Flip Graphs for Arrangements of Pseudocircles*

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Abstract

An arrangement of pseudocircles is a finite collection of simple closed curves in the plane such that every pair of curves is either disjoint or intersects in two crossing points. We study flip graphs of families of pseudocircle arrangements. We prove that triangle flips induce a connected flip graph (i) on *intersecting* arrangements and (ii) on *cylindrical intersecting* arrangements. Our constructions make essential use of variants of the sweeping lemma for pseudocircle arrangements due to Snoeyink and Hershberger (Proc. SoCG 1989: 354–363). We also study cylindrical arrangements in their own right and provide new combinatorial characterizations of this class of pseudocircle arrangements.

1 Introduction

Reconfiguration is a widely studied topic in discrete mathematics and theoretical computer science [9]. In many cases, reconfiguration problems can be stated in terms of a *flip graph*. For a class of objects, the flip graph has a vertex for each object and adjacencies are determined by a local flip operation, which transforms one object into another. Typically, the first question is whether a flip graph is connected. In the affirmative case, more refined questions regarding diameter, the degree of connectivity, or Hamiltonicity can be of interest. Hamiltonicity of flip graphs is related to Gray codes, cf. [8]. For further details on flip-graphs in general we also refer the reader to the survey [2].

Ringel [10] showed flip-connectivity for arrangements of *pseudolines* under triangle flips. There, an arrangement of pesudolines is a set of bi-infinite curves that pairwise intersect exactly once. A triangle flip then corresponds to moving a pseudoline incident to a triangular cell over the crossing of the two other pseudolines. If pseudolines are also allowed to be disjoint, flipping triangles is not enough, as one also needs to allow two pseudolines to become intersecting or non-intersecting. Snoeyink and Hershberger [11] showed flip-connectivity for such arrangements of pseudolines with these three operations.

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Similarly, in the context of arrangements of pseudocircles, which have been first studied by Grünbaum [7], flipping triangles is not enough if disjoint pseudocircles are allowed. In such cases, the set of flips is extended by *digon-create* to allow two initially disjoint pseudocircles to start intersecting in a digon and the reverse operation, called *digon-collapse* (see Section 2 for definitions).

With all the three flips, the flip-connectivity of arrangements of proper circles is evident since one can shrink all the circles until they have pairwise disjoint interiors. Essentially the same idea works for arrangements of pseudocircles. In this case, however, the fact that a pseudocircle can be shrunk is based on the sweeping lemma of Snoeyink and Hershberger [11]. Allowing only triangle flips, Felsner and Scheucher [5] showed flip-connectivity for classes of arrangements of proper circles and conjectured that the results persist for pseudocircles:

▶ **Conjecture 1** ([5, Conjecture 8.6]). *For every $n \in \mathbb{N}$:*

(1) The flip graph of intersecting arrangements of n pseudocircles is connected.
(2) The flip graph of digon-free intersecting arrangements of n pseudocircles is connected.

As the main result of this article we prove part (1) of Conjecture 1.

▶ **Theorem 1.1.** *The flip graph of arrangements of n pairwise intersecting pseudocircles is connected.*

For our proof of Theorem 1.1, we use *cylindrical arrangements*. These are arrangements of pseudocircles in the plane such that the bounded interiors of all the pseudocircles have a common intersection, which we call the *center*. We first show that every cylindrical intersecting arrangement can be flipped into a canonical arrangement by only using triangle flips and without leaving the class of cylindrical arrangements.

▶ **Theorem 1.2.** *The flip graph of cylindrical arrangements of n pairwise intersecting pseudocircles is connected.*

Showing that every intersecting arrangement $\mathcal{A}$ can be flipped into some cylindrical arrangement then completes the proof of Theorem 1.1. We further study the diameter of flip graphs. In each of the two considered settings (cylindrical / general arrangements of pairwise intersecting pseudocircles), we obtain asymptotically tight bounds for the diameter.

▶ **Proposition 1.3.** *The flip graph of cylindrical arrangements of n pairwise intersecting pseudocircles has diameter at least $2\binom{n}{3}$ and at most $4\binom{n}{3}$.*

▶ **Proposition 1.4.** *The flip graph of arrangements of n pairwise intersecting pseudocircles has diameter $\Theta(n^3)$.*

Last but not least, we present the following equivalent characterizations of cylindrical arrangements of pseudocircles (cf. the full version of this paper). Item (2) uses a special arrangement of three pseudocircles that we call NonKrupp(3) (see Section 2), for items (3-4), we orient pseudocircles counterclockwise, which induces an orientation on the edges of the arrangement, and the eccentricity in item (5) refers to the dual graph of the arrangement: The *eccentricity* of a face in an arrangement of pseudocircles is the maximum distance to any other face, where the *distance* between two faces z, z' is the minimum number of pseudocircles that a curve starting in the interior of z and ending in the interior of z' must cross.

▶ **Proposition 1.5.** *Let $\mathcal{A}$ be an arrangement of n pseudocircles with pairwise overlapping interiors. Then, the following five statements are equivalent:*

(1) $\mathcal{A}$ is cylindrical.
(2) $\mathcal{A}$ does not contain a NonKrupp(3) as a subarrangement.
(3) There is no clockwise oriented cycle in $\mathcal{A}$.
(4) There is no clockwise oriented face in $\mathcal{A}$.
(5) The unbounded face has eccentricity n.

2 Preliminaries

A *pseudocircle* is a simple closed curve C which partitions the plane into a bounded region, the interior $\text{int}(C)$, and an unbounded region, the exterior $\text{ext}(C)$. An *arrangement of pseudocircles* is a finite collection of pseudocircles such that every two pseudocircles either are disjoint or they intersect in two points, where the curves cross properly. Furthermore, no three pseudocircles intersect in a common point. An arrangement partitions the plane into *vertices* (the intersection points), *edges* (maximal contiguous vertex-free pieces of pseudocircles), and *faces* (connected components of the plane after removing all pseudocircles).

A face with k edges along its boundary is a *k-face*, a 2-face is a *digon* (some authors call it *empty lens*), and a 3-face is a *triangle*. It is an instructive exercise to verify that there are exactly four arrangements of three pairwise intersecting pseudocircles (shown in Figure 1).

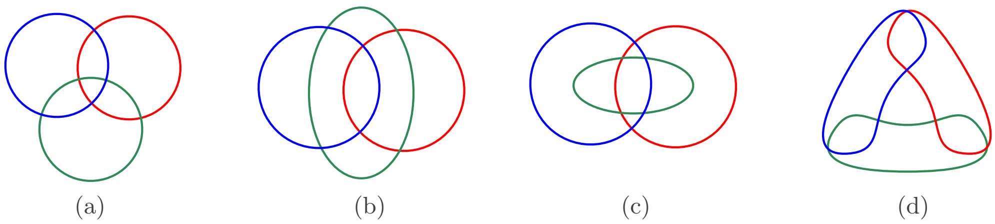

Figure 1 The four non-isomorphic arrangements of 3 pairwise intersecting pseudocircles in the plane. (a) shows the Krupp and (b)–(d) show the three types of NonKrupp arrangements.

Following [5], we call the arrangement, depicted in Figure 1(a), with 8 triangles the *Krupp* arrangement and the other ones *NonKrupp*. To make them distinguishable we write NonKrupp(k) to denote the NonKrupp arrangement whose unbounded face has complexity k, e.g., NonKrupp(2) is the arrangement shown in Figure 1(c). Note that among the arrangements of Figure 1, the arrangement (d), i.e., the NonKrupp(3), is the only non-cylindrical.

2.1 Sweeps and Flips

Snoeyink and Hershberger [11] studied continuous transformations of curves and, in particular, of pseudocircles. More precisely, they define the *sweep* of a pseudocircle as a continuous process to expand or shrink the pseudocircle. However, crucially, they also argue that this continuous process can be viewed as a discrete process as the combinatorics of the arrangement changes with one of the following operations, called *flips*:

- a pseudocircle moves over the crossing of two others (*triangle flip*),
- a pseudocircle gains two intersections with a pseudocircle (*digon-create*), or
- a pseudocircle loses its two intersections with a pseudocircle (*digon-collapse*).

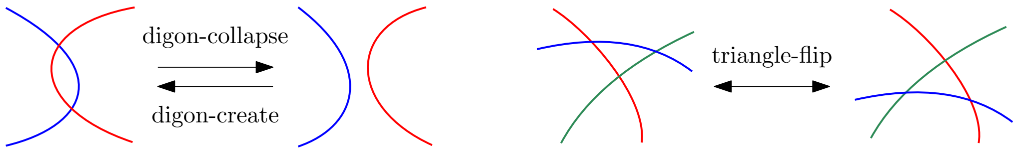

Figure 2 An illustration of the three flip operations.

Figure 2 depicts the three flip operations. Whenever we speak of *flips* without further specification we refer to these three flips, the term *digon flip* refers to the two flips involving a digon, and otherwise we use the precise term when referring to a specific type of flip.

Snoeyink and Hershberger prove a sweeping lemma (cf. [11, Lemma 3.2]) for families of simple curves that pairwise intersect at most twice and are either bi-infinite or closed (i.e., pseudocircles). The flip operations for bi-infinite curves are defined analogously.

▶ **Lemma 2.1** (Sweeping Lemma [11]). *Let $\mathcal{A}$ be an arrangement of pseudocircles or an arrangement of bi-infinite curves that pairwise intersect at most twice. Then $\mathcal{A}$ can be swept starting from any curve C in A by using three operations: triangle flips, digon-create, and digon-collapse.*

Note that in the context of pseudocircles we sweep towards the *inside* or *outside*, whereas in the context of bi-infinite curves we sweep *upwards* or *downwards*. It is straight-forward to verify that Lemma 2.1 implies the flip-connectivity for arrangements of pseudocircles.

It will be convenient to have a separate sweeping lemma for lenses. A *lens* in an arrangement is a maximally bounded region in a subarrangement formed by two intersecting pseudocircles. An *arc* is a contiguous subset of a pseudocircle, starting and ending at a vertex of the arrangement.

Let $\mathcal{A}$ be an arrangement of pseudocircles and let Q be (the closure of) a lens bounded by two pseudocircles C_L and C_R. We denote by $L = C_L \cap Q$ and $R = C_R \cap Q$ the two boundary arcs on Q belonging to C_L and C_R, respectively. An *arc of Q* is a maximal connected piece of the intersection of a pseudocircle C with Q, where $C \notin \{C_L, C_R\}$. In other words, an arc of Q is always a contiguous subset of C which has both endpoints on the boundary of Q and whose relative interior lies completely in the interior of Q. If an arc a of Q has both endpoints on L or both endpoints on R, then a forms a lens with L or R, respectively. Otherwise the arc has one endpoint on L and one on R, in this case we call the arc *transversal* (see Figure 3).

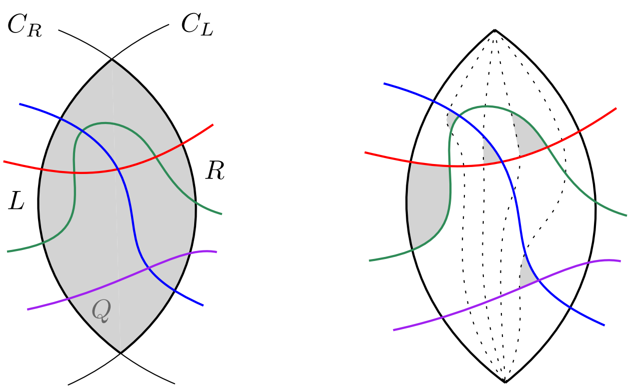

Figure 3 Illustration of Lemma 2.2: a lens Q with four transversal arcs and a sweep of Q.

▶ **Lemma 2.2.** *If Q is a lens and all the arcs of Q are transversal, then, using only triangle flips, L can be swept towards R until the interior of Q does not contain a vertex of the arrangement anymore.*

2.2 Cylindrical Arrangements

An *arrangement of pseudoparabolas* is a finite collection of x-monotone curves defined over a common interval such that every two curves are either disjoint or intersect in two points where the curves cross. The following result gives a reversible mapping from cylindrical arrangements to arrangements of pseudoparabolas. The result has been announced by Bultena et al. [3, Lemma 1.3] and a full proof has been given by Agarwal et al. [1, Lemma 2.11].

▶ **Proposition 2.3** ([3, 1]). *A cylindrical arrangement of pseudocircles $\mathcal{C}$ can be mapped to an arrangement of pseudoparabolas $\mathcal{A}$ in an axis-aligned rectangle B such that $\mathcal{C}$ is isomorphic to the arrangement obtained by identifying the two vertical sides of B and mapping the resulting cylindrical surface homeomorphically to a ring in the plane.*

3 Flip Graphs on Pseudocircle Arrangements

We here sketch the flip-connectivity (Theorem 1.2 and Theorem 1.1).

3.1 Proof Sketch of Theorem 1.2: Connectivity Cylindrical

We prove the flip-connectivity for cylindrical arrangements by showing that any given arrangement can be flipped to a *canonical* arrangement which is depicted in Figure 4.

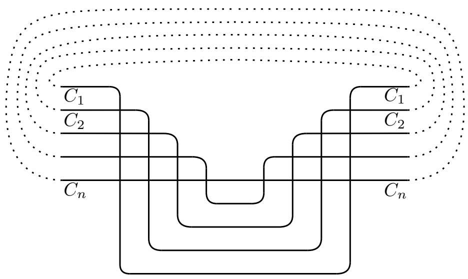

Figure 4 The canonical arrangement for cylindrical arrangements.

Let $\mathcal{A}$ be an intersecting, cylindrical arrangement of n pseudocircles. Using Proposition 2.3, we can represent $\mathcal{A}$ as a pseudoparabola arrangement, in which we label the curves from top to bottom by $C_1, \ldots, C_n$. The idea is to flip the pseudoparabolas downwards one by one in the order of increasing indices, as illustrated in Figure 5. The availability of suitable triangle flips follows from Lemma 2.1 and our construction to consider pseudoparabolas from top to bottom. This completes the proof sketch for Theorem 1.2.

3.2 Proof Sketch of Theorem 1.1: Connectivity Intersecting

Let $\mathcal{A}$ be an arrangement of n pairwise intersecting pseudocircles. We show by induction on n that $\mathcal{A}$ can be transformed into a cylindrical arrangement with a finite number of triangle flips. The flip-connectivity then follows from Theorem 1.2.

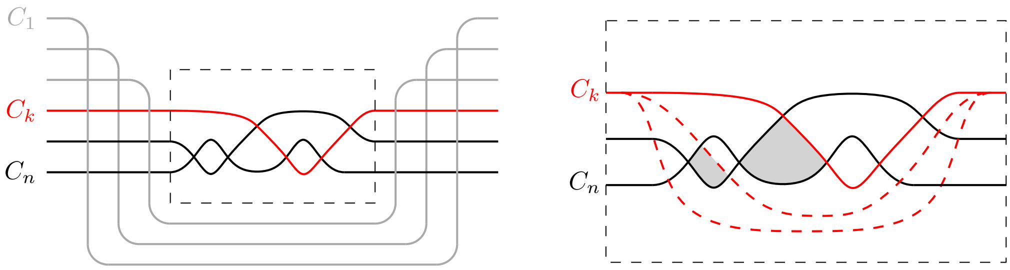

Figure 5 Illustration of the proof of Theorem 1.2: flipping the pseudoparabola C_k downwards.

The induction base is trivially fulfilled for $n = 2$. For the induction step, we choose a designated point p which lies inside a maximum number of pseudocircles. If p lies in the interior of all pseudocircles, then $\mathcal{A}$ is already cylindrical and we are done.

Hence, we may assume that there exists a pseudocircle C which does not contain p in its interior. We show how to expand C until containing p, using only triangle flips. First observe that Lemma 2.1 guarantees the existence of a flip to expand C. Since C already intersects all other pseudocircles, this must be a triangle or digon-collapse flip. As long as there exists a triangle flip expanding C, we perform it and transform $\mathcal{A}$ accordingly.

Suppose now that C does not yet contain p and can only be expanded by collapsing a digon formed with another pseudocircle C'. Then all remaining pseudocircles must intersect the lens $\text{int}(C) \cap \text{ext}(C')$ transversally (see Figure 6). Hence, using Lemma 2.2, we can expand C' until C and C' are parallel. We say that two pseudocircles C and C' are *parallel* in $\mathcal{A}$ if every vertex of $\mathcal{A} - \{C, C'\}$ lies in $(\text{int}(C) \cap \text{int}(C'))$ or in $(\text{ext}(C) \cap \text{ext}(C'))$.

Next consider the arrangement $\mathcal{A}' := \mathcal{A} - C'$ which is obtained by deleting C' from $\mathcal{A}$. By the induction hypothesis, $\mathcal{A}'$ can be transformed into a cylindrical arrangement by a finite sequence of triangle flips. We now carefully mimic this flip sequence on $\mathcal{A}$, while maintaining that C and C' are parallel. Suppose that a triangle T in $\mathcal{A}'$ is flipped. If none of the edges of T belongs to C, we can directly apply this triangle flip also in $\mathcal{A}$. If one of the edges e of T belongs to C and e is crossed by C' then the digon D is located along e. In this case, we apply two triangle flips to C' so that the digon is transferred to one of the two neighboring edges of C as illustrated in Figure 6, obtaining that e is not crossed by C' (without changing $\mathcal{A}'$). Finally, if e is not crossed by C' then we apply the according triangle flip twice, namely, once for C and once for C'. This completes the proof sketch.

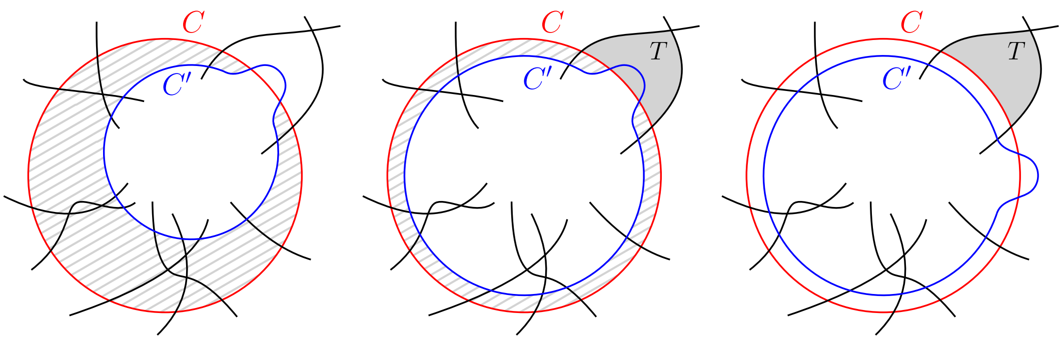

Figure 6 *Left:* C forms a digon with C' that is in $\text{ext}(C) \cap \text{int}(C')$. *Middle:* flip C' so that it becomes parallel to C. *Right:* flip C' so that T becomes also a triangle in $\mathcal{A}$.

4 Conclusion

While we have proven part (1) of Conjecture 1, part (2) remains a challenging open question. Also, several questions concerning the structure of the flip graphs remain open, such as Hamiltonicity or the degree of connectivity. The maximum degree Δ of the flip graph corresponds to the maximum number of triangles among all arrangements and $\Delta = \frac{4}{3}\binom{n}{2} + O(n)$ is known [6]. The minimum degree δ corresponds to the minimum number of triangles among all arrangements. For (not necessarily digon-free) intersecting arrangements $\frac{2n}{3} \leq \delta \leq n-1$ is known and $\delta = n-1$ is conjectured for $n \geq 3$. For digon-free intersecting arrangements $\delta = \max\{8, \lceil \frac{4n}{3} \rceil\}$ holds for $n \geq 3$ [4, 6].

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