The Number of Edges in Maximal 2-planar Graphs[*]

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──── **Abstract** ────────────────────────────────

A graph is 2-*planar* if it has local crossing number two, that is, it can be drawn in the plane such that every edge has at most two crossings. A graph is *maximal* 2-*planar* if no edge can be added such that the resulting graph remains 2-planar. A 2-planar graph on n vertices has at most $5n - 10$ edges, and some (maximal) 2-planar graphs—referred to as *optimal* 2-*planar*—achieve this bound. However, in strong contrast to maximal planar graphs, a maximal 2-planar graph may have fewer than the maximum possible number of edges. In this paper, we determine the minimum edge density of maximal 2-planar graphs by proving that every maximal 2-planar graph on $n \geq 5$ vertices has at least $2n$ edges. We also show that this bound is tight, up to an additive constant.

1 Introduction

Maximal planar graphs a.k.a. (combinatorial) triangulations are a rather important and well-studied class of graphs with a number of nice and useful properties. To begin with, the number of edges is uniquely determined by the number of vertices, as every maximal planar graph on $n \geq 3$ vertices has $3n - 6$ edges. It is natural to wonder if a similar statement can be made for the various families of near-planar graphs, which have received considerable attention over the past decade; see, e.g. [7, 8].

In this paper we focus on k-planar graphs, specifically for $k = 2$. These are graphs with local crossing number at most k, that is, they admit a drawing in $\mathbb{R}^2$ where every edge has at most k crossings. The maximum number of edges in a k-planar graph on n vertices increases with k, but the exact dependency is not known. A general upper bound of $O(\sqrt{k}n)$ is known due to Ackerman and Pach and Tóth [1, 11] for graphs that admit a *simple* k-plane drawing, that is, a drawing where every pair of edges has at most one common point. A 1-planar graph on n vertices has at most $4n - 8$ edges and there are infinitely many *optimal* 1-planar graphs that achieve this bound, as shown by Bodendiek, Schumacher, and Wagner [5]. A 2-planar graph on n vertices has at most $5n - 10$ edges and there are infinitely many *optimal* 2-planar graphs that achieve this bound, as shown by Pach and Tóth [11]. In fact, there are complete characterizations, for optimal 1-planar graphs by Suzuki [13] and for optimal 2-planar graphs by Bekos, Kaufmann, and Raftopoulou [4].

Much less is known about *maximal* k-planar graphs, that is, graphs for which adding any edge results in a graph that is not k-planar anymore. In contrast to planar graphs, where maximal and optimal coincide, the difference between maximal and optimal can be quite large for k-planar graphs, even—perhaps counterintuitively—maximal k-planar graphs for $k \geq 1$ may have fewer edges than maximal planar graphs on the same number of vertices. Hudák, Madaras, and Suzuki [9] describe an infinite family of maximal 1-planar graphs with only $8n/3 + O(1) \approx 2.667n$ edges. An improved construction with $45n/17 + O(1) \approx 2.647n$ edges was given by Brandenburg, Eppstein, Gleißner, Goodrich, Hanauer, and Reislhuber [6]

who also established a lower bound by showing that every maximal 1-planar graph has at least $28n/13 - O(1) \approx 2.153n$ edges. Later, this lower bound was improved to $20n/9 \approx 2.22n$ by Barát and Tóth [3].

Maximal 2-planar graphs were studied by Auer, Brandenburg, Gleißner, and Hanauer [2] who constructed an infinite family of maximal 2-planar graphs with n vertices and $387n/147 + O(1) \approx 2.63n$ edges.[1] We are not aware of any nontrivial lower bounds on the number of edges in maximal k-planar graphs, for $k \geq 2$.

Results.   In this paper, we give tight bounds on the number of edges in maximal 2-planar graphs, up to an additive constant.

▶ **Theorem 1.** *Every maximal 2-planar graph on $n \geq 5$ vertices has at least $2n$ edges.*

▶ **Theorem 2.** *There exists a constant $c \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists a maximal 2-planar graph on n vertices with at most $2n + c$ edges.*

2   Preliminaries

A drawing is *simple* if every pair of edges has at most one common point. A drawing is *k-plane*, for $k \in \mathbb{N}$, if every edge has at most k crossings. A graph is *k-planar* if it admits a k-plane drawing. A graph is *maximal k-planar* if no edge can be added to it so that the resulting graph is still k-planar.

To analyze a k-planar graph one often analyzes one of its k-plane drawings. It is, therefore, useful to impose additional restrictions on this drawing if possible. One such restriction is to consider a *crossing-minimal k-plane drawing*, that is, a drawing that minimizes the total number of edge crossings among all k-plane drawings of the graph. For small k, such a drawing is always simple; for $k \geq 4$ this is not the case in general [12, Footnote 112].

▶ **Lemma 3** (Pach, Radoičić, Tardos, and Tóth [10, Lemma 1.1]). *For $k \leq 3$, every crossing-minimal k-plane drawing is simple.*

In figures, we use the following convention to depict edges: Uncrossed edges are shown green, singly crossed edges are shown purple, doubly crossed edges are shown blue, and edges for which the number of crossings is undetermined are shown black.

3   The Lower Bound

In this section we briefly describe our lower bound on the edge density of maximal 2-planar graphs by analyzing the distribution of vertex degrees. As we aim for a lower bound of $2n$ edges, we want to show that the average vertex degree is at least four. Then, the density bound follows by the handshaking lemma. However, maximal 2-planar graphs may contain vertices of degree less than four. By the following property (whose proof is deferred to the full version), we know that the degree of every vertex is at least two. But degree two vertices, so-called *hermits*, may exist, as well as vertices of degree three.

▶ **Lemma 4.** *For $k \leq 2$, every maximal k-planar graph on $n \geq 3$ vertices is 2-connected.*

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[1] Maximality is proven via uniqueness of the 2-plane drawing of the graph. However, there is no explicit proof of the uniqueness in this short abstract.

In order to lower bound the average degree by four, we employ a charging scheme where we argue that every *low-degree* vertex, that is, every vertex of degree two and three claims a certain number of halfedges at an adjacent *high-degree* vertex, that is, a vertex of degree at least five. Claims are exclusive, that is, every halfedge at a high-degree vertex can be claimed at most once. We use the term *halfedge* because the claim is not on the whole edge but rather on its incidence to one of its high-degree endpoints. The incidence at the other endpoint may or may not be claimed independently (by another vertex). For an edge uv we denote by $\overrightarrow{uv}$ the corresponding halfedge at v and by $\overrightarrow{vu}$ the corresponding halfedge at u. Vertices of degree four have a special role, as they are neither low– nor high-degree. However, a vertex of degree four that is adjacent to a hermit is treated like a low-degree vertex. More precisely, our charging scheme works as follows:

(C1) Every hermit claims two halfedges at each high-degree neighbor.
(C2) Every degree three vertex claims three halfedges at some high-degree neighbor.
(C3) Every degree four vertex that is adjacent to a hermit h claims two halfedges at some neighbor v of degree ≥ 6. Further, the vertices h and v are adjacent, so h also claims two halfedges at v by (C1). If $\deg(v) = 6$, then v is adjacent to exactly one hermit.
(C4) At most one vertex claims (one or more) halfedges at a degree five vertex.

We state some useful properties of low-degree vertices. Then we present the proof of Theorem 1 in Section 3.3. The validity of our charging scheme is deferred to the full version.

3.1 Hermits and degree four vertices

▶ **Lemma 5.** *Let h be a hermit and let x, y be its neighbors in G. Then x and y are adjacent in G and all three edges xy, hx, hy are uncrossed in D. Further, $\deg(x) \geq 4$ and $\deg(y) \geq 4$.*

We refer to the edge xy as the *base* of the hermit h, which *hosts* h.

▶ **Lemma 6.** *Let G be a maximal 2-planar graph on $n \geq 5$ vertices. Every edge of G hosts at most one hermit. Further, a vertex of degree i in G is adjacent to at most $\lfloor i/3 \rfloor$ hermits.*

By Lemma 5, both neighbors of a hermit have degree at least four. A vertex is of type *T4-H* if it has degree four and it is adjacent to a hermit. The following lemma characterizes these vertices and ensures that every hermit has at least one high-degree neighbor.

▶ **Lemma 7.** *Let u be a T4-H vertex with neighbors h, v, w, x in G such that h is a hermit and v is the second neighbor of h. Then both uw and ux are doubly crossed in D, and the two faces of $D \setminus h$ incident to uv are triangles that are bounded by (parts of) edges incident to u and doubly crossed edges incident to v. Furthermore, we have $\deg(v) \geq 6$, and if $\deg(v) = 6$, then h is the only hermit adjacent to v in G.*

In our charging scheme, each hermit h claims two halfedges at each high-degree neighbor v: the halfedge $\overrightarrow{hv}$ and the halfedge $\overrightarrow{uv}$, where uv denotes the edge that hosts h. Each T4-H vertex u claims the two doubly crossed halfedges at v that bound the triangular faces incident to uv in D.
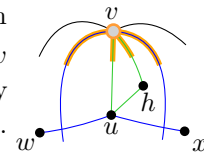
3.2 Degree three vertices

We distinguish four different types of degree three vertices in G, depending on their neighborhood and on the crossings on their incident edges in D. Consider a degree three vertex u in G. Every vertex is incident to at least one uncrossed edge in D (the proof is deferred to the full version).

T3-1: exactly one uncrossed edge. The two other edges incident to u are crossed.

▶ **Lemma 8.** *Let u be a T3-1 vertex with neighbors v, w, x in G such that the edge uv is uncrossed in D. Then the two faces of D incident to uv are triangles that are bounded by (parts of) edges incident to u and doubly crossed edges incident to v. Furthermore, we have $\deg(v) \geq 5$.*

In our charging scheme, each T3-1 vertex u claims three halfedges at its adjacent high-degree vertex v: the uncrossed halfedge $\overrightarrow{uv}$ along with the two neighboring halfedges at v, which are doubly crossed by Lemma 8.
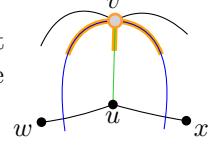
T3-2: exactly two uncrossed edges. The third edge incident to u is crossed.

▶ **Lemma 9.** *Let u be a T3-2 vertex with neighbors v, w, x s.t. the edge uv is crossed. Then uv is singly crossed by a doubly crossed edge wb in D, $\deg(w) \geq 5$ and $\min\{\deg(v), \deg(x)\} \geq 4$.*

A halfedge $\overrightarrow{wx}$ is *peripheral* for a vertex u of G if (1) u is a common neighbor of w and x; (2) $\deg(w) \geq 5$; and (3) $\deg(x) \geq 4$. In our charging scheme, every T3-2 vertex u claims three halfedges at the adjacent high-degree vertex w: the halfedge $\overrightarrow{uw}$, the doubly crossed halfedge $\overrightarrow{bw}$, and one of the uncrossed peripheral halfedges $\overrightarrow{vw}$ or $\overrightarrow{xw}$. While the former two are closely tied to u, the situation is more complicated for the latter two halfedges. Eventually, we argue that u can exclusively claim (at least) one of the two peripheral halfedges. But for the time being we say that it *assesses* both of them and these edges are depicted in lightblue.
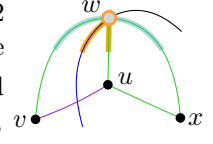
T3-3: all three incident edges uncrossed. We say that such a vertex is of type T3-3. As an immediate consequence of Lemma **??** each T3-3 vertex u together with its neighbors N(u) induces a plane K_4 in D. We further distiguish two subtypes of T3-3 vertices.

The first subtype accounts for the fact that there may be two adjacent T3-3 vertices in D. We refer to such a pair as an *inefficient hermit* and a T3-3 vertex that is part of an inefficient hermit is called a *T3-3 hermit*. T3-3 hermits behave similar to hermits, we defer the details of T3-3 hermits to the full version. The second subtype is formed by those T3-3 vertices that are not T3-3 hermits; we call them *T3-3 minglers*. All neighbors of a T3-3 mingler have degree at least four.

▶ **Lemma 10.** *Let u be a T3-3 mingler in D, and let v, w, x be its neighbors. Then each of v, w, x has degree at least four. Further, at least one vertex among v, w, x has degree at least six, or at least two vertices among v, w, x have degree at least five.*

Let Q denote the plane K_4 induced by u, v, w, x in D. The T3-3 mingler u claims the three halfedges of Q at one of its high-degree neighbors. That is, the vertex u assesses all of its peripheral halfedges at high-degree neighbors.
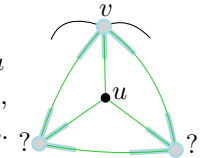
3.3   Proof of Theorem 1

Let G be a maximal 2-planar graph on $n \geq 5$ vertices, and let m denote the number of edges in G. We denote by v_i the number of vertices of degree i in G. By Lemma 4 we know that G

is 2-connected and, therefore, we have $v_0 = v_1 = 0$. Thus, we have

$$n = \sum_{i=2}^{n-1} v_i \text{ and by the Handshaking Lemma } 2m = \sum_{i=2}^{n-1} i \cdot v_i. \quad (1)$$

Vertices of degree four or higher can be adjacent to hermits. Let $v_i^{\mathrm{h}j}$ denote the number of vertices of degree i incident to j hermits in G. By Lemma 6 we have

$$v_i = \sum_{j=0}^{\lfloor i/3 \rfloor} v_i^{\mathrm{h}j} \qquad \text{for all } i \geq 3. \quad (2)$$

By Lemma 5 both neighbors of a hermit have degree at least four. Thus, double counting the edges between hermits and their neighbors we obtain

$$2v_2 \leq v_4^{\mathrm{h}1} + v_5^{\mathrm{h}1} + v_6^{\mathrm{h}1} + 2v_6^{\mathrm{h}2} + v_7^{\mathrm{h}1} + 2v_7^{\mathrm{h}2} + 2v_8 + v_9^{\mathrm{h}1} + 2v_9^{\mathrm{h}2} + 3v_9^{\mathrm{h}3} + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \quad (3)$$

If a vertex u claims halfedges at a vertex v, we say that v *serves* u. According to (C2), every vertex of degree three claims three halfedges at a high-degree neighbor. Every degree four vertex that is adjacent to a hermit together with this hermit claims four halfedges at a high-degree neighbor by (C3). We sum up the number of these claims and assess how many of them can be served by the different types of high-degree vertices.

In general, a high-degree vertex of degree $i \geq 5$ can serve at most $\lfloor i/3 \rfloor$ such claims. For $i \in \{5, 6, 7, 9\}$, we make a more detailed analysis, taking into account the number of adjacent hermits. Specifically, by (C3) and (C4) a degree five vertex serves at most one low-degree vertex, which is either a hermit or a degree three vertex. A degree six vertex can serve two degree three vertices but only if it is not adjacent to a hermit. If a degree six vertex serves a degree four vertex, it is adjacent to exactly one hermit by (C3). In particular, a degree six vertex that is adjacent to two hermits does not serve any degree three or degree four vertex. Altogether we obtain the following inequality:

$$v_3 + v_4^{\mathrm{h}1} \leq v_5^{\mathrm{h}0} + 2v_6^{\mathrm{h}0} + v_6^{\mathrm{h}1} + 2v_7^{\mathrm{h}0} + 2v_7^{\mathrm{h}1} + v_7^{\mathrm{h}2} + 2v_8 + 3v_9^{\mathrm{h}0} + 2v_9^{\mathrm{h}1} + 2v_9^{\mathrm{h}2} + v_9^{\mathrm{h}3} + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \quad (4)$$

The combination $((3) + (4))/2$ together with (2) yields

$$v_2 + \frac{1}{2}v_3 \leq \frac{1}{2}v_5 + v_6 + \frac{3}{2}v_7 + 2v_8 + 2v_9 + \sum_{i=10}^{n-1} \lfloor i/3 \rfloor v_i. \quad (5)$$

Now, using these equations and inequalities, we can prove that $m - 2n \geq 0$, to complete the proof of Theorem 1. Let us start from the left hand side, using (1).

$$
\begin{aligned}
m - 2n &= \frac{1}{2} \sum_{i=2}^{n-1} i v_i - 2 \sum_{i=2}^{n-1} v_i = \sum_{i=2}^{n-1} \frac{i-4}{2} v_i \\
&= -v_2 - \frac{1}{2}v_3 + \frac{1}{2}v_5 + v_6 + \frac{3}{2}v_7 + 2v_8 + \frac{5}{2}v_9 + \sum_{i=10}^{n-1} \frac{i-4}{2} v_i
\end{aligned}
$$

By (5) the right hand side is nonnegative, quod erat demonstrandum.

4   The Upper Bound: Proof outline of Theorem 2

We illustrate a family of maximal 2-planar graphs with $2n + c$ edges in Figure 1. The graphs can roughly be described as braided cylindrical grids where each layer consists of a cycle on ten vertices and every pair of consecutive layers have edges between them. The number of layers in the graph can be increased arbitrarily, and the gadget graph is attached to each of the green edges of the innermost and the outermost cycles. The graph is maximal 2-planar and has $2n + c$ edges, where $c = 350$. The details are deferred to the full version.
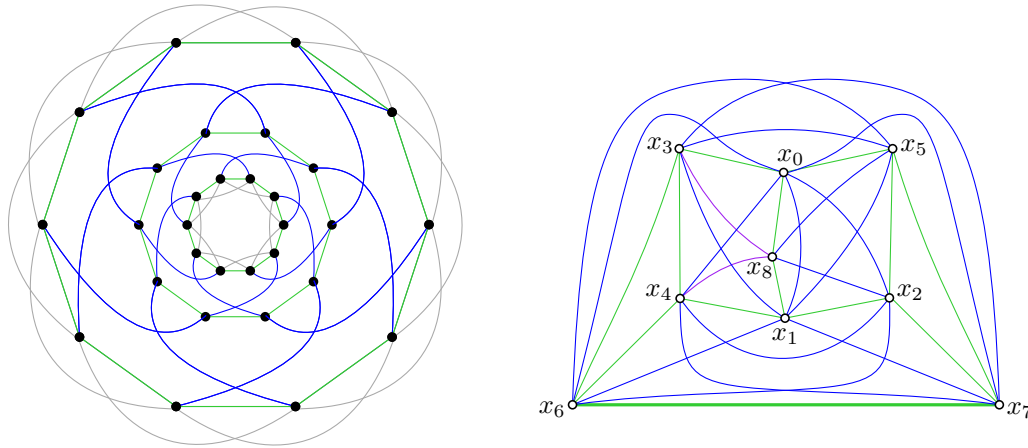

Figure 1 The layered graph (left); the gadget that we attach to the extreme green edges (right).

5   Conclusions

We have obtained tight bounds on the number of edges in maximal 2-planar graphs, up to an additive constant. Naturally, one would expect that our approach can also be applied to other families of near-planar graphs, specifically, to maximal 1- and 3-planar graphs. Intuitively, for k-planar graphs the challenge with increasing k is that the structure of the drawings gets more involved, whereas with decreasing k we aim for a higher bound.

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