Circling a Square: The Lawn Mowing Problem Is Algebraically Hard

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Abstract

For a given polygonal region P, the Lawn Mowing Problem (LMP) asks for a shortest tour T that gets within Euclidean distance 1 of every point in P; this is equivalent to computing a shortest tour for a unit-disk cutter D that covers all of P. We show that the LMP is algebraically hard: it is not solvable by radicals over the field of rationals, even for a simple case in which P is a 4×4 square. This implies that it is impossible to compute exact optimal solutions under models of computation that rely on elementary arithmetic operations and the extraction of kth roots.

1 Introduction

Long before the invention of computers, geometry already faced unsolvable algorithmic problems. This hardness was not rooted in the asymptotic complexity of finding the best of a finite number of candidates, but in the impossibility of obtaining solutions with a given set of construction tools: Computing the length of the diagonal of a square is not possible with only rational numbers; trisecting any given angle cannot be done with ruler and compass, nor can a square be computed whose area is equal to that of a given circle.

In the following, we consider the *Lawn Mowing Problem* (LMP), in which we are given a (not necessarily simple or even connected) polygonal region P and a disk cutter D of radius 1; the task is to find a closed roundtrip (a *tour*) of minimum Euclidean length, such that the cutter "mows" all of P, i.e., a shortest tour that moves the center of D within distance 1 from every point in P. The LMP naturally occurs in a wide spectrum of practical applications, such as robotics, manufacturing, farming, quality control, and image processing, so it is of both theoretical and practical importance. As a generalization of the classic Traveling Salesman Problem (TSP), the LMP is also NP-hard; however, while the TSP has shown to be amenable to exact methods for computing provably optimal solutions even for large instances, the LMP has defied such attempts, with only some moderate recent progress [20].

The main result of this paper is to establish a fundamental reason for this perceived difficulty: Computing an optimal lawn mowing tour is not only NP-hard, but also algebraically hard, even for the seemingly harmless case of mowing a 4×4 square by a unit-radius disk, as it requires computing zeroes of high-order irreducible polynomials. As a consequence, computing even near-optimal solutions for the LMP requires dealing with algebraic issues of numerical approximation and accuracy, making the LMP fundamentally more challenging than its special case, the discrete (Euclidean) TSP.

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Related Work There is a wide range of practical applications for the LMP, including manufacturing [3, 23, 24], cleaning [9], robotic coverage [10, 11, 22, 27], inspection [15], CAD [14], farming [4, 12, 30] and pest control [6]. In Computational Geometry, the Lawn Mowing Problem was first introduced by Arkin et al. [1], who later gave the currently best approximation algorithm with a performance guarantee of $2\sqrt{3}\alpha_{\mathrm{TSP}} \approx 3.46\alpha_{\mathrm{TSP}}$ [2], where α_{TSP} is the performance guarantee for an approximation algorithm for the TSP.

Optimally covering even relatively simple regions by a set of n unit disks has received considerable attention, but is excruciatingly difficult; see [7, 8, 21, 25, 28, 29]. As recently as 2005, Fejes Tóth [16] established optimal values for $n = 8, 9, 10$. Progress on covering by (not necessarily equal) disks has been achieved by Fekete et al. [17, 18].

A seminal result for understanding algebraic aspects of geometric optimization problems was achieved by Bajaj [5], who established algebraic hardness for the Fermat-Weber problem of finding a point in $\mathbb{R}^2$ that minimizes the sum of Euclidean distances to all points in a given set. Note, however, that the Fermat-Weber problem is relatively benign in practical difficulty, as it amounts to minimizing a smooth, convex function over a compact set, which can be achieved with high accuracy by using a numerical approach such as Newton's method. This was exploited for algorithmic purposes by Fekete et al. [19].

The use of straight-edge and compass is known to be equivalent to the use of $(+, -, *, /, \sqrt{})$ over $\mathbb{Q}$ [13]. Our main result implies that the Lawn Mowing Problem is not solvable by radicals over $\mathbb{Q}$, i.e., a solution is not expressible in terms of $(+, -, *, /, \sqrt[k]{})$ over $\mathbb{Q}$.

2 Optimal Tours in Rectangles

Recent work by Fekete et al. [20] shows that when mowing a triangle, optimal tours may need to contain vertices with irrational coordinates. In the following we show even if P is a 4×4 square, an optimal tour may involve coordinates that cannot be described with radicals.

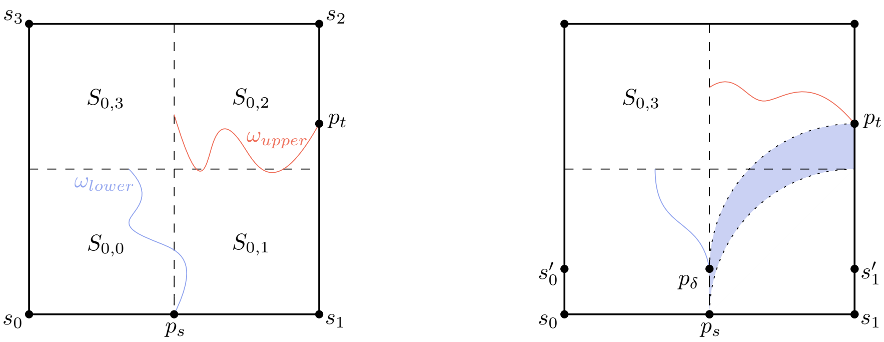
▶ **Theorem 2.1.** *For any rational height $h \geq 4$, there are rectangles P with height h and rational vertex coordinates for which the Lawn Mowing Problem is not solvable by radicals.*

See Figure 3 for the structure of optimal trajectories. A key observation is that covering each of the four corners $(0, -2), (4, -2), (4, 2), (0, 2)$ of a 4×4 square S requires the disk center to leave the subsquare λ with vertices $\lambda_0 = (1, -1)$, $\lambda_1 = (3, -1)$, $\lambda_2 = (3, 1)$, $\lambda_3 = (1, 1)$, obtained by offsetting the boundary of P by the unit radius of D, which is the locus of all disk centers for which λ stays inside P. However, covering the area close to the center of P also requires keeping the center of D within λ; as we argue in the following, this results in a trajectory as shown in Figure 3a, with a "long" portion (shown vertically in the figure) for which the disk covers the center of P and the boundary of D traces the boundary of P, and a "short" portion for which D only dips into λ without tracing the boundary of P.

We start our proof by considering an optimal lawn mowing tour for a rectangle and then argue why no solution can be obtained in terms of $(+, -, *, /, \sqrt[k]{})$ over $\mathbb{Q}$.

2.1 Properties of Optimal Tours

For the 4×4 square S, consider the upper left 2×2 subsquare S_0 with corners $(0, 0)$, $(0, 2)$, $(2, 2)$, $(0, 2)$, further subdivided into four 1×1 quadrants $S_{0,0}, \ldots, S_{0,3}$, as shown in Figure 1a, and an optimal path ω that enters S_0 at the bottom and leaves it to the right. Let $p_s = (p_s^x, 0), p_t = (2, p_t^y)$ be the points where ω enters and leaves S_0, respectively. For the following lemmas, we assume that a covering path exists that obeys the above conditions. We will later determine that path and show that it covers S_0.



(a) The upper left 2×2 subsquare S_0 of S. **(b)** Visualization of Lemma 2.3.

Figure 1 Computing an optimal path ω through the square S_0.

▶ **Lemma 2.2.** *$p_s^x \leq 1$ and $p_t^y \geq 1$ and either $p_s^x = 1$ or $p_t^y = 1$.*

Proof. To cover s_1, ω must intersect a unit circle centered in s_1. Any path with p_s right of $(1, 0)$ or p_t below $(2, 1)$ can be made shorter by shifting the point p_s to $(1, 0)$ or p_t to $(2, 1)$. Any path with p_s left of $(1, 0)$ and p_t above $(2, 1)$ must enter $S_{0,1}$, resulting in a detour. ◀

Without loss of generality, we assume that $p_s^x = 1$. The next step is to find the optimal position of p_t. As an optimal path ω must enter the quadrant $S_{0,3}$ once, we can subdivide the path into two parts. We denote the part from p_s to $S_{0,3}$ as the *lower portion* and from $S_{0,3}$ to p_t as the *upper portion* of ω. For some $\delta > 0$, let $p_t^y = 1 + \delta$ and $p_\delta = (1, \delta)$.

▶ **Lemma 2.3.** *For any $\delta > 0$, ω has a subpath $p_s p_\delta$.*

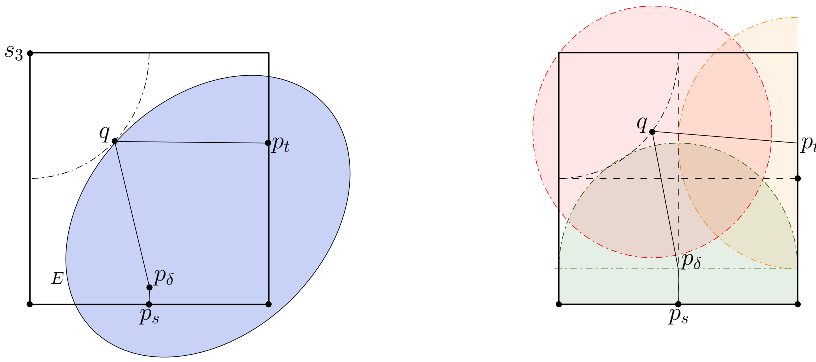
Proof. Let $s'_1 = (2, \delta)$ and $\varepsilon = s_1 s'_1$. Segment ε must be covered by ω. We distinguish two cases; (i) ε is covered by the lower portion of ω or (ii) ε is covered by the upper portion of ω. For case (i), let us assume that ε is covered by the lower portion of ω. When ω would enter $S_{0,1}$ it would also have to enter $S_{0,0}$ to cover the left side of $S_{0,0}$. It is clear that traversing the segment $p_s p_\delta$ of length δ is the best way to cover the lower portion of $S_{0,0}, S_{0,1}$, as any other path would need additional segments in x-direction, see Figure 1b. Any path that obeys case (ii) is suboptimal, as it has to cover ε from within $S_{0,2}$, for a detour of at least 2δ. ◀

We can now define the optimal path ω, which has four vertices. The exact coordinates are defined in the proof of Lemma 2.4.

▶ **Lemma 2.4.** *The unique optimal lawn mowing path between two adjacent sides of S_0 is $\omega = (p_s, p_\delta, q, p_t)$ and has length $L_{S_0} \approx 2.618$.*

Proof. We now identify a shortest path for visiting one point q in the unit circle C centered in s_3 dependent on δ, which is a necessary condition for a feasible path. Let $c = d(p_\delta, q) + d(q, p_t)$ be the distance from both points to C. Consider an ellipse E with foci p_δ, p_t which touches C in a single point, see Figure 2a. By the definition of an ellipse, the intersection point q minimizes the distance c. For any $\delta \in [0, 1]$ we want to find a minimum distance c that allows E to have a single intersection point with C. Let $p_c = (p_c^x, p_c^y)$ be the center point of E and d_E be the distance from the center point of E and a, b the major/minor axis.

$$p_c^x = \frac{3}{2} \quad p_c^y = \frac{1}{2} + \delta \quad d_E = d(p_\delta, p_c) = \frac{1}{\sqrt{2}} \quad a = \frac{1}{2} d_E \quad b = \sqrt{a^2 - d_E^2} \tag{1}$$



(a) Any $0 \leq \delta \leq 1$ defines p_δ, p_t, q and ellipse E. **(b)** The optimal path ω through S_0.

Figure 2 Visualizations for Lemma 2.4.

The ellipse can now be defined with its center point p_c, the major/minor axis a, b and the angle θ, which is the angle between a line through p_δ, p_t and the x-axis. We formulate the shortest path problem as a minimization problem while inserting Equation (1).

$$\begin{aligned}
\min \quad & c + \delta \\
\text{s.t.} \quad & x^2 + (y-2)^2 - 1 && = 0 \\
& \frac{((x - p_c^x)\cos(\theta) + (y - p_c^y)\sin(\theta))^2}{a^2} + \frac{((x - p_c^x)\sin(\theta) - (y - p_c^y)\cos(\theta))^2}{b^2} && = 1 \\
& \sqrt{(x-1)^2 + (y-\delta)^2} + \sqrt{(x-2)^2 + (y-1-\delta)^2} - c && = 0
\end{aligned}$$

The objective minimizes the total length of the path ω with variables that encode the exact coordinates of p_δ, q, p_t. An intersection point of E and C with center $s_3 = (0, 2)$ is a solution to the first and second constraints, respectively. An exact optimization approach using Mathematica reveals that δ, q^x, q^y can only be expressed as the first, third, and first roots of three irreducible high-degree polynomials $f_\delta, f_{q^x}, f_{q^y}$, see Equations (2) to (4).

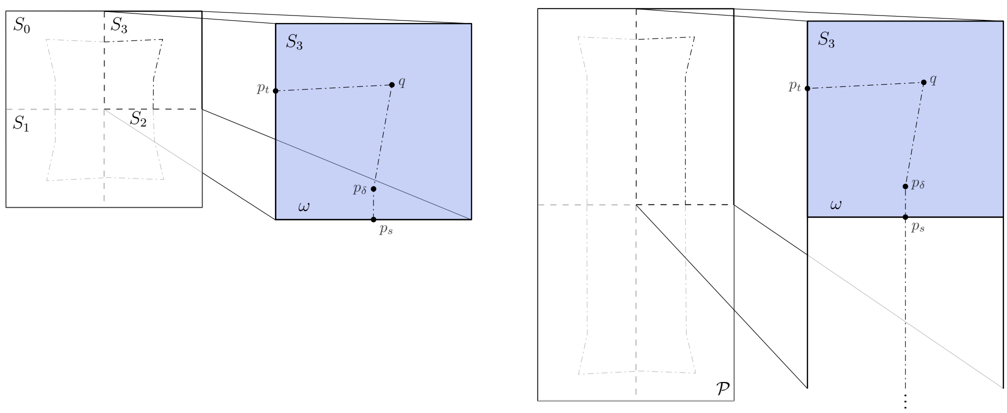
$$\begin{aligned}
f_\delta(x) =& 9x^{16} - 216x^{15} + 2514x^{14} - 18\,846x^{13} + 101\,755x^{12} - 418\,512x^{11} + \\
& 1\,350\,994x^{10} - 3\,475\,302x^9 + 7\,165\,772x^8 - 11\,828\,976x^7 + 15\,512\,224x^6 - \\
& 15\,916\,002x^5 + 12\,459\,638x^4 - 7\,145\,094x^3 + 2\,800\,022x^2 - 656\,964x + 67\,417
\end{aligned} \tag{2}$$

$$\begin{aligned}
f_{q^x}(x) =& 256x^{16} - 1792x^{14} + 5312x^{12} - 8768x^{10} + 384x^9 + 8544x^8 - 1632x^7 - \\
& 3648x^6 + 1200x^5 - 152x^4 + 288x^3 + 252x^2 - 324x + 81
\end{aligned} \tag{3}$$

$$\begin{aligned}
f_{q^y}(x) =& 256x^{16} - 8192x^{15} + 122\,624x^{14} - 1\,139\,712x^{13} + 7\,361\,472x^{12} - \\
& 35\,034\,880x^{11} + 127\,069\,376x^{10} - 358\,188\,736x^9 + 792\,777\,952x^8 - \\
& 1\,381\,642\,752x^7 + 1\,888\,549\,824x^6 - 2\,001\,789\,968x^5 + 1\,611\,461\,512x^4 - \\
& 951\,341\,552x^3 + 387\,921\,820x^2 - 97\,469\,232x + 11\,350\,269
\end{aligned} \tag{4}$$

The value for $\delta \approx 0.335752$ defines the points p_δ and p_t. Together with the values for q^x, q^y, we can define all points in ω as follows:

$$p_s = (1, 0) \quad p_\delta = (1, \delta) \approx (1, 0.336) \quad q \approx (0.772, 1.365) \quad p_t = (2, 1 + \delta) \approx (2, 1.336) \tag{5}$$



(a) Optimal lawn mowing tour for a 4×4 square. **(b)** Optimal lawn mowing tour for a rectangle.

Figure 3 Optimal lawn mowing tours for a 4×4 square and a $4 \times h$ rectangle.

The combined length of the path is $\delta + c \approx 2.617676448$. As ω contains a subpath that crosses the full height of $S_{0,0}$ and another subpath that crosses the full width of $S_{0,2}$, both quadrants are covered by ω, see Figure 2b. By construction, the bottom right quadrant is covered by the segment $p_s p_\delta$ and the point p_t. The top left quadrant is covered by q, because $S_{0,3}$ is fully contained in a unit disk centered in q. Therefore, ω is a feasible path between two adjacent edges of S_0 with a length of $L \approx 2.618$. ◀

▶ **Lemma 2.5.** *A square P of side length 4 has a unique optimal lawn mowing tour T of length $L = 4L_{S_0}$, where $L_{S_0} \approx 2.618$.*

Proof. We start by subdividing P by its vertical and horizontal center line into four quadrants (squares) $S_0, \dots, S_3$ with side length 2. To cover the center point of each quadrant, a lawn mowing tour has to intersect it at least once. As P is convex, T cannot leave P at any point. Finally, T is symmetric with respect to the vertical and horizontal lines because otherwise, the quadrant subpaths could be replaced by the shortest one. By Lemma 2.4, there is a unique optimal lawn mowing path through each quadrant yielding an optimal tour of length $L = 4L_{S_0} \approx 4 \cdot 2.618 \approx 10.472$, see Figure 3a. ◀

The optimal path from Lemma 2.4 can be used more extensively on rectangles with fixed width 4 and arbitrary height $h \geq 4$. For this, we extend the path from p_s outwards perpendicular to the 2×2 square S_0. One can use a similar construction as in Lemma 2.5 to obtain optimal tours for arbitrary rectangles, refer to Figure 3b.

▶ **Corollary 2.6.** *Any rectangle P with width 4 and height $h \geq 4$ has a unique optimal lawn mowing tour T of length $L = 4L_{S_0} + 2h - 8$.*

2.2 Algebraic Hardness of the LMP

As our next step, we show that the coordinates of the optimal path ω are not solvable by radicals. For this, we employ a similar technique as Bajaj [5] for the generalized Weber problem. A *field* K is said to be an *extension* (written as $K/\mathbb{Q}$) of $\mathbb{Q}$ if K contains $\mathbb{Q}$. Given a polynomial $f(x) \in \mathbb{Q}[x]$, a finite extension K of $\mathbb{Q}$ is a *splitting field* over $\mathbb{Q}$ for $f(x)$ if it can be factorized into linear polynomials $f(x) = (x - a_1) \cdots (x - a_k) \in K[x]$ but not over any proper subfield of K. Alternatively, K is a splitting field of $f(x)$ of degree n over $\mathbb{Q}$

if K is a minimal extension of $\mathbb{Q}$ in which $f(x)$ has n roots. Then the *Galois group* of the polynomial f is defined as the Galois group of $K/\mathbb{Q}$. In principle, the Galois group is a certain permutation group of the roots of the polynomial. From the fundamental theorem of Galois theory, one can derive a condition for solvability by radicals of the roots of $f(x)$ in terms of algebraic properties of its Galois group. We now state three additional theorems from Galois theory and Bajaj's work. The proofs can be found in [26, 5].

▶ **Lemma 2.7** ([26]). *$f(x) \in \mathbb{Q}[x]$ is solvable by radicals over $\mathbb{Q}$ iff the Galois group over $\mathbb{Q}$ of $f(x)$, $Gal(f(x))$, is a solvable group.*

▶ **Lemma 2.8** ([26]). *The symmetric group S_n is not solvable for $n \geq 5$.*

▶ **Lemma 2.9** ([5]). *If $n \equiv 0 \mod 2$ and $n > 2$ then the occurrence of an $(n-1)$-cycle, an n-cycle, and a permutation of type $2 + (n-3)$ on factoring the polynomial $f(x)$ modulo primes that do not divide the discriminant of $f(x)$ establishes that $Gal(f(x))$ over $\mathbb{Q}$ is the symmetric group S_n.*

▶ **Theorem 2.1.** *For any rational height $h \geq 4$, there are rectangles P with height h and rational vertex coordinates for which the Lawn Mowing Problem is not solvable by radicals.*

Proof. It suffices to show that f_δ is not solvable by the radicals as it describes the y-coordinates of two points in the solution. We provide three factorizations of f_δ modulo three primes that do not divide the discriminant $disc(f_\delta(x))$.

$$\begin{aligned}
f_\delta(x) \equiv\, & 9(x^{16} + 22x^{15} + 11x^{14} + 22x^{13} + 8x^{12} + 20x^{11} + 15x^{10} + 10x^9 + 11x^8 + 12x^7 + \\
& 9x^6 + 10x^5 + x^4 + 7x^3 + 13x^2 + 6x + 3) \mod 23 \\
f_\delta(x) \equiv\, & 9(x + 41)(x^2 + 21x + 15)(x^{13} + 8x^{12} + 4x^{11} + 46x^{10} + 4x^9 + 14x^8 + 32x^7 + 14x^5 + \\
& 31x^4 + 41x^3 + 37x^2 + 32x + 41) \mod 47 \\
f_\delta(x) \equiv\, & 9(x + 19)(x^{15} + 16x^{14} + 54x^{13} + 7x^{12} + 9x^{11} + 36x^{10} + 45x^9 + x^8 + 45x^7 + 3x^6 + \\
& 22x^5 + 36x^4 + 26x^3 + 22x^2 + 54x + 23) \mod 59
\end{aligned}$$

For the good primes $p = 23, 47$, and 59 the degrees of the irreducible factors of $f_\delta(x)$ mod p gives us an $16 - cycle$, a $2 + 13$ permutation and a 15-cycle, which is enough to show with Lemma 2.9 and $n = 16$ that $Gal(f_\delta) = S_{16}$. By Lemma 2.8, S_{16} is not solvable; with Lemma 2.7, this proves the theorem. ◀

3 Conclusion

We have shown that the Lawn Mowing Problem is algebraically hard, even when mowing a 4×4 square P by a unit-radius cutter D. This implies that computing provably optimal tours (such as for the TSP) would involve complicated coordinates; even good approximations (such as a PTAS) require good numerical approximations of the involved algebraic terms.

While our proof makes intricate use of the underlying structure of optimal tours, it is conceivable that similar techniques may help to better understand the difficulty of other excruciatingly hard optimization problems, such as disk packing or covering.

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