Theory, Series B*, 65(2):269–272, 1995. URL: <https://www.sciencedirect.com/science/article/pii/S0095895685710544/pdf?md5=cb7b57b58c203666ae179c8de5e87e19&pid=1-s2.0-S0095895685710544-main.pdf>.
- 13 Hein Van der Holst, László Lovász, and Alexander Schrijver. The Colin de Verdière graph parameter. *Graph Theory and Computational Biology (Balatonlelle, 1996)*, pages 29–85, 1999.