# Hamiltonian Cycles and Matchings in 1-planar Graphs*

**Michael Hoffmann, Meghana M. Reddy, and Emanuel Seemann**

**Department of Computer Science, ETH Zürich, Switzerland,**
{hoffmann, meghana.mreddy}@inf.ethz.ch, emanuel.seemann@me.com

## Abstract

A graph is 1-*planar* if it can be drawn in the plane such that every edge has at most one crossing. A 1-planar graph is triangulated if it has a 1-plane drawing where every face is a triangle, i.e., every face contains either exactly three distinct vertices or exactly two distinct vertices and one crossing. We investigate the Hamiltonicity and matching properties of 1-planar graphs. We show that there always exists a Hamiltonian cycle with one prescribed edge in a 4-connected triangulated 1-planar graph. The result also holds for 4-connected maximal 1-planar graphs.

A connected graph with an even number of vertices is $k$-extendable if any matching of size $k$ can be extended to a perfect matching. We prove that all 4-connected triangulated 1-planar graphs are 1-extendable and all 5-connected triangulated 1-planar graphs are 2-extendable. We also construct an infinite family of 7-connected triangulated 1-planar graphs that have a unique 1-plane drawing.

## 1 Introduction

Planar graphs are graphs that can be drawn in the plane without any crossings. The class of 1-planar graphs was discovered by Ringel [8] in 1961 while trying to prove the 4-color theorem. A graph is 1-planar if it has a local crossing number of one, i.e., it can be drawn in the plane such that every edge has at most one crossing. Over time, these graphs found a new home as generalizations of planar graphs. Consequently, a sizeable part of research on 1-planar graphs is concerned with generalizing properties of planar graphs. For instance, while planar graphs on $n$ vertices can have at most $3n-6$ edges, 1-planar graphs on $n$ vertices can have at most $4n-8$ edges. Not all results generalize this well, probably since 1-planar graphs are NP-hard to recognize [6] while their planar counterparts can be recognized in polynomial time [2].

In this article, we show generalizations of results concerning Hamiltonian cycles and matching extensions in planar graphs to 1-planar graphs. For hamiltonicity, we generalize a result by Thomassen [9], which states that 4-connected planar graphs contain a Hamiltonian cycle through any prescribed edge $e$. A connected graph with an even number of vertices is *$k$-extendable* if any matching of size $k$ can be extended to a perfect matching. For matching extensions, we generalize results by Plummer [7] which showed that 4-connected planar graphs are 1-extendable and identified conditions under which they are 2-extendable.

A graph is called *triangulated* if there exists a drawing of the graph that is triangulated, i.e., every face is a triangle. A graph is called *internally triangulated* if there exists a drawing of the graph where every internal face is a triangle (which implies that the outer face is not necessarily a triangle). A planar (resp. 1-planar) graph is called *optimal* if it contains the maximum number of edges possible. A planar (resp. 1-planar) graph is called *maximal* if no edge can be added to the graph such that it remains planar (resp. 1-planar). A 1-plane

* M. Hoffmann and M. M. Reddy were supported by the Swiss National Science Foundation within the collaborative DACH project *Arrangements and Drawings* as SNSF Project 200021E-171681.

39th European Workshop on Computational Geometry, Barcelona, Spain, March 29–31, 2023.
This is an extended abstract of a presentation given at EuroCG'23. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

drawing $D$ is *locally-maximal* if for each crossing of two edges, the end vertices of these edges induce a $K_4$ in $D$. A graph is called locally-maximal 1-planar if it admits such a drawing. A 1-plane drawing $D$ is *near-optimal* if the subdrawing $H$ of $D$ induced by uncrossed edges contains faces of sizes three or four only, such that the vertices of faces of size four induce a $K_4$ in $D$, and no two triangular faces of $H$ share an edge. A graph is called near-optimal 1-planar if it admits such a drawing.

## 2 Hamiltonian cycles

Fabrici et al. [4] proved that every 4-connected locally-maximal 1-planar graph is Hamiltonian. Independently, Biedl [1] studied Hamiltonian cycles in 4-connected 1-planar graphs and showed that there exist 4-connected 1-planar graphs without a Hamiltonian cycle. However, she proved that 4-connected 1-planar graphs that are triangulated always contain a Hamiltonian cycle. Both the theorems are a generalization of a theorem by Tutte [10], which states that 4-connected planar graphs are Hamiltonian.

▶ **Theorem 1** ([1]). *Any 4-connected triangulated 1-planar graph $G$ has a Hamiltonian cycle.*

The main idea behind the proof by Biedl [1] is to pick a 4-connected triangulated 1-plane drawing of $G$, and for each crossing, one of the two edges involved in the crossing is removed. The edge that is removed is chosen such that the resulting graph remains 4-connected. Once we obtain a 4-connected plane graph, Tutte's theorem [10] can be applied to construct a Hamiltonian cycle.

We study the generalization of Theorem 1 where a prescribed edge $e$ must be part of the Hamiltonian cycle. For 4-connected planar graphs, this generalization of Tutte's theorem is known as Thomassen's theorem [9]. If the prescribed edge is one of the edges that is removed to obtain a 4-connected plane graph, then the proof technique of Biedl [1] cannot be used to prove Hamiltonicity. Nevertheless, we can still prove that there exists a Hamiltonian cycle that contains a prescribed edge $e$ of a 4-connected triangulated 1-planar graph.

▶ **Theorem 2.** *Let $G$ be a 4-connected triangulated 1-planar graph. Let $e$ be an arbitrary edge of $G$. Then there exists a Hamiltonian cycle in $G$ that contains $e$.*

We fix a triangulated 1-plane drawing of $G$. The main idea is to re-use Biedl's edge-removing strategy whenever possible. If the prescribed edge $e$ is not removed by Biedl's edge-removing strategy, the proof is trivial and follows immediately. In the case where the prescribed edge $e$ would be removed, we need to be more careful. In this case the edge $e$ must be involved in a crossing, say with the edge $f$. Instead of removing the edge $e$ we remove the edge $f$, and remove all the other edges as directed by Biedl's edge-removing strategy to obtain a graph $G'$. At this point we can no longer guarantee that $G'$ is 4-connected. However we can now guarantee that $G'$ is a maximal planar graph, so every 3-cut in $G'$ must be a separating triangle. Furthermore, we know that every separating triangle of $G'$ contains the edge $e$. This motivates the following definition.

▶ **Definition 3** (Bishop's hat). Let $G$ be a plane graph. A bishop's hat $H$ over the edge $e = (v_1, v_2)$ is a set of vertices $\{c_1, ..., c_k\}$ such that the triangle $(v_1, v_2, c_i)$ exists in $G$ and is separating for all $i \in [k]$.

We call the edge $e$ the brim of the bishop's hat and refer to each $c_i$ as a tip. Refer to Figure 1 for an illustration. For any plane drawing of a bishop's hat $H$ we say that $H$ is one-sided if there is a triangle $\Delta = (v_1, v_2, c_i)$ that contains all other separating triangles

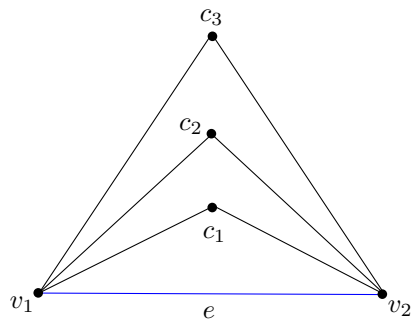


**Figure 1** A bishop's hat on 3 vertices with brim $e$.

of $H$ in its interior. Otherwise we say that $H$ is two-sided. For one-sided bishop's hats we label the tips $c_i$ in the order they appear under set inclusion, i.e. $c_i$ is the tip such that $\Delta = (v_1, v_2, c_i)$ contains all tips $c_j$ in its interior, where $1 \leq j < i \leq k$. For a two-sided bishop's hat, we can label the tips analogously on the two sides independently. We refer to the number of tips $k$ of the bishop's hat as the height of the bishop's hat. We also refer to $c_1$ as the lowest tip and $c_k$ as the highest tip for brevity.

Using this definition, we know that all separating triangles of $G'$ are contained in a bishop's hat $H$. We prove Theorem 2 using the following theorem for maximal planar graphs.

▶ **Theorem 4.** *Let $G$ be a maximal planar graph that is free of separating triangles apart from a bishop's hat of arbitrary height. Then $G$ contains a Hamiltonian cycle.In particular, one can construct a Hamiltonian cycle that goes through the brim $e$ of the bishop's hat.*

The proof of this theorem relies on tools developed by Whitney [11] and later by Chen [3].

▶ **Definition 5** ([3])**.** Let $G$ be an internally triangulated plane graph and let $A$ and $B$ be two vertices on the outer face of an internally triangulated drawing of $G$. $(G, A, B)$ is said to satisfy Whitney's condition if it satisfies both

- (W1) $G$ has no separating triangles, and
- (W2) for the two paths $(A = a_0, a_1, ..., a_m = B)$ and $(B = b_0, b_1, ..., b_n = A)$ along the outer face there are no chords of the type $(a_i, a_j)$ or $(b_i, b_j)$

▶ **Lemma 6** ([11])**.** *Let $G$ be an internally triangulated plane graph and let $A$ and $B$ be two vertices on the outer face of an internally triangulated drawing of $G$. If $(G, A, B)$ satisfies Whitney's condition, then there is a Hamiltonian path from $A$ to $B$.*

▶ **Theorem 7** ([3], Theorem 6)**.** *Let $G$ be a 4-connected maximal planar graph. Let $e$ and $f$ be two edges that lie on the same face of $G$. Then there is a Hamiltonian cycle that goes through both $e$ and $f$.*

Chen [3] proved the above theorem using Whitney's result detailed in Lemma 6. Using these tools we can prove the following lemma.

▶ **Lemma 8.** *Let $G$ be a maximal plane graph with a one-sided bishop's hat $H$. Let $c_i$, $c_{i+1}$ be two consecutive tips of $H$. Then there is path from $c_i$ to $c_{i+1}$ whose interior consists of all the vertices contained inside of the cycle $\{c_i, v_2, c_{i+1}, v_1\}$ in the plane drawing of $G$.*

We refer to the paths from Lemma 8 as local Hamiltonian paths. We prove a similar lemma for the lowest tip $c_1$ of the bishop's hat.

▶ **Lemma 9.** *Let $G$ be a maximal plane graph with a one-sided bishop's hat $H$. Let $c_1$ be the lowest tip of $H$. Then there is path from $v_2$ to $c_1$ whose interior consists of all the vertices contained inside of the cycle $\{v_1, v_2, c_1\}$ in the plane drawing of $G$.*

The proofs of Lemmata 8 and 9 are deferred to the full version.

**Proof of Theorem 4.** As we can freely choose which face of $G$ is considered as the outer face, it suffices to prove the theorem for one-sided bishop's hats.

Assume $G$ has a one-sided bishop's hat of height $k$. Let $c_k$ be the highest tip of this hat and $e = (v_1, v_2)$ the brim. We consider the subgraph $G'$ of $G$, which is obtained from $G$ by removing all vertices and edges contained strictly inside the bishop's hat. In $G'$ the triangle $\Delta = (v_1, v_2, c_k)$ is a facial triangle. $G'$ is also 4-connected, as the triangle $\Delta$ is no longer separating. Therefore we can apply Theorem 7 on the edges $e = (v_2, c_k)$ and $f = (v_1, v_2)$ and obtain a Hamiltonian cycle $C$ that contains both the edges $e$ and $f$, and all the vertices of $G'$. Inside the bishop's hat, we concatenate the local Hamiltonian paths obtained by applying Lemma 8 to every pair of consecutive tips, along with the path from $v_2$ to $c_1$ obtained by Lemma 9. We then create a Hamiltonian cycle in $G$ from $C$ by replacing the edge $(v_2, c_k)$ with the path constructed using the local Hamiltonian paths. This cycle contains the edge $e = (v_1, v_2)$. This construction can be seen in Figure 2. ◀

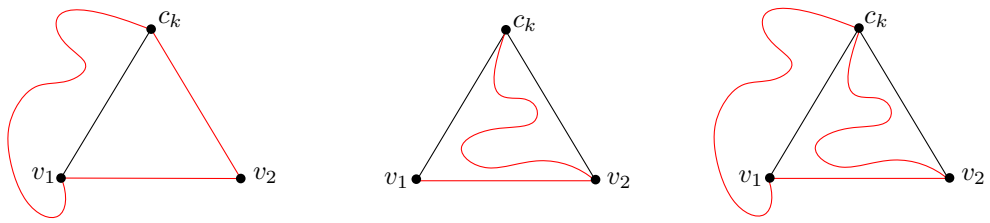


**Figure 2** Building the Hamiltonian cycle through $e = (v_1, v_2)$ in $G$ with a one-sided bishop's hat of height $k$. (left) the cycle $C$ given by Chen's Theorem containing all the exterior vertices and edges $(v_1, v_2)$ and $(v_2, c_k)$. (middle) the concatenated local Hamiltonian paths. (right) the cycle and path combined to obtain the desired Hamiltonian cycle.

**Proof of Theorem 2.** We start with a 4-connected triangulated 1-plane graph $G$. We apply Biedl's algorithm to every crossing that does not involve $e$ to get a 4-connected subgraph $G'$ of $G$. If $e$ is not crossed, $G'$ is plane. In this case we can directly apply Thomassen's Theorem to $G'$ to get a Hamiltonian cycle that contains $e$. If $e$ is crossed, we instead remove the edge that crosses $e$. This can potentially cause $G'$ to become 3-connected with a bishop's hat of arbitrary height with brim $e$. Note that all separating triangles of $G'$ are part of this bishop's hat by construction. We apply Theorem 4 to get a Hamiltonian cycle passing through $e$. ◀

We can further make the following observation about maximal 1-planar graphs, which then immediately implies Corollary 11. The proof is deferred to the full version.

▶ **Lemma 10.** *Let $G$ be a 4-connected maximal 1-planar graph. Then every 1-plane drawing of $G$ is triangulated.*

▶ **Corollary 11.** *Let $G$ be a 4-connected maximal 1-planar graph. Let $e$ be an arbitrary edge of $G$. Then there exists a Hamiltonian cycle in $G$ that contains $e$.*

## 3 Matching extendability

Theorem 2 immediately implies that for any given edge $e$ of a 4-connected triangulated 1-planar graph, there exists a perfect matching that contains $e$.

▶ **Corollary 12.** *Every 4-connected triangulated 1-planar graph of even order is 1-extendable.*

Further using Lemma 10, we get:

▶ **Corollary 13.** *Every 4-connected maximal 1-planar graph of even order is 1-extendable.*

Plummer [7] and Fujisawa et al. [5] studied 2-extendability of planar and 1-planar graphs. Plummer [7] proved the 2-extendability of 4-connected maximal planar graphs with forbidden substructures called *generalized butterflies*. Let $G$ be a graph, and let $e = (u, v)$, $f = (w, x)$ be two edges of $G$. If the graph $G \setminus \{u, v, w, x\}$ contains a component $C$ with an odd number of vertices, the induced subgraph $G[V(C) \cup \{u, v, w, x\}]$ is called a generalized butterfly. Fujisawa et al. [5] proved that any optimal 1-planar graph of even order with no generalized butterfly is 2-extendable. Since trivially 5-connected graphs do not contain generalized butterflies, both the papers were able to extend their results to 5-connected planar graphs and 5-connected optimal 1-planar graphs, respectively. We prove that 5-connected triangulated planar graphs are 2-extendable. It is not difficult to see that optimal 1-planar graphs are triangulated, thus, our result is a strengthening of Fujisawa et al.'s result. The proof of the theorem is deferred to the full version.

▶ **Theorem 14.** *Every 5-connected triangulated 1-planar graph of even order is 2-extendable.*

## 4 1-planar graphs with high connectivity

Biedl [1] constructed a class of 5-connected 1-plane graphs that are non-Hamiltonian. Interestingly, these graphs contained a matching of size $\lfloor \frac{n}{2} \rfloor - 1$, which are one edge away from being a perfect matching. This motivated Biedl [1] to question whether 1-planar graphs with higher connectivity are Hamiltonian. In particular, if we can prove that 6- or 7-connected 1-planar graphs are always triangulated, then we can use techniques similar to Theorem 2 to prove hamiltonicity for these graphs. Unfortunately, we discovered that there exist 7-connected 1-planar graphs that are neither maximal nor triangulated. The counterexample suggests that new tools that do not depend on a triangulated drawing are required to address the problem of Hamiltonicity of 1-planar graphs with high connectivity.

▶ **Theorem 15.** *For each $k \in \mathbb{N}$, there exists a 7-connected triangulated near-optimal 1-planar graph with $n = 24 + 8k$ vertices. If $k \geq 3$, this graph is maximal 1-planar.*

▶ **Theorem 16.** *There exist 7-connected 1-planar graphs that are neither maximal nor triangulated.*

To this end, we first prove that there exist many 7-connected triangulated near-optimal 1-planar graphs. We then show that these graphs have many edges that can be removed such that the resulting graph remains 7-connected. We briefly describe the construction of the graph. We start with the *double stop-sign* graph given by Fabrici and Madaras, illustrated in Figure 3. Using computer-based testing, we verified that the double stop-sign graph is 7-connected. We then increase the number of 8-cycles around the central $K_4$. The resulting graph is called a *k-layered double stop-sign* graph, where $k$ refers to the number of newly added 8-cyles. A 1-layered double stop-sign graph is illustrated in Figure 3. We can prove the

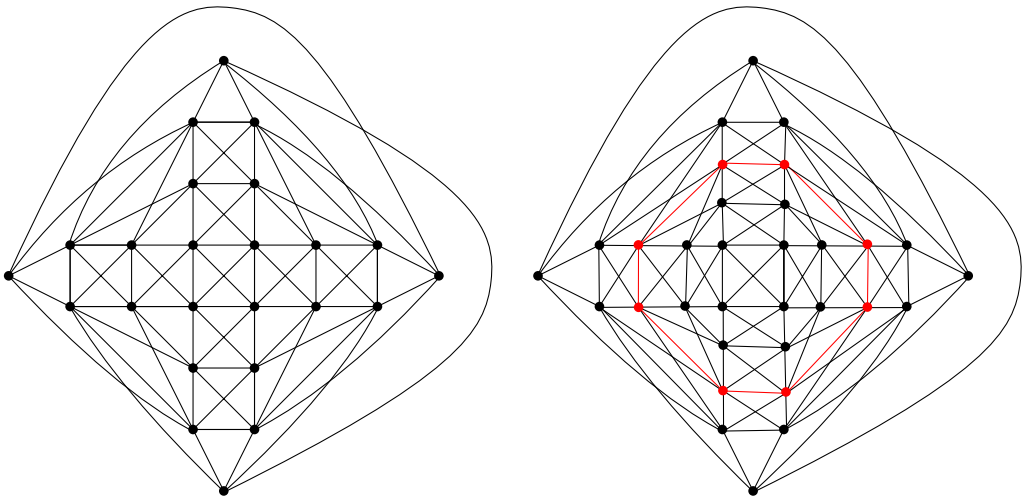

**Figure 3** (left) The double stop-sign graph. (right) The 1-layered double stop-sign graph, where the new layer is depicted in red.

7-connectedness of the new graph by using the fact that the double stop-sign is 7-connected and thus there must exist seven internally vertex-disjoint paths between any two vertices, which can be extended to obtain seven new paths in the new graph. Finally, we can observe that one of the diagonal edges between two consecutive layers can be removed, and the graph remains 7-connected, thereby proving Theorem 16.

## 5 Conclusion

There are quite a few properties of 1-planar graphs that still need to be investigated. Based on our work, we list the most interesting open questions below.

▶ **Open Question 17.** *Let $G$ be a 4-connected triangulated 1-planar graph that contains no generalized butterflies. Is $G$ then 2-extendable?*

▶ **Open Question 18.** *For a 4-connected triangulated 1-planar graph, is there a Hamiltonian cycle through any two of its edges?*

▶ **Open Question 19.** *Are 6- or 7- connected 1-planar graphs Hamiltonian?*

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