Sometimes Two Irrational Guards are Needed

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Abstract

In the art gallery problem, we are given a closed polygon P, with rational coordinates and an integer k. We are asked whether it is possible to find a set (of guards) G of size k such that any point $p \in P$ is seen by a point in G. We say two points p, q see each other if the line segment pq is contained inside P. It was shown by Abrahamsen, Adamaszek, and Miltzow that there is a polygon that can be guarded with three guards, but requires four guards if the guards are required to have rational coordinates. In other words, an optimal solution of size three might need to be irrational. We show that an optimal solution of size two might need to be irrational. Note that it is well-known that any polygon that can be guarded with one guard has an optimal guard placement with rational coordinates.

Hence, our work closes the gap on when irrational guards are possible to occur.

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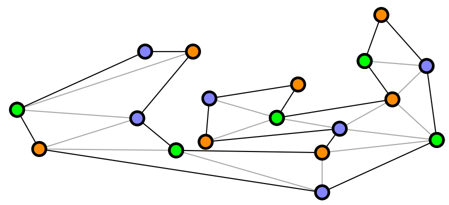

Figure 1 Any triangulation of a simple polygon can be three-colored. At least one of the color classes has at most $\lfloor n/3 \rfloor$ vertices. This color class also guards the entire polygon, as every triangle is incident to all three colors [7].

1 Introduction

In the art gallery problem, we are given a closed polygon P, on n vertices, with rational coordinates and an integer k. We are asked whether it is possible to find a set (of guards) G of size k such that any point $p \in P$ is seen by a point in G. We say two points p, q see each other if the line segment pq is contained inside P.

We show that an optimal solution of two guards might need to have irrational coordinates. In such a case, we say a polygon has *irrational guards*. Specifically, we construct a polygon that can be guarded by two irrational guards but requires three rational guards.

The art gallery problem was formulated in 1973 by Victor Klee. See, for example, the book by O'Rourke [8, page 2]. One of the earliest results states that every simple polygon on n vertices can always be guarded with $\lfloor n/3 \rfloor$ guards [4, 7].

Interestingly, it is actually very tough to find any positive algorithmic results on the art gallery problem. It seems like the art gallery problem is almost impenetrable. For instance, only in 2002, Micha Sharir pointed out that the problem was even decidable [5, 6, see acknowledgments]. The decidability of the art gallery problem is actually easy once you know methods from real algebraic geometry [3]. The idea is to reduce the problem to the first-order theory of the reals. We encode guard positions by variables, and then we check if every point in the polygon is seen by at least one guard. Note that this is easy to encode in the first-order theory of the reals, as we are allowed to use existential $(\exists g_1, g_2, \ldots)$ and universal quantifiers $(\forall p = (x, y))$. Since then, despite much research on the art gallery problem, no better algorithm appeared, as far as worst-case complexity is concerned. The underlying reason for the difficulty to find better algorithms can be explained by the fact that the art gallery problem is $\exists\mathbb{R}$-complete [9, 2]. In a nutshell, $\exists\mathbb{R}$-completeness precisely entails that there is no better method for the worst-case complexity of the problem. ($\exists\mathbb{R}$ can be defined as the class of problems that are equivalent to finding a real root to a multivariate polynomial with integer coordinates. See the full version for an introduction.) More specifically, it was shown that arbitrary algebraic numbers may be needed to describe an optimal solution to the art gallery problem. This may come as a surprise to some readers, and was clearly a surprise back then. Specifically, "in practice", it seems very rare that irrational guards are ever needed. The reason is that a typical situation is one of the following two. Either the guards have some freedom to move around and still see the entire polygon. Or if a guard has no freedom, it is forced to be on a line defined by vertices of the polygon. As the vertices of the polygon are at rational coordinates, the guards will be at rational coordinates in that case as well. Indeed, only in 2017, the first polygon requiring irrational guards was found [1]. Even though $\exists\mathbb{R}$-reductions exhibit an infinite number of polygons that require irrational guards, those polygons are not "concrete" in the naive sense of the word. And up to this day, this is the only "concrete" polygon [1] that we know does require irrational guards. In this work, we find a second polygon. It is superior to the first one in the sense that it shows that two guards are already enough to enforce irrational guards. As a single guard can always be chosen to have rational coordinates, we settle the question of the minimum number of guards required to have irrational guards. We summarize our results in the following theorem.

▶ **Theorem 1.1.** *There exists a polygon with rational coordinates, such that there is only one way of guarding this polygon optimally with two guards. Those two guards have irrational coordinates.*

Organization.

We provide background information in the full version. There we discuss our results from different angles, we give a selected overview of related research on the art gallery problem, and we add some background on the existential theory of the reals. In Section 2, we give an overview of how we constructed the polygon and what is the intuition behind the different parts. In Section 3, we give the polygon with coordinates of all vertices. Finally, in the full version we also provide a formal proof of correctness and explain how we constructed the polygon and what technical challenges we had to overcome.

2 Preparation

We aim to construct a polygon. This polygon should be guarded by two guards at irrational coordinates but requires three guards at rational coordinates. We must restrict the possible coordinates the guards can be positioned. In this section, we will explore the tools to restrict the possible positions of the two guards within the polygon.

2.1 Basic Definitions

Each guard g will be able to guard some region of the polygon: we call this region its *visibility polygon* $\text{vis}(g)$. The visibility polygon includes all points for which the line segment between the guard and the point is included in the polygon P. Notably, the union of the visibility polygons of the two guards must be the art gallery. Otherwise, the art gallery is not completely guarded.

A *window* is an edge of the visibility polygon $\text{vis}(g)$ that is not part of the boundary of P. We can find windows in the guard g's visibility polygon, by shooting rays from g to reflex vertices (the vertices of the polygon, with an interior angle larger than π). If these rays do not leave the polygon at the reflex vertex, a window will exist between the reflex vertex and the position where the ray does intersect the boundary of the polygon. Let the *window's end* be the intersection of the ray with an edge of the polygon.

Our final polygon consists of the core and a number of pockets, as shown in Figure 2. The *core* of the polygon is the square in the center. We will enforce that both guards are located in the core. As a square is a convex shape, this implies that both guards will guard the core. The *pockets* are all regions outside the core. We will use pockets that are either quadrilateral or triangular. Pockets are *attached* to either the core or another pocket: they have one edge that lies on the boundary of the core or on the boundary of another pocket. Quadrilateral pockets will always be attached to the core. Each quadrilateral pocket has one edge that is not on the boundary of the core, nor adjacent to it. We will call this edge the *wall* of a quadrilateral pocket. Similarly, triangular pockets will be attached to either the core or a quadrilateral pocket. We will use pockets as a tool to limit the locations of the two guards.

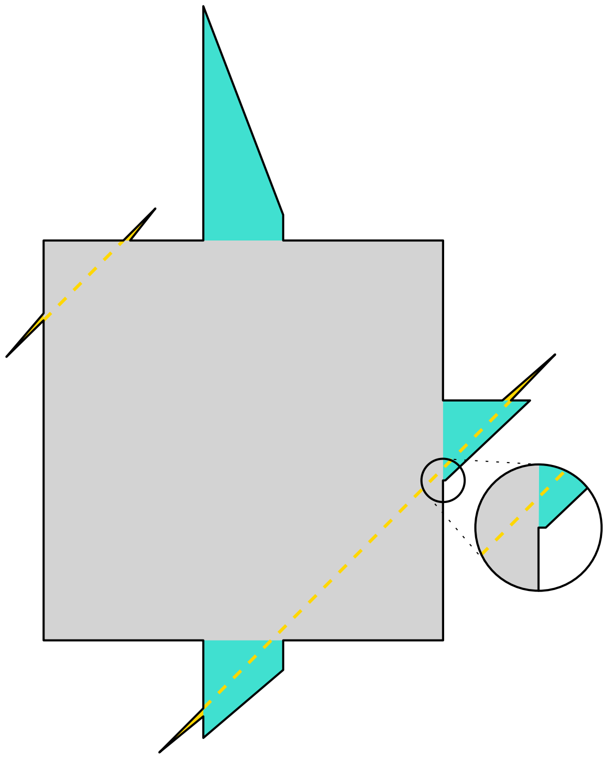

Figure 2 Our final polygon: it has a core (gray), three quadrilateral pockets (blue), and four narrow triangular pockets (yellow).

2.2 Guard Segments

We can force a guard to be positioned on a line segment within the polygon. Such a line segment is called a *guard segment*. Guard segments are commonly used in the context of the art gallery problem [1, 9]. In this section, we will describe how we construct a guard segment. We denote by s the segment and by ℓ its supporting line.

To make s a guard segment, we add two triangular pockets where ℓ intersects ∂P. Each of the triangular pockets has an edge on ℓ. Besides this one edge, the pockets lay on different sides of ℓ. Only a guard on the line segment between the two pockets can guard both triangular pockets at the same time.

We have two guards in our polygon and both will be on distinct guard segments. If the two guard segments are not intersecting, we can enforce that there must be one guard on each of them as follows. First, we introduce only four triangular pockets. Second, we make the triangular pockets sufficiently narrow. In this way, it is impossible to guard two of the triangular pockets outside of a guard segment. Thus at least one guard must be on each guard segment. A simple construction with two non-intersecting guard segments is shown in Figure 3.

2.3 Guarding Quadrilateral Pockets

We will now describe how given the position of guard l and a quadrilateral pocket Q will limit the position of guard t. See Figure 4 for an illustration of the following description. First, note that if l will not guard Q completely then there will remain some unguarded

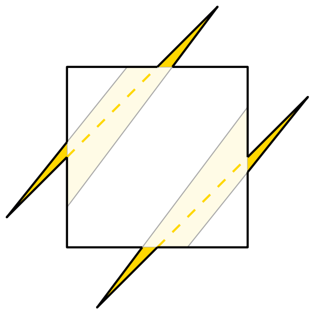

Figure 3 A small polygon that can only be guarded by two guards, because each guard segment (yellow dashed line) must contain a guard. The region where a guard could guard at least one pocket is shaded in light yellow.

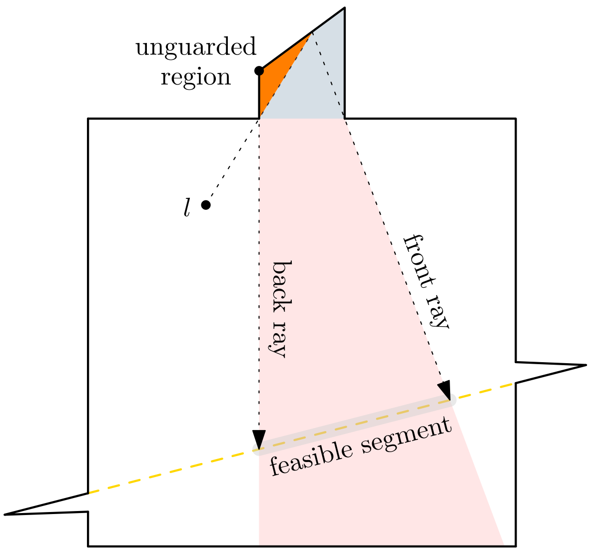

Figure 4 A polygon with guard l. The guard l defines an unguarded region in the quadrilateral pocket, a front ray and a back ray, and a feasible segment.

region (orange) in Q. The part of the guard segment of t where the unguarded region is visible is referred as the feasible segment. It is bounded from the back ray and the front ray. It is clear that t must be on the feasible segment.

We can compute the front ray by first computing the window end's s from l to the wall of Q and then shooting a ray from s in the direction of the second reflex vertex of Q.

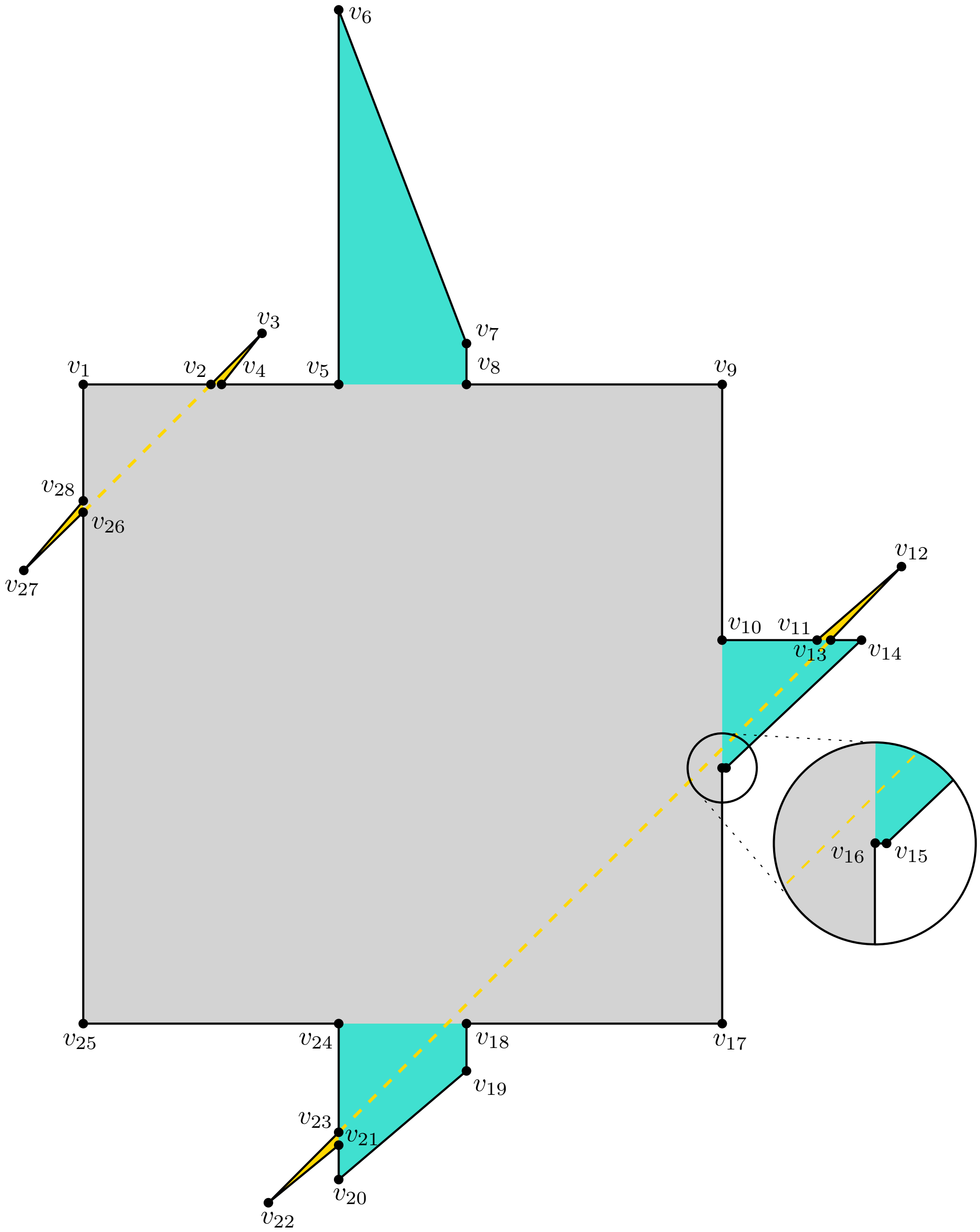

Figure 5 Our complete polygon. The art gallery is shaded according to the function of each region: gray is the core, yellow is the pockets used to create guard segments, and turquoise are other pockets. The yellow dashed lines represent the guard segments. The coordinates of important vertices are given.

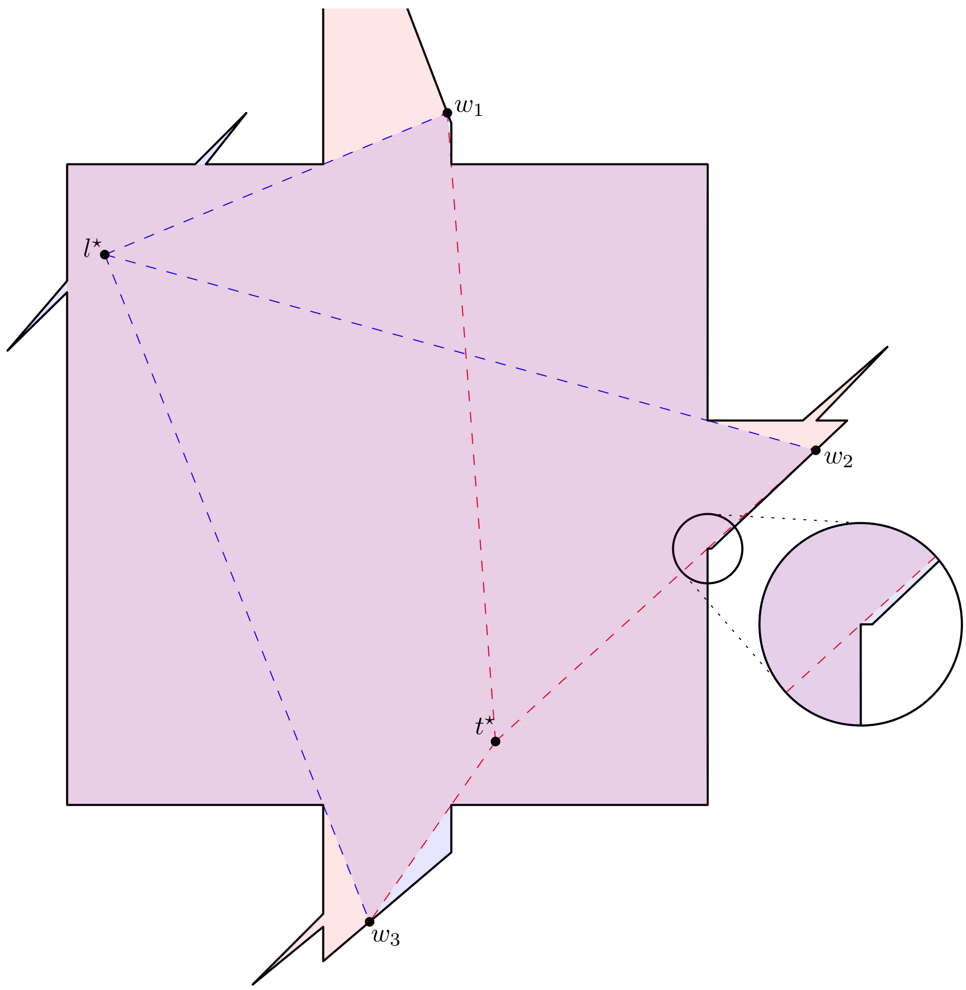

Figure 6 Our complete polygon. The optimal solution has two guards at irrational coordinates is shown. The light blue regions are guarded by the upper left guard; the light red regions are guarded by the bottom right guard; the purple (overlay of red and blue) regions are guarded by both. The dashed lines are rays shot from the guards through reflex vertices. For each pocket, these windows meet at a point on the art gallery's wall, of which the coordinates are also given.

Table 1 Coordinates of the vertices of the polygon ($v_1, \dots, v_{28}$), the guards (l and t), and the window's ends (w_1, w_2, w_3).

v_1	$(0, 10)$	v_{12}	$(12.7, 7)$	v_{23}	$(4, -1.7)$
v_2	$(2, 10)$	v_{13}	$(11.7, 6)$	v_{24}	$(4, 0)$
v_3	$(3, 11)$	v_{14}	$(\frac{1230422}{101007}, 6)$	v_{25}	$(0, 0)$
v_4	$(2.3, 10)$	v_{15}	$(\frac{1016072}{101007}, 4)$	v_{26}	$(0, 8)$
v_5	$(4, 10)$	v_{16}	$(10, 4)$	v_{27}	$(-1, 7)$
v_6	$(4, \frac{465522}{29357})$	v_{17}	$(10, 0)$	v_{28}	$(0, 8.3)$
v_7	$(6, \frac{312388}{29357})$	v_{18}	$(6, 0)$	l^*	$(3.7 - 2.2 \cdot \sqrt{2}, 11.7 - 2.2 \cdot \sqrt{2})$
v_8	$(6, 10)$	v_{19}	$(6, \frac{-25442}{34407})$	t^*	$(7.4 - 0.5 \cdot \sqrt{2}, 1.7 - 0.5 \cdot \sqrt{2})$
v_9	$(10, 10)$	v_{20}	$(4, \frac{-84128}{34407})$	w_1	$(\frac{293570 \cdot \sqrt{2} + 8052346}{1425913}, \frac{-765670 \cdot \sqrt{2} + 16485384}{1425913})$
v_{10}	$(10, 6)$	v_{21}	$(4, 2)$	w_2	$(\frac{1071750 \cdot \sqrt{2} + 29733818}{2673483}, \frac{1010070 \cdot \sqrt{2} + 13370606}{2673483})$
v_{11}	$(11.4, 6)$	v_{22}	$(3, -2.7)$	w_3	$(\frac{344070 \cdot \sqrt{2} + 3108526}{760803}, \frac{293430 \cdot \sqrt{2} + 1804526}{760803})$

3 Complete Polygon

In this section, we will present our complete polygon: a polygon that can be guarded by two guards if and only if both guards are situated at irrational points.

3.1 The Polygon

As we described in Section 2 and displayed in Figure 5, the polygon consists of a core and some pockets. The polygon has four triangular pockets defining two guard segments. The two guard segments lie on the lines $y = x + 8$ and $y = x - 5.7$. Furthermore, the polygon has three quadrilateral pockets. In Table 1, the coordinates of the vertices of the polygon, the coordinates of the two guards, and the coordinates of the window's ends are given.

The walls of the three quadrilateral pockets have the supporting lines:

1. Top pocket: $y = \frac{-76567 \cdot x + 771790}{29357}$.
2. Right pocket: $y = \frac{101007 \cdot x - 587372}{107175}$.
3. Bottom pocket: $y = \frac{29343 \cdot x - 201500}{34407}$.

In the full version, we prove that this polygon can be guarded by two guards, if and only if the guards are at irrational coordinates in and we discuss the difficulties we encountered while searching for this polygon.

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References

1 Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. Irrational guards are sometimes needed. In *SoCG 2017*, pages 3:1–3:15, 2017. Arxiv 1701.05475.

2 Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The Art Gallery Problem is $\exists\mathbb{R}$-complete. *Journal of the ACM*, 69(1):1–70, 2022. doi:10.1145/3486220.

3 Sauguta Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in Real Algebraic Geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer, Berlin, Heidelberg, 2006. `doi:10.1007/3-540-33099-2`.

4 Václav Chvátal. A combinatorial theorem in plane geometry. *Journal of Combinatorial Theory, Series B*, 18(1):39–41, 1975.

5 Alon Efrat and Sariel Har-Peled. Guarding galleries and terrains. In *IFIP 2002*, pages 181–192, 2002.

6 Alon Efrat and Sariel Har-Peled. Guarding galleries and terrains. *Inf. Process. Lett.*, 100(6):238–245, 2006. `doi:10.1016/j.ipl.2006.05.014`.

7 Steve Fisk. A short proof of Chvátal's watchman theorem. *J. Comb. Theory, Ser. B*, 24(3):374, 1978. URL: `http://dx.doi.org/10.1016/0095-8956(78)90059-X`, `doi:10.1016/0095-8956(78)90059-X`.

8 Joseph O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

9 Jack Stade. Complexity of the boundary-guarding art gallery problem. *arXiv preprint arXiv:2210.12817*, 2022.