A Lower Bound on the Number of Triangulations of Planar Point Sets *

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Abstract

We show that the number of straight-edge triangulations exhibited by any set of n points in general position in the plane is bounded from below by $\Omega(2.33^n)$.

1 Introduction

A triangulation of a finite planar point set S is a maximal non-crossing straight-edge graph with vertex set S. Efficiently counting the number of triangulations of S is an intriguing open problem. The currently fastest method is based on the recent concept of triangulation path [1], which follows a divide and conquer approach. But still the running time shows exponential growth and in general computations are limited to $n \leq 40$ where n denotes the cardinality of S. It is an open problem whether it is possible to count the number of triangulations of a given point set in polynomial time. A well known method to enumerate all triangulations of S is based on the reverse-search technique of Avis and Fukuda [8]. Here the running time is at least proportional to the number of triangulations of S.

So far no tight asymptotic upper or lower bounds for the number of triangulations of point sets of fixed cardinality are known. On maximizing the number of triangulations there exist point sets with as much as $8^{n-\Theta(\log(n))}$ triangulations [14, 17]. The currently best upper bound is much larger, although it has recently been improved from approximately 256^n [9] to $59^{n-\Theta(\log(n))}$, see [17].

In the opposite direction we conjecture that, for fixed cardinality n, the minimum number of triangulations is always obtained by a special structure,

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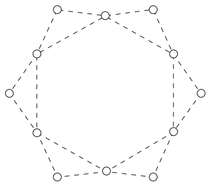

Figure 1: Double circle, conjectured to minimize the number of triangulations. 3-segments are dashed.

the so-called double circle, see Figure 1. The double circle contains $h = \lfloor n/2 \rfloor$ extreme points forming a regular h-gon. The remaining $\lceil n/2 \rceil$ interior points are placed sufficiently close to the edges of the h-gon, such that the set of interior edges, that are not crossed by any other edge, forms a star shaped region. (If n is odd an additional interior point can be placed in a way that the number of triangulations is still minimized, i.e., the double circle is well defined in this case, too. See [11] for a detailed discussion.) By an inclusion-exclusion argument ([15, 11]) the number of triangulations of the double circle can be shown to be $\sum_{i=0}^{h}(-1)^{h-i}\binom{h}{i}C_{h-2+i}$, where $C_n = \Theta(4^n n^{-3/2})$ denotes the n-th Catalan number. Asymptotically the sum gives $\sqrt{12}^{\,n-\Theta(\log(n))}$ [17] and thus the double circle constitutes the first known structure with $o(C_n)$ triangulations. From exhaustive computations [5, 19] it is known that the double circle is the only point set (i.e., order type) which minimizes the number of triangulations for $n \leq 11$.

These results have to be seen in contrast to related structures, where more information has already been obtained. For example it is known that the number of crossing-free perfect matchings as well as the number of crossing-free spanning trees is minimized by point sets in convex position [14]. For special point sets (so-called wheels) an interesting relation between the number of triangulations and the number of pointed pseudo-triangulations is given by Randall et al. [16]. Most recently it has been shown that, in fact, point sets in convex position also minimize the number of pointed pseudo-triangulations [3].

Surprisingly, up to now no good general lower bound on the number of triangulations is known, although it is commonly assumed that there are 'always exponentially many' triangulations. In this paper we quantify this 'common assumption'. More precisely let $t(n)$ be the minimum number of triangulations that every set of n points in the plane[1] exhibits. In [13] it is shown that any triangulation on n points contains at least $(n-4)/6$ edges that can be flipped simultaneously. This immediately yields a first lower bound $t(n) \geq 2^{(n-4)/6} \geq 1.12^n$.

We will present a general scheme based on induction, leading to the inequality $t(n) \geq c \cdot \tau^n$ for constants $c > 0$ and $\tau > 1$. The best value for τ achieved up to now is $\tau = 2.33$ for sufficiently large n. The main result of this paper can be stated as follows.

Theorem 1 *Let $t(n)$ denote the least number of straight-edge triangulations of any set of n points in general position in the plane. Then $t(n) \geq 0.092 \cdot 2.33^n$ for $n \geq 1212$.*

[1] All point sets considered in this paper are assumed to be in general position, i.e., no three points are collinear. Note that the general position assumption is crucial to avoid trivialities.

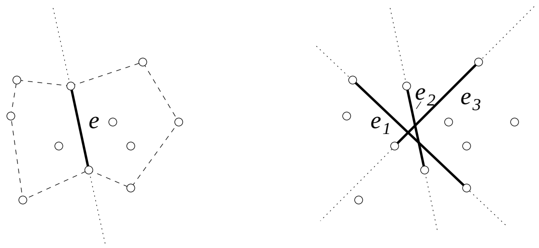

Figure 2: (a) Segment e splitting the point set into subsets of cardinalities 6 and 7. (b) Three pairwise crossing segments splitting 6:7, 6:7, and 5:8.

The proof of the theorem will be given in the remaining sections. To the knowledge of the authors this constitutes the first non-obvious lower bound.

2 Recurrence Relations

Throughout this paper, let S be a set of n points in the plane in general position with h extreme points. Let $E(S)$ be the set of interior edges (straight-line segments) spanned by points in S, that is, the set of edges spanned by S without the h edges forming the boundary of the convex hull of S. Two edges of $E(S)$ are said to cross each other if they properly intersect in their interior.

Let $t(S)$ be the number of triangulations of S and for $e \in E(S)$ let $t_e(S)$ be the number of triangulations of S that contain the edge e. Moreover let S'_e and S''_e be the two subsets of S contained in the two closed halfplanes bounded by the straight-line supported by e. Because e is an interior edge we get $3 \leq |S'_e|, |S''_e|$ and symmetrically $|S'_e|, |S''_e| \leq n-1$, where $|A|$ denotes the cardinality of a point set A. Note that since the two points spanning e are counted in both subsets we have $|S'_e| + |S''_e| = n + 2$, see Figure 2.

Obviously $t(S) \geq t(|S|)$ and $t(S) \geq t_e(S) \geq t(S'_e) \cdot t(S''_e) \geq t(|S'_e|) \cdot t(|S''_e|)$. For any set S of cardinality $n \geq 4$ and any given n' with $3 \leq n' \leq n-1$ we can find an appropriate segment $e \in E(S)$ such that $|S'_e| = n'$ and $|S''_e| = n + 2 - n'$. Therefore we get

Lemma 1 *For $n \geq 4$ and all $n', n'' \geq 3$ with $n' + n'' = n + 2$ we have $t(n) \geq t(n') \cdot t(n'')$.*

A solution of the recurrence relation of Lemma 1 is given by

$$t(n) \geq \frac{1}{\tau^2} \cdot \tau^n \text{ for } n \geq 4 \tag{1}$$

for any constant $\tau > 1$. The inequality of Lemma 1 is rather loose, since only triangulations including a certain edge are considered. A generalization of Lemma 1 can be achieved by the following observation, cf. Figure 2 (b). Assume that $e', e'' \in E(S)$ are two mutually crossing edges and let $T_{e'}(S)$ and $T_{e''}(S)$ be the sets of triangulations of S including e' and e'', respectively. Then $T_{e'}(S) \cap T_{e''}(S) = \varnothing$, i.e., no triangulation of S belongs to both sets. Thus if $E(S)$ contains $k \geq 2$ pairwise crossing edges, also called a *crossing family* of size k (k-family for short), we can apply the recurrence of Lemma 1 k times, but have no longer control over the cardinalities of the resulting subsets.

Lemma 2 *Let S be a set of n points which admits a set of $k \geq 2$ pairwise crossing edges. Then there exist values n_i', n_i'', $1 \leq i \leq k$, with $n_i', n_i'' \geq k+1$ and $n_i' + n_i'' = n+2$ such that $t(n) \geq \sum_{i=1}^{k} t(n_i') \cdot t(n_i'')$.*

Note that the lower bound on n_i', n_i'' stems from the fact that any involved segment is crossed by at least $k-1$ other segments, giving $k-1$ points on each side of its supporting line.

Solving the above recurrence relation gives

$$t(n) \geq \frac{1}{k\tau^2} \cdot \tau^n \text{ for } n \geq 2k. \tag{2}$$

Asymptotically this is similar to equation (1), but for the task of determining τ for small instances this leads to an improvement as will be pointed out in the following subsection. Moreover equation (1) can be seen as the case $k = 1$ of equation (2). Note that k pairwise crossing segments are only needed for sets of cardinality *larger than* the induction base, not for instances of the induction base itself.

2.1 Getting started

To make use of equation (2) we need to determine both, the range of the induction base and the value of τ the recurrence relation (2) can 'start' with. Let us first assume a fixed range for the induction base, say $a \leq n \leq b$. Assume that within this range a lower bound for $t(n)$, denoted by $t^-(n)$, is given, that is, $t(n) \geq t^-(n)$ for $a \leq n \leq b$. From equation (2) we can compute a lower bound for τ for some fixed cardinality n, denoted by $\tau(n)$, by $\tau(n) \geq (k \cdot t^-(n))^{\frac{1}{n-2}}$. Thus from the induction base we get

$$\tau \geq \min_{a \leq n \leq b} (k \cdot t^-(n))^{\frac{1}{n-2}}. \tag{3}$$

So it remains to determine a suitable range $a \leq n \leq b$ for the induction base. Intuitively speaking it should be clear that we have to avoid point sets of too small cardinality, since they exhibit only very few triangulations. Table 1 in the next section provides concrete values supporting this observation. In other words, for an edge e used in the relation (2) we have to guarantee that $|S_e'|$ and $|S_e''|$ can be bounded from below. To this end for $2 \leq \ell \leq (n+2)/2$ we call e an *ℓ-segment* if $\min\{|S_e'|, |S_e''|\} = \ell$. In a similar way we define an ℓ^+-segment if $\min\{|S_e'|, |S_e''|\} \geq \ell$. Note that our definition of ℓ-segments differs from the standard definition of k-set edges [10] or j-edges [6] in two ways. On one hand we also count the points which span our segment and on the other hand ℓ always corresponds to the subset with the smaller cardinality.

Summarizing the obtained results we get the following central theorem.

Theorem 2 *Let $a < b$ and k be integers. If for every n with $a \leq n \leq b$ we have $t(n) \geq \frac{1}{k\tau^2} \cdot \tau^n$, and if for every set of $n > b$ points there exists a crossing family of size k entirely consisting of a^+-segments, then for all $n \geq a$ the bound $t(n) \geq \frac{1}{k\tau^2} \cdot \tau^n$ holds.*

To make use of Theorem 2 we have to guarantee that all sets of cardinality larger than b contain k pairwise crossing a^+-segments. For $k = 1$ this is obviously the case for $b \geq 2a - 2$. For $k = 2$ we get the bound from the following Lemma.

Lemma 3 *For a point set S of cardinality $n \geq 9$ there always exist two crossing $\lceil \frac{n}{2} - 2 \rceil^+$-segments.*

Proof. For a point p of S consider the set of segments from $E(S)$ having p as one endpoint. Sort these segments by the circular order of their supporting lines around p. By continuity it follows that p exhibits a (halving) $\frac{n+2}{2}$-segment if n is even and two (neighboring in the circular order) $\frac{n+1}{2}$-segments if n is odd.

For even n we take the $\frac{n+2}{2}$-segment together with the next three segments clockwise and counterclockwise, respectively. We thus obtain a set of seven ℓ^+-segments for p with $\ell \geq \frac{n}{2} - 2$. Similarly for odd n we take two neighbors for each $\frac{n+1}{2}$-segment (chosen in circular order opposite to the other $\frac{n+1}{2}$-segment) and get a set of six ℓ^+-segments with $\ell \geq \frac{n+1}{2} - 2$.

Repeating this process for all points p of S we get a set of $7n/2$ ℓ^+-segments for even n ($6n/2$ for n odd), too much for this set to be planar. Thus we must have two crossing $\lceil \frac{n}{2} - 2 \rceil^+$-segments. $\square$

From Lemma 3 it follows that for 2-families the base has to cover a range of $a, \ldots, b = 2a + 2$. For $k \geq 3$ pairwise crossing segments the situation is more involved.

Lemma 4 *For integers k and ℓ, $3 \leq k < \ell$, let $c(k)$ be the smallest number such that for any set of size $c(k)$ a crossing family of size k exists. Then for any set S of size $n \geq c(k) + 2\ell^2 - 2\ell k - 5\ell + 3k + 3$ there exists a crossing family of size k entirely consisting of ℓ^+-segments.*

Proof. Any convex subset of S of size $2\ell - 2$ contains a crossing family of size $\ell - 1$ which entirely consists of ℓ-segments . So assume that all convex subsets have size at most $2\ell - 3$.

Consider the $\ell - 1 - k$ outermost convex layers of S. We obtain them by iteratively taking the extreme points of S, removing them, taking the extreme points of the remaining set and so on. Repeat this $\ell - 1 - k$ times. In this way we get an onion-like structure which consists of at most $(2\ell - 3)(\ell - k - 1) = 2\ell^2 - 2\ell k - 5\ell + 3k + 3$ points. The remaining set S_c of at least $c(k)$ points exhibits a crossing family of size k, each segment having at least $k + 1$ points of S_c on each side of its supporting line. In addition, each of the $\ell - k - 1$ convex layers (note that they are 'around' S_c) adds at least one point on both sides of a segment, thus all segments are $(k + 1) + (\ell - k - 1)^+ = \ell^+$-segments. $\square$

To determine bounds for $c(k)$ several known relations might be used. For example we could take the Erdos-Szekeres Theorem on convex sets [18]: among any $\binom{2m-5}{m-2} + 2$ points there are at least m points in convex position, providing a crossing family of size $\lfloor \frac{m}{2} \rfloor$, that is, $c(\lfloor \frac{m}{2} \rfloor) \leq \binom{2m-5}{m-2} + 2$. This gives $c(3) \leq 37$. In [7] the existence of crossing families of size $\sqrt{n/12}$ for every set of n points is proven. For $k = 3$ this gives the weaker bound $c(3) \leq 108$ but for larger k it is superior to the bounds in [18]. In [5] it is shown that every set S of $n \geq 10$ points admits a crossing family of size 3. Since there exist sets of 9 points without 3-families we have $c(3) = 10$. From Lemma 4 we thus obtain for 3-families:

Lemma 5 *For any set S of size $n \geq 2\ell^2 - 11\ell + 22$ there exists a crossing family of size 3 entirely consisting of ℓ^+-segments.*

We summarize the discussion on k-families for $k \leq 3$ in the following corollary. It can be seen that there is a tradeoff between the size of the base range and the obtainable base of the exponential.

Corollary 1

$$\text{If for} \left\{ \begin{array}{c} a \leq n \leq 2a-2 \\ a \leq n \leq 2a+2 \\ a \leq n \leq 2a^2 - 11a + 21 \end{array} \right\} \text{the bound} \left\{ \begin{array}{c} t(n) \geq \tau^{n-2} \\ t(n) \geq \frac{1}{2}\tau^{n-2} \\ t(n) \geq \frac{1}{3}\tau^{n-2} \end{array} \right\} \text{holds,}$$

then this bound holds for all $n \geq a$.

3 Induction Base

Let us again stress the fact that we are here in the remarkable situation that any improvement of lower bounds for the number of triangulations for small sets will yield an improvement in the asymptotics of lower bounds for $t(n)$. Therefore the present and the next section are devoted to derive a good induction base.

For $n \leq 11$, the minimum number of triangulations can be determined exactly by counting them in an exhaustively way for each possible order type. A data base for all order types realizable as point sets in the plane has recently been developed, see [2] for details. A similar project has been carried out in [12], however they did not obtain realizations of the order types as point sets nor did they exclude non-realizable ones. Thus their results seem less suitable for applications like counting triangulations.

n	$t(n)$	$k=1$	$k=2$	$k=3$	$k=4$
3	1	1.00000	2.00000	3.00000	4.00000
4	1	1.00000	1.41421	1.73205	2.00000
5	2	1.25992	1.58740	1.81712	2.00000
6	4	1.41421	1.68179	1.86121	2.00000
7	11	1.61539	1.85560	2.01235	2.13153
8	30	1.76273	1.97860	2.11693	2.22091
9	89	1.89882	2.09647	2.22149	2.31469
10	250	1.99408	2.17456	2.28761	2.37137
11	776	2.09457	2.26226	2.36651	2.44338

Table 1: Values of $\tau(n)$ for small instances and $k = 1, \ldots, 4$.

The second column of Table 1 for values of $n \leq 10$ is taken from [2, 5] and is extended by the recently obtained result for $n = 11$ [19]. The column shows exact lower bounds on the number of triangulations of n points, that is, the value of $t(n)$, for $n = 3, \ldots, 11$. The table reflects the known fact that n points in convex position, whose number of triangulations is given by the Catalan numbers C_{n-2}, do not lead to the minimum. This is in contrast to other structures like e.g. crossing-free matchings, crossing-free spanning trees [14], and pointed pseudo-triangulations [3] where this happens to be true. For $n > 11$ the currently best examples minimizing (and maximizing, respectively) the numbers of triangulations can be found on the web [19].

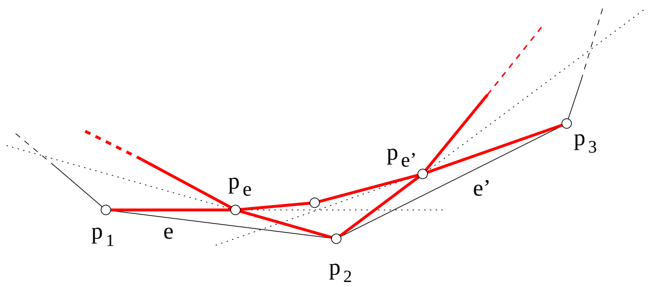

Figure 3: Proof of Lemma 6: All 3-segments are drawn in red (bold).

The remaining entries of Table 1 give lower bounds for $\tau(n)$ obtained by $\tau(n) \geq (k \cdot t^{-}(n))^{\frac{1}{n-2}}$, cf. the discussion for equation (3). Note that entries $n < 2k$ still make sense since k-families are only required for sets larger than the induction base.

4 Extended Induction Base

In the following we give some recursive relations for $t(n)$ to extend the induction base beyond $n = 11$. Let us point out that none of these relations leads to a direct improvement of the asymptotics of the lower bound for $t(n)$. Instead they are used to compute concrete values to bound $t(n)$ for (small, constant) values of $n > 11$.

From Lemma 1 we immediately get

$$t(n) \geq \max_{3 \leq i \leq n-1} \{t(i) \cdot t(n+2-i)\} \text{ for } n \geq 4 \tag{4}$$

For the next relation we first need a lemma on the cardinality of the subsets of two crossing edges.

Lemma 6 *For any set S of $n \geq 6$ points there always exist two mutually crossing edges which are either two 3-segments or one 3- and one 4-segment.*

Proof. If two crossing 3-segments exist we are done, so for the remaining proof suppose that no two 3-segments cross.

Let $L(S)$ denote the second convex layer of S, that is, the set of points and segments of the boundary of the convex hull of the interior points of S. We first claim that all segments of $L(S)$ are 3-segments.

Let $e = (p_1, p_2)$ be an edge of the boundary of the convex hull of S. Then the two segments from $E(S)$ emanating from p_1 and p_2, respectively, minimizing the angles to e are 3-segments. Since no two 3-segments cross they must have an endpoint p_e in common, i.e., together with e they form a triangle. Similarly for a convex hull edge $e' = (p_2, p_3)$ adjacent to e we get a point $p_{e'}$. Note that $p_{e'} \neq p_e$ since $n \geq 6$, and that both points belong to $L(S)$, cf. Figure 3. If the segment $(p_e, p_{e'})$ is also part of $L(S)$ then it is a 3-segment, since on one side of $(p_e, p_{e'})$ there lies only p_2. Otherwise with the same argument the convex chain of $L(S)$ between p_e and $p_{e'}$ entirely consists of 3-segments, proving our claim.

In a similar way as above consider now the two adjacent angle-minimizing 4-segments of e. Since the two 3-segments of e have an endpoint in common it follows that these two 4-segments either cross each other or also have an endpoint in common. In both cases at least one of them has to cross an edge of the second convex layer $L(S)$, i.e., we have a crossing pair consisting of one 3- and one 4-segment. $\square$

The supporting line of a 3-segment splits S in a way that the subset with larger cardinality contains $n-1$ points. Analogously we get a subset of $n-2$ points for a 4-segment. Since $t(n-1) \geq t(n-2)$ Lemma 6 immediately leads to a Fibonacci-like relation:

$$t(n) \geq t(n-1) + t(n-2) \text{ for } n \geq 6 \tag{5}$$

Note that from the existence of two crossing 3-segments we could directly derive $t(n) \geq 2 \cdot t(n-1)$ and thus $t(n) \geq 2^n$. However, such structures need not always exist, consider the double circle of Figure 1 for an example. Surprisingly up to now no simple proof for $t(n) \geq 2 \cdot t(n-1)$ seems to be known.

Next consider $\mathcal{P}$, the number of pairs (e, Δ), where Δ is a triangulation of S, and $e \in E(S)$ is an interior edge of Δ. Since every triangulation of S contains exactly $3n-2h-3$ interior edges we get $\mathcal{P} = (3n-2h-3) \cdot t(S)$. Counting the number of pairs of $\mathcal{P}$ in an edge-based way leads to $\mathcal{P} = \sum_{e \in E(S)} t_e(S)$, yielding

$$t(S) = \frac{\sum_{e \in E(S)} t_e(S)}{3n-2h-3}. \tag{6}$$

Our goal is now to find a general lower bound for the sum $\sum_{e \in E(S)} t_e(S)$ which implies a lower bound for $t(S)$ and thus $t(n)$, respectively.

For a point $p \in S$ let E_p be the edges in $E(S)$ that are incident to p. We get $\sum_{e \in E(S)} t_e(S) = \frac{1}{2} \sum_{p \in S} \sum_{e \in E_p} t_e(S)$. If p is one of the h extreme points of S, then analogously as for equation (4) we get

$$\sum_{EXT} := \sum_{e \in E_p} t_e(S) \geq \sum_{i=3}^{n-1} t(i) \cdot t(n+2-i) \text{ for } n \geq 4.$$

If p is an interior point of S the situation is more involved. To simplify the argumentation assume that $|S|$ is even, the case where $|S|$ is odd can be handled similarly.

Order the edges of E_p cyclically around p by supporting line, and number them in this order by $e_{-(n/2-1)}, \ldots, e_{-1}, e_0, e_1, \ldots, e_{n/2-1}$ with $e_0 \in E_p$ some edge with $|S'_{e_0}| = |S''_{e_0}| = \frac{n}{2}+1$. The existence of e_0 follows from continuity in the cyclical ordering. For the same reason for each edge e_i, $-(n/2-1) \leq i \leq n/2-1$, we get $\max\{|S'_{e_i}|, |S''_{e_i}|\} \leq \min\{\frac{n}{2}+1+|i|, n-1\}$ and $\min\{|S'_{e_i}|, |S''_{e_i}|\} \geq \max\{\frac{n}{2}+1-|i|, 3\}$. From this we conclude for $n \geq 4$ even

$$\sum\nolimits_{INT} := \sum_{e \in E_p} t_e(S) \geq t(\frac{n}{2}+1)^2 +$$

$$+2 \cdot \sum_{i=1}^{\frac{n}{2}+1} \min_j \left\{ t(\frac{n}{2}+1+j) \cdot t(\frac{n}{2}+1-j) \mid 0 \leq j \leq \min\{i, \frac{n}{2}-2\} \right\}$$

The first term is related to e_0. Each element of the sum bounds the number of triangulations for edges $e_{\pm i}$ for $i = 1, \ldots, \frac{n}{2} - 1$. In a similar way we get

$$\sum_{INT} \geq 2 \cdot \sum_{i=0}^{\frac{n-3}{2}} \min_j \left\{ t(\frac{n+3}{2} + j) \cdot t(\frac{n+1}{2} - j) \mid 0 \leq j \leq \min\{i, \frac{n-5}{2}\} \right\}$$

for $n \geq 5$ odd.

Combining the cases of extremal and interior points, respectively, together with equation (6) we get the lower bound

$$t(S) \geq \left(h \cdot \sum_{EXT} + (n-h) \cdot \sum_{INT} \right) / (2(3n - 2h - 3)).$$

Since $\sum_{EXT} \geq \sum_{INT}$ this expression is minimized when h is minimized, that is for $h = 3$, yielding $t(S) \geq (3 \cdot \sum_{EXT} + (n-3) \cdot \sum_{INT})/(6n - 18)$. Since this is true for any set S of cardinality n we finally get

$$t(n) \geq \left\lceil \left(3 \cdot \sum_{EXT} + (n-3) \cdot \sum_{INT} \right) / (6n - 18) \right\rceil \text{ for } n \geq 4. \tag{7}$$

To see that this is indeed a recursive inequality observe that $\sum_{EXT}$ and $\sum_{INT}$ denote expressions that involve values of $t(k)$ only for $k < n$.

n	$t(n) \geq$	eqn	$k = 1$	$k = 2$	$k = 3$	$k = 4$
12	1 026	(5)	2.000	2.144	2.233	2.298
13	1 802	(5)	1.977	2.105	2.185	2.242
14	2 828	(5)	1.939	2.055	2.125	2.177
15	6 423	(7)	1.963	2.070	2.136	2.184
16	14 560	(7)	1.983	2.084	2.145	2.190
17	35 015	(7)	2.009	2.104	2.162	2.203
18	81 947	(7)	2.028	2.118	2.172	2.212
19	200 206	(7)	2.050	2.136	2.187	2.225
20	602 176	(4)	2.095	2.177	2.226	2.262
21	$1.0 \cdot 10^6$	(7)	2.071	2.148	2.194	2.227
22	$2.1 \cdot 10^6$	(7)	2.071	2.144	2.188	2.220
23	$4.9 \cdot 10^6$	(7)	2.082	2.152	2.194	2.224
24	$1.1 \cdot 10^7$	(7)	2.089	2.156	2.196	2.225
25	$2.2 \cdot 10^7$	(7)	2.086	2.150	2.188	2.216
26	$4.1 \cdot 10^7$	(7)	2.076	2.137	2.173	2.199
27	$9.5 \cdot 10^7$	(7)	2.086	2.144	2.179	2.204
28	$2.2 \cdot 10^8$	(7)	2.095	2.151	2.185	2.209
29	$5.0 \cdot 10^8$	(7)	2.101	2.155	2.188	2.211
30	$1.1 \cdot 10^9$	(7)	2.105	2.158	2.189	2.212
31	$2.7 \cdot 10^9$	(7)	2.115	2.166	2.197	2.219
32	$6.4 \cdot 10^9$	(7)	2.124	2.173	2.203	2.224
33	$1.5 \cdot 10^{10}$	(7)	2.132	2.181	2.209	2.230
34	$3.5 \cdot 10^{10}$	(7)	2.136	2.183	2.211	2.231
35	$8.3 \cdot 10^{10}$	(7)	2.143	2.188	2.215	2.235
36	$1.9 \cdot 10^{11}$	(7)	2.149	2.193	2.220	2.239
37	$4.5 \cdot 10^{11}$	(7)	2.154	2.197	2.222	2.241
38	$9.5 \cdot 10^{11}$	(7)	2.152	2.193	2.218	2.236
39	$2.1 \cdot 10^{12}$	(7)	2.155	2.196	2.220	2.238
40	$4.9 \cdot 10^{12}$	(7)	2.158	2.198	2.221	2.238

Table 2: Extended induction base: lower bounds for $t(n)$ and $\tau(n)$, $k = 1, ..., 4$.

Formulas (4), (5) and (7) are used to compute lower bounds on the number of triangulations for constant values of $n \geq 12$. For any n the maximum among the obtained values is taken. The results for $n \leq 40$ are shown in Table 2 together with the reference number of the equations used to derive the bound, and values of $\tau(n)$ for $k = 1, ..., 4$. For $n > 40$ the best bounds for $t(n)$ are always obtained from relation (7). From Table 2 we see that for $k = 2$ and a base range of $17, \ldots, 36$ we get $\tau \geq 2.1$. Extending Table 2 and fixing $k = 2$ we get $\tau \geq 2.2$ for a base range of $41, \ldots, 84$ and $\tau \geq 2.3$ for a base range of $231, \ldots, 464$. Finally a base of range $1212, \ldots, 2426$ and using 2-families provides $\tau \geq 2.330037$. Together with Corollary 1 this proves Theorem 1. For even larger ranges of the induction base, the base of the exponential seems to converge to a value less than 2.34 (the best value we got so far is $\tau \approx 2.33817$ for a base of $6635, \ldots, 13272$ with $k = 2$).

5 Discussion

Using the results of Sections 3 and 4, the extended induction base might be further expanded, e.g. up to cardinalities of several millions. On the other hand crossing families of size 4 or more might be considered, again leading to a larger induction base. Both would significantly increase the computational complexity of the approach. For example to obtain a lower bound of $\tau \geq 2.2$ using 4-families the induction base would already have a range of $27, \ldots, 1313$, cf. Table 2. Although for small values of n crossing families of size larger than two seem to be promising, see Table 2, computer investigations showed no improvements for $k > 2$. This is due to the large ranges of the induction base required for $k > 2$. In fact for $k = 1$ the numerical results are only slightly worse than the results for 2-families. Using 3-families the best result has been $\tau \geq 2.2249$ for a base range of $42, \ldots, 3087$.

Most promising lines of attack to improve on τ are to determine the exact lower bounds for sets of size $n = 12, 13, ...$ or to show the existence of k-families for $k \geq 4$ consisting of ℓ-segments for relatively small point sets. Together with an improvement of Lemma 4 this would help to avoid huge induction base ranges.

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