Primal-Dual Cops and Robber

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Abstract

Cops and Robber is a family of two-player games played on graphs in which one player controls a number of cops and the other player controls a robber. In alternating turns, each player moves (all) their figures. The cops try to capture the robber while the latter tries to flee indefinitely. In this paper we consider a variant of the game played on a planar graph where the robber moves between adjacent vertices while the cops move between adjacent faces. The cops capture the robber if they occupy all incident faces. We prove that a constant number of cops suffices to capture the robber on any planar graph of maximum degree Δ if and only if $\Delta \leq 4$.

Related Version *Full Version*: `https://arxiv.org/abs/2301.05514` [9]

1 Introduction

Cops and Robber is probably the most classical combinatorial pursuit-evasion game on graphs. The robber models an intruder in a network that the cops try to capture. Two players play with complete information on a fixed finite graph $G = (V, E)$. The cop player controls a set of k cops, each occupying a vertex of G (possibly several cops on the same vertex), while the robber player controls a single robber that also occupies a vertex of G. The players take alternating turns, where the cop player in his turn can decide for each cop individually whether to stay at its position or move the cop along an edge of G onto an adjacent vertex. Similarly, the robber player on her turn can leave the robber at its position or move it along an edge of G. The cop player starts by choosing starting positions for his k cops and wins the game as soon as at least one cop occupies the same vertex as the robber, i.e., when the robber is captured. The robber player, seeing the cops' positions, chooses the starting position for her robber and wins if she can avoid capture indefinitely. The least integer k for which, assuming perfect play on either side, k cops can always capture the robber, is called the *cop number* of G, usually denoted by $c(G)$.

In this paper, we introduce *Primal-Dual Cops and Robber* which is played on a plane graph G, i.e., with a fixed plane embedding. Here, the cops occupy the faces of G and can move between adjacent faces (i.e., faces that share an edge), while the robber still moves along edges between adjacent vertices of G. In this game, the robber is captured if *every* face incident to the robber's vertex is occupied by at least one cop. Analogously, we call the least integer k for which k cops can always capture the robber in the Primal-Dual Cops and Robber game the *primal-dual cop number* of G and denote it by $c^*(G)$.

An obvious lower bound for $c^*(G)$ is the maximum number of faces incident to any vertex in G: The robber can choose such a vertex as its start position and just stay there indefinitely (note that there is no *zugzwang*, i.e., no obligation to move during one's turn). In particular, if G has maximum degree $\Delta(G)$ and there exists a vertex v with $\deg(v) = \Delta(G)$, which is not a cut-vertex, then $c^*(G) \geq \Delta(G)$. E.g., $c^*(K_{2,n}) = \Delta(K_{2,n}) = n$ for any $n \geq 2$.

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This is an extended abstract of a presentation given at EuroCG'23. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

Our contribution. We investigate whether the primal-dual cop number $c^*(G)$ is bounded in terms of $\Delta(G)$ for all plane graphs G. The answer is 'Yes' if $\Delta(G) \leq 4$ and 'No' otherwise.

▶ **Theorem 1.1.** *Each of the following holds.*

1. *For every plane graph G with $\Delta(G) \leq 3$ we have $c^*(G) \leq 3$.*
2. *For every plane graph G with $\Delta(G) \leq 4$ we have $c^*(G) \leq 12$.*
3. *For some n-vertex plane graphs G with $\Delta(G) = 5$ we have $c^*(G) = \Omega\big(\sqrt{\log(n)}\big)$.*

Related work. Let us just briefly mention that Cops and Robber was introduced by Nowakowski and Winkler [11] and Quillot [13] for one cop and Aigner and Fromme [1] for k cops 40 years ago. Since then numerous results and variants were presented, see e.g., [2, 3]. Perhaps most similar to our new variant are the recent surrounding variant of Burgess et al. [5] with vertex-cops and the containment variant of Crytser et al. [6, 12] with edge-cops. In these variants the robber is captured if every adjacent vertex, respectively every incident edge, is occupied by a cop. The smallest number of cops that always suffices for any planar graph G is 3 in the classical variant [1], 7 in the surrounding variant [4], $7\Delta(G)$ in the containment variant [6] and 3 when both, cops and robber, move on edges [7].

2 Cops win always if the maximum degree is at most four

We start with an observation that simplifies the proofs of statements 1 and 2 in Theorem 1.1.

▶ **Observation 2.1.** *Let the robber be on a vertex u with a neighbor v of degree 1. Then the robber is never required to move to v to evade the cops.*

This is true because the set of faces required to capture the robber at v is a subset of the faces required to capture him at u. Further, his only possible moves at v are either staying there or moving back to u. As there is no zugzwang, he could just stay at u all along.

In both of the following proofs we assume that the graph contains only degree-3-vertices (respectively degree-4-vertices) and degree-1-vertices. This can always be achieved by adding leaves to vertices not yet having the correct degree.

Proof of statement 1 in Theorem 1.1. We give a winning strategy for three cops c_1, c_2, c_3 in a planar graph G with $\Delta(G) \leq 3$. First the cops choose arbitrary faces to start on. Then the robber chooses its start vertex u, which we assume to be of degree 3 by Observation 2.1 (it is trivial to capture him if all vertices have degree 1). Let $\angle_1^u, \angle_2^u, \angle_3^u$ be the three angles incident to u. We denote the face containing an angle $\angle$ by $f(\angle)$ and define for each cop c_i a *target face* f_i, $i = 1, 2, 3$. Initially we set $f_i = f(\angle_i^u)$. The goal of each cop is to reach his target face, thereby capturing the robber when all three cops arrive. If the robber moves, each cop updates his target face. Our strategy guarantees that the total distance of all three cops to their target faces decreases over time, so it reaches zero after finitely many turns.

Clearly, in every game the robber has to move at some point to avoid being captured. Assume that the robber moves from vertex u to vertex v (both of degree 3 by Observation 2.1). Without loss of generality the angles around u and v are labeled as in Figure 1 with $f_i = f(\angle_i^u)$ being the current target face of cop c_i, $i = 1, 2, 3$.

First assume that c_3 (or symmetrically c_2) has not reached his target face yet. In this case we assign the new target faces $f_1 = f(\angle_1^v)$, $f_2 = f(\angle_2^v)$ and $f_3 = f(\angle_3^v)$. Note that for $i = 1, 2$ faces $f(\angle_i^u)$ and $f(\angle_i^v)$ are adjacent, so cop c_i can keep his distance to his target face unchanged (or even decrease it) during his next turn. Further note that $f(\angle_3^u) = f(\angle_3^v)$,

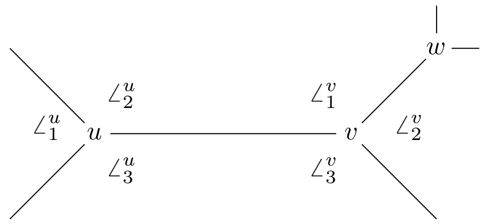

Figure 1 Labeling of the angles for a robber move from u to v (and possibly further to w).

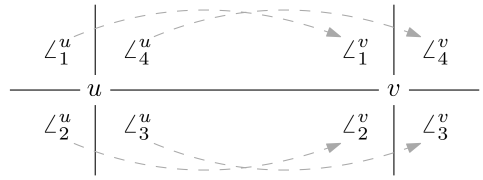

Figure 2 A vertex cop and its four accompanying face-cops moving from u to v.

so cop c_3 can even decrease his distance by one during the next turn. Thus the total distance of the three cops to their target faces decreased by at least one.

It remains the case that c_2 and c_3 have already reached their target faces (but c_1 has not, as the game would be over otherwise). In this case we move c_1 one step towards his target face $f_1 = f(\angle_1^u)$ and c_2, c_3 both to $f(\angle_2^v)$. Now its the robber's turn again. If she does not move, we assign target faces $f_i = f(\angle_i^v)$, $i = 1, 2, 3$, and the total distance decreases after the cops' next turn. If she moves back to u, we assign target faces $f_i = f(\angle_i^u)$, $i = 1, 2, 3$, and the total distance decreases after the cops' next turn. The last possibility for the robber is to move towards another neighbor w of v, see Figure 1. Then we assign $f_1 = f(\angle_1^v)$ and f_2, f_3 to be the faces containing the other two angles at w. In their next turn, c_2 and c_3 can again reach their target faces, while c_1 can decrease his distance to his target face $f(\angle_1^v)$ by one compared to the initial situation with the robber at vertex u. Again, the total distance is decreased, which concludes the proof. ◀

To prove statement 2 in Theorem 1.1, we reduce our Primal-Dual Cops and Robber to the classical Cops and Robber with cops on vertices of G and then use a result from the literature.

▶ **Lemma 2.2.** *In a plane graph G with $\Delta(G) \leq 4$, four face-cops can simulate a vertex-cop.*

Proof. Let c be a vertex-cop starting at a vertex $u \in V(G)$ with up to four incident angles $\angle_i^u$ (for $i \in \{1, 2, 3, 4\}$). We place four face-cops on the (up to) four faces $f(\angle_i^u)$. If the vertex-cop moves to an adjacent vertex v, the four face cops around it can in one step also move to faces containing the angles incident to v, see Figure 2 for the case that u and v both have degree 4. For vertices of degree less then 4 it only gets easier for the face-cops. ◀

An immediate corollary of Lemma 2.2 is that $c^*(G) \leq 4 \cdot c(G)$ for planar graphs G with $\Delta(G) \leq 4$. With $c(G) \leq 3$ for all planar graphs G [1], statement 2 in Theorem 1.1 follows.

3 Robber wins sometimes if the maximum degree is at least five

In this section we prove statement 3 in Theorem 1.1, i.e., that $c^*(G) = \Omega\left(\sqrt{\log(n)}\right)$ for some n-vertex plane graphs G with $\Delta(G) \geq 5$. We utilize a result of Nisse and Suchan [10]

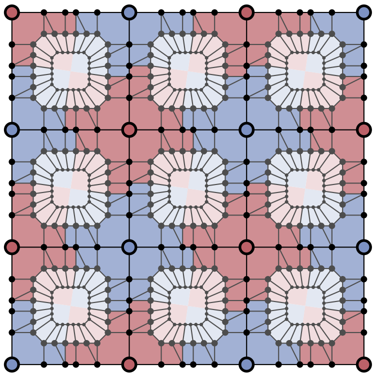

Figure 3 $G_{4,2,2}$: An $n \times n$ grid with each edge subdivided four times and two rings. Faces are colored according to their closest grid vertex. Deep and shallow faces are light and dark, respectively.

about the cop number $c_{p,q}(G)$ for a different variant of Cops and Robber for any graph G and positive integers p and q. Here (as in the classical variant) the cops and the robber are on the vertices of G. However, in each turn the cops may traverse up to p edges of G, while the robber may traverse up to q edges of G. We refer to p and q as the *velocities* of the cops and the robber, respectively.

▶ **Theorem 3.1** ([8, 10]). *Let G_n be the $n \times n$ grid graph, p be the velocity of the cops and q be the velocity of the robber. If $p < q$, then $c_{p,q}(G_n) = \Omega\big(\sqrt{\log(n)}\big)$.*

The idea to prove statement 3 in Theorem 1.1 is to construct a "grid-like" graph $G_{n,s,r}$ for positive integers n, s, r in which the robber in the primal-dual variant can move around faster than the cops. Then she can simulate the evasion strategy of the robber in the variant of Nisse and Suchan.

We start with the $n \times n$ grid graph G_n, $n \geq 3$, with a planar embedding such that the 4-faces are the inner faces. We call the vertices of G_n the *grid vertices*. Then, each edge of G_n is subdivided by $2s$ new vertices, called *subdivision vertices*, to obtain $G_{n,s}$. Two grid vertices are called *neighboring* if they are adjacent in G_n. Further, inside each inner face of $G_{n,s}$ we add r nested cycles, called *rings*, of length $12s$ each and call their vertices the *ring vertices*. Between any two consecutive rings we add a planar matching of $12s$ edges. Each inner face of $G_{n,s}$ has $8s$ subdivision vertices on its boundary and $12s$ ring vertices on its outermost ring. At last, we add (in a crossing-free way) three edges from each subdivision vertex to the outermost ring vertices in the two incident faces of $G_{n,s}$ such that two edges go to one ring, the third edge to the other ring, and every ring vertex receives exactly one such edge. Along the $2s$ vertices of each subdivision path in $G_{n,s}$ the side with two edges to the ring should always switch. Thus each inner face of $G_{n,s}$ receives $12s$ edges which are connected to the $12s$ vertices of the outermost ring such that the drawing remains planar.

Call the resulting graph $G_{n,s,r}$ and note that $\Delta(G_{n,s,r}) = 5$. See also Figure 3. We shall use a robber strategy in which she only focuses on grid vertices and moves between these through the paths of subdivision vertices, i.e., only plays on $G_{n,s}$. The purpose of the additional rings in $G_{n,s,r}$ is to slow down the cops and force them to stay close to grid and subdivision vertices, too, thereby simulating the game of Nisse and Suchan on G_n.

Formally, we call an inner face of $G_{n,s,r}$ *shallow* if it is incident to some subdivision vertex, and *deep* otherwise. Lemma 3.2 below implies that, due to the number of rings, cops should not use deep faces. Omitted proofs of statements marked with ($\star$) can be found in the full version [9].

▶ **Lemma 3.2** (⋆). *Let a_1, a_2 be two shallow faces of $G_{n,s,r}$ inside the same inner face A of G_n. If $r > 3s$, then any cop moving from a_1 to a_2 along a shortest path without leaving A uses only shallow faces.*

We have to hinder the cops from taking shortcuts through the outer face f_0 of $G_{n,s,r}$. To this end let $G'_{n,s,r}$ be a copy of $G_{n,s,r}$ with outer face f'_0. Change the outer face of $G'_{n,s,r}$ such that f'_0 is an inner face (while not changing the cyclic ordering of the edges around the vertices) and define $\overline{G}_{n,s,r}$ to be the graph obtained from gluing $G_{n,s,r}$ into face f'_0 of $G'_{n,s,r}$ and identifying corresponding vertices. The robber will always stay on vertices of $G_{n,s,r}$ and whenever a cop uses a vertex v' of $G'_{n,s,r}$ she acts as if he was on the corresponding vertex v of $G_{n,s,r}$. Without loss of generality, we can therefore assume below that the game is played on $G_{n,s,r}$ with the cops being prohibited to enter the outer face.

For a face $f \in F$, we denote by v_f the grid vertex closest to f, breaking ties arbitrarily.

▶ **Lemma 3.3** (⋆). *Let a, b be two shallow faces whose closest grid vertices v_a, v_b have distance d in G_n. If $r > 3s$, then in $G_{n,s,r}$ the robber moving from v_a to v_b needs at most $(2s+1)d$ steps, while any cop moving from a to b needs at least $3s(d-4)$ steps.*

Proof of statement 3 in Theorem 1.1. Nisse and Suchan [10] (see also [8] for the omitted proofs) describe an evasion strategy for a robber with velocity q that requires $\Omega\big(\sqrt{\log(n)}\big)$ vertex-cops with velocity p to capture him in G_n, provided $q > p$; see Theorem 3.1. We describe how a robber with velocity 1 in $G_{n,s,r}$ (for sufficiently large n, s, r) can simulate this strategy against face-cops with velocity 1.

We choose $p = 15$, $q = 16$ and consider the game of Nisse and Suchan for these velocities. For their graph G_n in which the robber can win against $k = \Omega\big(\sqrt{\log(n)}\big)$ vertex-cops, we then consider $G_{n,s,r}$ with $s = 16$ and $r = 3s + 1 = 49$. Now we copy the evasion strategy $\mathcal{S}$ for the robber as follows: Whenever it is the robber's turn and the face-cops occupy faces $f_1, f_2, \ldots, f_k$ in $G_{n,s,r}$, consider the corresponding situation in G_n where the vertex-cops occupy $v_{f_1}, v_{f_2}, \ldots, v_{f_k}$. Based on these positions, $\mathcal{S}$ tells the robber to go to a vertex v at distance $d \leq q = 16$ from the current position of the robber in G_n. By Lemma 3.3, the robber in $G_{n,r,s}$ can go to v in at most $(2s+1)d \leq (2 \cdot 16 + 1) \cdot 16 = 528$ turns.

In the meantime, each face-cop also makes up to 528 moves in $G_{n,r,s}$, traveling from some face a to some face b, which is interpreted in G_n as the corresponding vertex-cop traveling from v_a to v_b. For v_a and v_b to be at distance $d' \geq 16$ in G_n, by Lemma 3.2 the face-cop needs at least $3s(d'-4) \geq 3 \cdot 16 \cdot 12 = 576$ turns, which is strictly more than 528. Thus, after 528 turns, each vertex-cop made at most $p = 15$ steps in G_n, as required for strategy $\mathcal{S}$.

Hence, the robber can evade k face-cops in $G_{n,s,r}$, proving $c^*(G_{n,s,r}) > k$. Since $G_{n,s,r}$ for $s, r \in O(1)$ has $O(n^2)$ vertices, this completes the proof. ◀

4 Conclusions

Let c^*_Δ denote the largest primal-dual cop number among all plane graphs with maximum degree Δ. We have shown that $c^*_3 = 3$, $c^*_4 \leq 12$ (this bound is certainly not optimal), and $c^*_5 = \infty$, while it is easy to see that $c^*_1 = 1$, $c^*_2 = 2$, and $c^*_\Delta = \infty$ for all $\Delta > 5$. Let us remark that our proof for $\Delta = 5$ also holds for a variant of the game where the robber is already captured when one cop is on one incident face. On the other hand, our proof for $\Delta = 3$ holds verbatim to prove that three cops also suffice in a variant of the game where the graph is embedded without crossings in any other surface, which makes it is interesting to consider $\Delta = 4$ here.

Another interesting direction would be to identify classes of plane graphs with unbounded maximum degree in which $c^*(G) \leq f(\Delta(G))$ for some function f. For example, what about plane 3-trees, also known as stacked triangulations?

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